Journal of Algebraic Combinatorics **9** (1999), 271–280

An Adjacency Criterion for Coxeter Matroids

ALEXANDRE BOROVIK
Department of Mathematics, UMIST, P.O. Box 88, Manchester M60 1QD, UK

ANDREW VINCE
Department of Mathematics, University of Florida, Gainesville FL 32611

Received October 8, 1996

Abstract. A coxeter matroid is a generalization of matroid, ordinary matroid being the case corresponding to the family of Coxeter groups A_n, which are isomorphic to the symmetric groups. A basic result in the subject is a geometric characterization of Coxeter matroid in terms of the matroid polytope, a result first stated by Gelfand and Serganova. This paper concerns properties of the matroid polytope. In particular, a criterion is given for adjacency of vertices in the matroid polytope.

Keywords: matroid, Coxeter matroid, Coxeter group, matroid polytope

1. Introduction

This paper continues a series of investigations [2, 3, 6–8, 10, 19] devoted to the systematic development of the theory of Coxeter matroids. The main result of the present paper (Theorem 1.2) concerns a geometric characterization of Coxeter matroids. It is used in the subsequent paper [9] and has inspired the rather unexpected results of [4].

Let W be a finite Coxeter group, P a standard parabolic subgroup in W, and $\leq$ the Bruhat ordering on the factor set W/P. For definitions concerning Coxeter groups and complexes, the representation of Coxeter groups as reflection groups, and Bruhat ordering, refer to [15] or [16]. For each element $w \in W$ define the w-Bruhat ordering $\leq^w$ of W/P by setting $A \leq^w B$ if $w^{-1}A \leq w^{-1}B$. A subset $\mathcal{M} \subseteq W/P$ is called a *Coxeter matroid* (for W and P) if it satisfies the following maximality property.

Maximality Property. For every $w \in W$ the set $\mathcal{M}$ contains a unique element A maximal with respect to the w-Bruhat ordering on W/P.

This means that $B \leq^w A$ for all $B \in \mathcal{M}$. The elements of a Coxeter matroid are referred to as *bases*.

Coxeter matroids were introduced (under the name of *WP*-matroids) by Gelfand and Serganova [13, 14]. The motivation for the definition is that when $W = A_n$ and P is a maximal parabolic subgroup, Coxeter matroid is equivalent to the classical concept of a matroid. Non-maximal parabolic subgroups P yield flag matroids and gaussian greedoids [13, 14]. Wenzel [22] has shown that the case $W = B_n$, with a particular choice of the parabolic subgroup P, gives rise to symmetric matroids in the sence of Bouchet [11]. More generally, Coxeter matroids for B_n and a maximal parabolic subgroup P are the symplectic matroids introduced and studied in [8]. The paper [7] contains examples of new results on matroids which were originally proven in the more general context of Coxeter matroids and then specialised for the classical situation.

To motivate our result, first recall a well-known theorem by Gelfand et al. [12] on convex polytopes associated with (ordinary) matroids. Let $\mathcal{B}$ be the collection of bases of a matroid on the set $[n] = \{1, 2, \ldots, n\}$. Two bases A and B of $\mathcal{B}$ are obtained from each other by an *elementary exchange* if $B = A\backslash\{a\} \cup \{b\}$ for some elements $a \in A\backslash B$ and $b \in B\backslash A$. Let $\mathbb{R}^n$ be the n-dimensional real vector space with the canonical base $\epsilon_1, \ldots, \epsilon_n$. For each basis $B \in \mathcal{B}$ set

$$\delta_B = \sum_{i \in B} \epsilon_i,$$

and let $\Delta = \Delta(\mathcal{B})$ be the convex hull of $\{\delta_B \mid B \in \mathcal{B}\}$. The polytope Δ is known as the *matroid polytope of* $\mathcal{B}$. Two vertices of a polytope are said to be *adjacent* if they are connected by an edge.

Theorem 1.1 ([12, Theorem 4.1]) *The points $\{\delta_B \mid B \in \mathcal{B}\}$ form the vertex set of Δ. Two vertices δ_A, δ_B are adjacent if and only if the bases A and B of the matroid $\mathcal{B}$ can be obtained from each other by an elementary exchange.*

In this paper, Theorem 1.1 is generalised to Coxeter matroids for an arbitrary finite Coxeter group W and a standard parabolic subgroup P. Let V be the space in which W is represented as a reflection group, and let δ be a point in V such that $Stab_W(\delta) = P$. Then the W-orbit $W \cdot \delta$ of δ is in one-to-one correspondence with the set $\mathcal{W}^P = W/P$. If $A \in \mathcal{W}^P$, denote by δ_A the corresponding point of $W \cdot \delta$, so that $\delta_P = \delta$. Associate with every subset $\mathcal{M}$ of $\mathcal{W}^P$ the convex hull Δ of $\delta(\mathcal{M}) = \{\delta_A \mid A \in \mathcal{M}\}$. It is easy to see that $\delta(\mathcal{M})$ is the set of vertices of the convex polytope Δ. If $\mathcal{M}$ is a Coxeter matroid, then Δ is called the *matroid polytope of* $\mathcal{M}$ and, up to combinatorial type, does not depend on the point δ (Theorem 5.5). The matroid polytope plays a fundamental role in the subject of Coxeter matroids; this will become apparent later in this paper. The main result of the paper is the following criterion for adjacency in the matroid polytope.

Theorem 1.2 *Let $\mathcal{M}$ be a Coxeter matroid for W and P. Then two vertices δ_A and δ_B of Δ are adjacent if and only if there is $w \in W$ such that the basis A immediately precedes B in $\mathcal{M}$ with respect to the ordering $\leq^w$, i.e., $B \leq^w A$ and there is no basis $C \in \mathcal{M}$ such that $B <^w C <^w A$.*

It is instructive to sketch how Theorem 1.1 follows from Theorem 1.2. In fact, Theorem 1.2 is applied to the special case $W = A_{n-1} = Sym_n$, where Sym_n is the symmetric group acting on $[n] = \{1, 2, \ldots, n\}$, with the the set of adjacent transpositions as generators. Let

$$P = \langle (12)(23), \ldots, (k-1k), (k+1k+2), \ldots, (n-1n) \rangle.$$

Then the parabolic subgroup P is the stabiliser in Sym_n of the set $[k] = \{1, \dots, k\}$; so the factor set W/P can be identified with the set $\mathcal{P}_k$ of all k-element subsets in $[n]$. The group $W = Sym_n$ acts on $\mathbb{R}^n$ by permuting coordinates. The group is generated by reflections; reflections correspond in W to transpositions (ij). The stabiliser in W of the point $\delta = \epsilon_1 + \cdots + \epsilon_k$ is P. Thus the setting of Theorems 1.1 and 1.2 coincide. The Bruhat ordering on $\mathcal{P}_k$ turns out to be the following: if two k-subsets

$$A = \{a_1 < a_2 < \cdots < a_k\}$$

and

$$B = \{b_1 < b_2 < \cdots < b_k\}$$

are listed in increasing order of elements, then $A \leq B$ if and only if

$$a_1 \leq b_1, \dots, a_k \leq b_k.$$

Though it is difficult to find a proof of this result in the literature, it is well-known; see for example, [14] or [16, p. 119]. An elementary proof will appear in [5] and a proof of a more general statement will appear in [21].

We first show, assuming Theorem 1.2, that if the bases A and B of the matroid $\mathcal{B}$ can be obtained from each other by an elementary exchange, then the vertices δ_A and δ_B are adjacent. If $\mathcal{B} \subseteq \mathcal{P}_k$ is a matroid of rank k and A and B are two bases of matroid $\mathcal{B}$ related by an elementary exchange, then they can be written $A = \{a_1, \dots, a_k\}$ and $B = \{a_1, \dots, a_{k-1}, b_k\}$. This corresponds to the bases being related by the transposition $(a_k b_k)$. Each permutation $w \in Sym_n$ defines an ordering on $[n]$. If we choose a permutation $w \in Sym_n$ which gives an ordering $\leq^w$ of $[n]$ in which

$$a_1 <^w a_2 <^w \cdots <^w a_{k-1} <^w b_k <^w a_k$$

are the top elements in $[n]$, then obviously A immediately preceeds B in the induced ordering $\leq^w$ of the set $\mathcal{P}_k$. Therefore the vertices δ_A and δ_B are adjacent by Theorem 1.2.

Because a transposition in Sym_n acting on $[n]$ corresponds to a reflection acting on $\mathbb{R}^n$, the converse implication of Theorem 1.1 takes the following form.

(*) *If δ_A and δ_B are adjacent then there is a reflection $t \in W$ such that $A = tB$.*

Statement (*) is part of an important geometric realization theorem on Coxeter matroids and their associated matroid polytopes originally stated by Gelfand and Serganova [14]; also see [19, 23] or Theorem 5.1 below.

For Coxeter matroids in general, the converse of statement (*) does not hold. If, in a Coxeter matroid $\mathcal{M}$, $A = tB$ for a reflection t, then the vertices δ_A and δ_B of Δ are not necessarily adjacent. An example is provided by $W = Sym_3$ and $P = 1$. Here W itself is a Coxeter matroid. (Notice that this does not fall under conditions for Theorem 1.1 because P is not maximal.) It is easy to see that the Coxeter polytope Δ is a planar hexagon. Two opposite vertices δ_1 and $\delta_{(13)}$ are interchanged by the reflection $t = (13)$ but are not adjacent.

Section 2 of this paper contains basic notions about Coxeter groups and their associated Coxeter complexes. Section 3 concerns matroid maps, a concept that provides an equivalent definition of Coxeter matroid. Combinatorial adjacency in a Coxeter matroid is defined in Section 4 and is characterized in terms of the matroid map. This is used in Section 5 to prove the main result, Theorem 1.2, and its corollaries.

2. Coxeter matroids and Coxeter complexes

Throughout this and the next two sections, W is a, possibly infinite, Coxeter group and P a finite standard parabolic subgroup of W. It is convenient in this paper to take a geometric view, regarding the Coxeter group in terms of the associated Coxeter complex. We refer to Tits [21] or Ronan [17] for the definitions of chamber systems, galleries, geodesic galleries, residues, panels, walls and half-complexes. Other useful sources on Coxeter complexes are Hiller [15] and Scharlau [18]. A short review of these concepts can be also found in [2, 3, 10] and in the forthcoming book [5]. A standard reference for root systems is Humphreys [16]. The Coxeter group W will be identified with the collection of chambers, denoted by $\mathcal{W}$, of the Coxeter complex, and, more generally, the collection W/P of cosets with the set of residues, denoted by $\mathcal{W}^P$. The Bruhat ordering on $\mathcal{W}^P$ is denoted by the same symbol $\leq$ as the Bruhat ordering on $\mathcal{W}$. The *w-Bruhat ordering* $a \leq^w b$ is defined by $w^{-1}a \leq w^{-1}b$. The notation $\geq^w$, $<^w$, and $>^w$ have the obvious meaning. The Bruhat ordering on $\mathcal{W}$ has a geometric interpretation as given in [10, Theorem 5.7].

For an infinite Coxeter group, the definition of Coxeter matroid must be modified slightly from the form given in the introduction. A subset $\mathcal{M} \subseteq \mathcal{W}^P$ is a *Coxeter matroid* if $\mathcal{M}$ satisfies the maximality principle, and every element of $\mathcal{M}$ is w-maximal in $\mathcal{M}$ with respect to some $w \in \mathcal{W}$. Again the elements of a Coxeter matroid $\mathcal{M}$ are called *bases*. A notion equivalent to Coxeter matroid is that of a *matroid map* $\mu : \mathcal{W} \to \mathcal{W}^P$, defined by the property that μ satisfies the *matroid inequality*

$$\mu(u) \leq^v \mu(v) \quad \text{for all } u, v \in \mathcal{W}.$$

The image $\mathcal{M} = \mu[\mathcal{W}]$ is obviously a Coxeter matroid. Conversely, given a Coxeter matroid $\mathcal{M}$, a matroid map μ can be obtained by setting $\mu(w)$ equal to the w-maximal element of $\mathcal{M}$. Thus there is natural bijection between matroid maps and Coxeter matroids.

Notice, however, that for an infinite Coxeter group, the second condition in the definition, that every element of $\mathcal{M}$ is w-maximal in $\mathcal{M}$ with respect to some $w \in \mathcal{W}$, is necessary. Otherwise the image $\mathcal{M}' = \mu[\mathcal{W}]$ of the matroid map associated with a set $\mathcal{M}$ satisfying the maximality property may happen to be a proper subset of $\mathcal{M}$ (the set of all 'extreme' or 'corner' chambers of $\mathcal{M}$). For example, take for $\mathcal{M}$ a large rectangular block of chambers in the Coxeter complex for the affine Coxeter group $\tilde{C}_2$ (the group of symmetries of the tiling of the plane by squares).

3. Characterisation of matroid maps

Two subsets A and B of $\mathcal{W}$ are called *adjacent* if there are two adjacent chambers $a \in A$ and $b \in B$, the common panel of a and b also being called a *common panel* of A and B.

A set A of chambers is *convex* if it contains, with any two chambers $a, b \in A$, all geodesic galleries connecting a and b.

Lemma 3.1 *If A and B are two adjacent convex subsets of $\mathcal{W}$, then all their common panels belong to the same wall σ.*

Proof: By [18, Proposition 5.1.3] (see also [10, Theorem 5.5]) every convex set A is the intersection of half complexes containing A. From this observation the result is obvious. □

In this situation, σ is called the *common wall* of A and B. The following result is to appear in [1]. We have included the proof here because its principal idea is similar to that used in the proof of Theorem 4.1.

Theorem 3.2 *A map $\mu : \mathcal{W} \to \mathcal{W}^P$ is a matroid map if and only if the following two conditions are satisfied.*
(1) *Each fiber $\mu^{-1}[A]$, $A \in \mathcal{W}^P$, is a convex subset of $\mathcal{W}$.*
(2) *If two fibers $\mu^{-1}[A]$ and $\mu^{-1}[B]$ of μ are adjacent then their images A and B are symmetric with respect to the common wall of $\mu^{-1}[A]$ and $\mu{-}1[B]$, and the residues A and B lie on the opposite sides of the wall σ from the sets $\mu^{-1}[A]$, $\mu^{-1}[B]$, respectively.*

Proof: If μ is a matroid map then the fact that conditions (1) and (2) are satisfied is the main result of [10].

Now assume that μ satisfies conditions (1) and (2). For any two adjacent fibers $\mu^{-1}[A]$ and $\mu^{-1}[B]$ of the map μ, denote by σ_{AB} the wall separating them, and let Σ be the set of all such walls σ_{AB}. Now take two arbitrary residues $A, B \in \mu[\mathcal{W}]$ and chambers $u \in \mu^{-1}[A]$ and $v \in \mu^{-1}[B]$. It suffices to prove that $A \geq^u B$.

Consider a geodesic gallery

$$\Gamma = (x_0, x_1, \ldots, x_n), \quad x_0 = u, x_n = v$$

connecting the chambers u and v. As a chamber x moves along Γ from u to v, the corresponding residue $\mu(x)$ moves from $A = \mu(u)$ to $B = \mu(v)$. Since the geodesic gallery Γ intersects every wall no more than once [17, Lemma 2.5], the chamber x crosses each wall σ in Σ no more than once and, if it crosses σ, it moves from the same side of σ as u to the opposite side. But, by the assumptions of the theorem, this means that the residue $\mu(x)$ crosses each wall σ no more than once and moves from the side of σ opposite u to the side containing u. According to the geometric interpretation of the Bruhat order [10, Theorem 5.7], this means that $\mu(x)$ decreases with respect to the u-Bruhat order at every such step, ultimately resulting in $A = \mu(u) \geq^u \mu(v) = B$. □

4. Adjacency

Let $\mathcal{M} \subseteq \mathcal{W}^P$ be a Coxeter matroid. We say that two bases $A, B \in \mathcal{M}$ are *combinatorially adjacent* in $\mathcal{M}$ if there exists a chamber $w \in \mathcal{W}$ with the property that A is maximal in $\mathcal{M}$

with respect to the w-Bruhat ordering and B immediately preceeds A in $\mathcal{M}$ with respect to the w-Bruhat ordering, i.e., there is no basis $C \in \mathcal{M}$ with $B <^w C <^w A$.

Theorem 4.1 *Let $\mathcal{M} \subseteq \mathcal{W}^P$ be a Coxeter matroid and $\mu : \mathcal{W} \to \mathcal{W}^P$ the corresponding matroid map. Two bases A and B of $\mathcal{M}$ are combinatorially adjacent in $\mathcal{M}$ if and only if their preimages $\mu^{-1}[A]$ and $\mu^{-1}[B]$ are adjacent.*

Proof: Assume that A and B are two combinatorially adjacent elements of $\mathcal{M}$. Select a chamber $w \in \mathcal{W}$ such that A is the w-maximal basis of $\mathcal{M}$ and B immediately preceeds A in $\mathcal{M}$ with respect to the w-ordering. Let $u \in \mu^{-1}[A]$, $v \in \mu^{-1}[B]$ and let

$$\Gamma = (x_0, x_1, \ldots, x_n), \quad x_0 = u, x_n = v$$

be the geodesic gallery connecting u and v. We can repeat the argument from the previous proof. As the chamber x moves from u to v along the gallery Γ, the corresponding basis $\mu(x)$ of $\mathcal{M}$ moves over $\mathcal{M}$ from $A = \mu(u)$ to $B = \mu(v)$ decreasing with respect to the ordering $\leq^w$. Since B is an immediate predecessor of A, the image $\mu(x)$ of x can take only two values, A and B. Therefore the gallery Γ is entirely contained in the union of two fibers $\mu^{-1}[A] \cup \mu^{-1}[B]$; so these two fibers are obviously adjacent.

Conversely, let $\mu^{-1}[A]$ and $\mu^{-1}[B]$ be two adjacent fibers of the matroid map μ. Take two chambers $u \in \mu^{-1}[A]$ and $v \in \mu^{-1}[B]$ which are adjacent, i.e., have a common panel (belonging to the wall σ separating $\mu^{-1}[A]$ and $\mu^{-1}[B]$). Then $u = vr$ for some standard generator r of W. We claim that B is an immediate predecessor of A in $\mathcal{M}$ with respect to the u-Bruhat ordering.

Indeed, assume the contrary and let C be a basis in $\mathcal{M}$ distinct from A and B and with the property $B <^u C <^u A$. Denote by b the smallest chamber in the residue B with respect to the u-Bruhat ordering; similarly for the u-minimal chambers $c \in C$ and $a \in A$. Then by [17, Theorem 2.9]

$$b <^u c <^u a.$$

Denote by $d(x, y)$ the distance $l(x^{-1}y)$ between elements x and y in the group W, i.e., the gallery distance in the Coxeter complex $\mathcal{W}$. Then

$$d(u, a) > d(u, c) > d(v, b)$$

and, for this reason, $d(u, a) - 2 \geq d(u, b)$. Since the chambers u and v are adjacent, we have

$$d(u, a) - 1 \geq d(u, b) + 1 \geq d(v, b).$$

Since B is the v-maximal basis in $\mathcal{M}$ we have $A <^v B$. If a' denotes the v-minimal chamber of A and b' the v-minimal chamber of B, then

$$b \geq^v b' >^v a'.$$

The crucial observation now is that the wall σ separates all chambers $a'' \in A$ and the chamber v from the chamber u, and this implies [17, Proposition 2.6] that $d(u, a'') = d(v, a'') + 1$. Being the u-minimal chamber of A, a is the chamber in A with the minimal possible distance $d(u, a)$ to u. Hence a also has the minimal possible distance $d(v, a)$ to v and therefore is the v-minimal element of A. But then $b \geq^v b' >^v a$ and

$$d(u, a) - 1 \geq d(v, b) > d(v, a)$$

and therefore

$$d(u, a) - 1 \geq d(v, a) + 1.$$

But we already know that

$$d(u, a) = d(v, a) + 1,$$

a contradiction. □

5. The matroid polytope

In this section the Coxeter group W is finite; hence the space V in which W is represented as a group generated by reflections is Euclidean. Let Φ be the root system of this Weyl group, and denote by Σ the collection of all mirrors of reflections in W, i.e., the collection of hyperplanes normal to roots in Φ. The walls of $\mathcal{W}$ can be interpreted as the hyperplanes Σ. The chambers of the Coxeter complex $\mathcal{W}$, in this finite case, are connected components of $V \setminus \bigcup_{H \in \Sigma} H$. Vectors in $V \setminus \bigcup_{H \in \Sigma} H$ are called *regular*.

Let P be a standard parabolic subgroup in W and δ a point in V such that $Stab_W(\delta) = P$. Then the W-orbit $W \cdot \delta$ of δ is in one-to-one correspondence with the set $\mathcal{W}^P$. If $A \in \mathcal{W}^P$, denote by δ_A the corresponding point of $W \cdot \delta$ so that $\delta_P = \delta$. Associate with every subset $\mathcal{M}$ of $\mathcal{W}^P$ the convex hull $\Delta = \Delta(\mathcal{M}, \delta)$ of $\delta(\mathcal{M}) = \{\delta_A \mid A \in \mathcal{M}\}$. It is easy to see that $\delta(\mathcal{M})$ is the set of vertices of the convex polytope Δ. If $\mathcal{M}$ is a Coxeter matroid, then $\Delta = \Delta(\mathcal{M}, \delta)$ is called the *matroid polytope of* $\mathcal{M}$. It is shown later in this section that the combinatorial type of Δ is independent of the choice of δ.

The following result generalises a classical geometric characterization of Coxeter matroids originally due to Gelfand and Serganova [14]. It is an abridged version of the main theorem in [19].

Theorem 5.1 *In the notation above, the following conditions are equivalent.*
(a) *$\mathcal{M}$ is a Coxeter matroid.*
(b) *Every edge of Δ is perpendicular to one of the mirrors in Σ.*
(c) *For any two adjacent vertices α, β of Δ there is a reflection $s \in W$ such that $s\alpha = \beta$.*
(d) *For any regular vector $\xi \in V$, the linear functional $\alpha \mapsto (\alpha, \xi)$ reaches its minimum on Δ at a unique point.*

The main result can now be proved.

Theorem 5.2 *Let $\mathcal{M}$ be a Coxeter matroid for a finite Coxeter group W and a parabolic subgroup P, and Δ its matroid polytope. Then two vertices δ_A and δ_B of Δ are adjacent if and only if there is $w \in W$ such that the basis A immediately preceeds B in $\mathcal{M}$ with respect to the ordering $\leq^w$, i.e., $B \leq^w A$ and there is no basis $C \in \mathcal{M}$ such that $B <^w C <^w A$.*

Proof: For each basis $A \in \mathcal{M}$ let

$$\Gamma_A = \{\xi \in V \mid (\alpha, \xi) \geq (\delta(A), \xi) \text{ for all } \alpha \in \Delta\}.$$

Then Γ_A is a closed convex polyhedral cone. It immediately follows from the previous theorem that the proper faces of Γ_A belong to hyperplanes in Σ and that the system of cones $\mathcal{G} = \{\Gamma_A \mid A \in \mathcal{M}\}$ forms the fan of cones dual to the polytope Δ. In particular, any two cones in $\mathcal{G}$ intersect along a common face and two cones Γ_A and Γ_B are adjacent, i.e., intersect along the common face of maximal dimension, if and only if the corresponding vertices δ_A and δ_B are adjacent.

It is easy to see that a set $\mathcal{X}$ of chambers in $\mathcal{W}$ is convex in the sense of the theory of Coxeter complexes if and only if the union of their closures $\bigcup_{X \in \mathcal{X}} \bar{X}$ is convex in the usual geometric meaning of this word. This follows, for example, from the characterization of convex subsets of $\mathcal{W}$ as intersections of half complexes [18, Proposition 5.1.3]. So, given the basis $A \in \mathcal{M}$, the set of chambers contained in the cone Γ_A is convex. Therefore the map $\mu : \mathcal{W} \to \mathcal{M}$, defined by the rule $\mu(w) = A$ if the chamber w belongs to Γ_A, has convex fibers $\mu^{-1}[A]$. Moreover, if two fibers $\mu^{-1}[A]$ and $\mu^{-1}[B]$ are adjacent, then the cones Γ_A and Γ_B are adjacent. Therefore these cones have in common a face of maximal dimension, which is exactly the mirror σ of symmetry of the edge $[\delta_A, \delta_B]$ of the convex polytope Δ. If s is the reflection in σ, then $A = sB$. Moreover, A and B lie on the opposite sides of σ from the chambers in $\mu^{-1}[A]$ and $\mu^{-1}[B]$, respectively. Therefore μ is a matroid map by Theorem 3.2, and, obviously, it is the matroid map associated with the matroid $\mathcal{M}$.

Hence two vertices δ_A and δ_B of Δ are adjacent if and only if the cones Γ_A and Γ_B are adjacent if and only if the convex sets of chambers $\mu^{-1}[A]$ and $\mu^{-1}[B]$ are adjacent if and only if, in view of Theorem 4.1, the bases A and B are combinatorially adjacent in $\mathcal{M}$. □

We shall draw two useful corollaries about matroid polytopes from Theorem 5.2.

Theorem 5.3 *Let $\mathcal{M}$ be a Coxeter matroid. Up to isomorphism, the graph of the matroid polytope $\Delta(\mathcal{M}, \delta)$ is independent of the choice of the point δ. Moreover, if δ and δ' are two points such that $Stab_W(\delta) = Stab_W(\delta') = P$, then corresponding edges of $\Delta(\mathcal{M}, \delta)$ and $\Delta(\mathcal{M}, \delta')$ are parallel.*

Proof: The first statement follows directly from Theorem 5.2. Concerning the second statement, let (α, β) and (α', β') be corresponding edges of $\Delta(\mathcal{M}, \delta)$ and $\Delta(\mathcal{M}, \delta')$, respectively. Corresponding means that α and α' (resp. β and β') are associated with the same coset of $\mathcal{W}^P$, say U (resp. V). By Theorem 5.1 there is a reflection t such that $tU = V$. Notice that it is enough to prove that there is a unique such reflection. Indeed, the uniqueness of t implies that $t\alpha = \beta$ and $t\alpha' = \beta'$; hence the edges $[\alpha, \beta]$ and $[\alpha', \beta']$ are parallel. Therefore Theorem 5.3 is reduced to the following lemma. □

Lemma 5.4 *If U and V are two residues in a Coxeter complex $\mathcal{W}$ for the Coxeter group W, then there is at most one reflection $t \in W$ such that $U = tV$.*

Proof of Lemma: Select chambers $u \in U$ and $v \in V$ such that the distance $d(u, v)$, is minimized, and let

$$\Gamma = (x_1, \ldots, x_n), \quad x_1 = u, x_n = v,$$

be a geodesic gallery connecting u and v.

Let τ be the wall of the reflection t. Because u and v lie on opposite sides of τ, this wall is the common wall of two adjacent chambers x_k and x_{k+1} in Γ. (By [17, Lemma 2.5] the geodesic gallery Γ intersects the wall τ only once; thus the chambers x_k and x_{k+1} are uniquely determined chambers in Γ.) We know that U and V, being residues, are *gated sets*; this means that for every $w \in \mathcal{W}$ there is a unique chamber in U (resp. in V) at the minimal distance from w [17, Theorem 2.9], [18, Theorem 5.1.7]. Applying the gated property of residues U and V shows that u and v are uniquely determined as the chambers in U and V at the minimal distance from x_k and x_{k+1}, respectively. Since the reflection t maps U into V and x_k into x_{k+1}, it must be the case that $d(x_k, u) = d(x_{k+1}, v)$ and hence t maps u onto v. But any element of W is uniquely determined by its action on a single chamber; therefore t is uniquely determined. □

Theorem 5.5 *The combinatorial type of the matroid polytope $\Delta(\mathcal{M}, \delta)$ for the Coxeter matroid $\mathcal{M}$ does not depend on the choice of the point δ.*

Proof: Let δ and δ' be two points such that $Stab_W(\delta) = Stab_W(\delta') = P$, and $\Delta = \Delta(\mathcal{M}, \delta)$ and $\Delta' = \Delta(\mathcal{M}, \delta')$ the corresponding matroid polytopes. It will suffice to prove that the correspondence between the vertex sets of Δ and Δ' which preserves adjacency also preserves the faces of Δ and Δ'.

Let Γ be a face of Δ and $\{\alpha_1, \ldots, \alpha_m\}$ its set of vertices. Denote by $\mathcal{N} = \{A_1, \ldots, A_m\}$ the corresponding set of cosets in $\mathcal{W}^P$. We wish to prove that the corresponding set $\{\alpha'_1, \ldots, \alpha'_m\}$ of vertices of Δ' also forms the vertex set of a face of Δ'. First notice that $\mathcal{N}$ is, by Theorem 5.1, a Coxeter matroid. By Theorem 5.3, $\mathcal{N}'$ is also a Coxeter matroid and, therefore, the convex hull Γ' of $\{\alpha'_1, \ldots, \alpha'_m\}$ is a matroid polytope. Moreover, the dimension of Γ equals of the dimension of the vector space spanned by the vectors $\overline{\alpha_i\alpha_j}$ corresponding to all pairs of adjacent vertices α_i, α_j in Γ. Therefore Γ' has the same dimension as Γ.

Now let π be a supporting hyperplane of Δ which contains the face Γ. Then π is perpendicular to all the mirrors of reflection for all edges of Γ. But, by Theorem 5.3, these mirrors are exactly the mirrors of reflection of edges of Γ', and therefore we can find a hyperplane π' parallel to π and containing the convex polytope Γ'.

To show that Γ' is a face of Δ', it now suffices to prove that π' is a supporting hyperplane of Δ'. If α, α' and β, β' are corresponding vertices of Δ and Δ' (i.e., α and α' correspond to the same coset in $\mathcal{M} \subseteq W^P$), then the proof of Theorem 5.3 implies, not only that the edges $[\alpha, \beta]$ and $[\alpha', \beta']$ are parallel, but that they have the same direction. Now if β is any vertex of $\Delta \backslash \Gamma$ adjacent to a vertex α of Γ, then the vector $\overline{\alpha\beta}$ points to the halfspace of π

containing Δ. Therefore all vectors $\overline{\alpha'\beta'}$ for adjacent vertices $\alpha' \in \Gamma'$ and $\beta' \in \Delta' \backslash \Gamma'$ point into the same halfspace determined by the hyperplane π'. This means that π' is a supporting hyperplane for Δ'. □

References

1. A.V. Borovik, "Matroid maps," Comm. Omsk University, 1997, No. 1, pp. 12–13.
2. A.V. Borovik and I.M. Gelfand, "Matroids on chamber systems," *Publ. LaCIM* **14** (1993), 25–62.
3. A.V. Borovik and I.M. Gelfand, "*WP*-matroids and thin Schubert cells on Tits systems," *Advances Math.* **103** (1994), 162–179.
4. A.V. Borovik, I.M. Gelfand, A. Vince, and N. White, "The lattice of flats of a matroid and its underlying flag matroid polytope," *Annals of Combinatorics* **1** (1997), 17–26.
5. A.V. Borovik, I.M. Gelfand, and N. White, *Coxeter Matroids*, Birkhauser, Boston, in preparation.
6. A.V. Borovik, I.M. Gelfand, and N. White, "Boundaries of Coxeter matroids," *Advances Math.* **120** (1996), 258–264.
7. A.V. Borovik, I.M. Gelfand, and N. White, "Exchange properties of Coxeter matroids and oriented matroids," *Discrete Math.* **179** (1998), 59–72.
8. A.V. Borovik, I.M. Gelfand, and N. White, "Symplectic matroids," *J. Alg. Comb.*, to appear.
9. A.V. Borovik, I.M. Gelfand, and N. White, "Coxeter matroid polytopes," *Annals of Combinatorics* **1** (1997), 123–134.
10. A.V. Borovik and K.S. Roberts, "Coxeter groups and matroids," in *Groups of Lie Type and Geometries*, W.M. Kantor and L.Di Martino (Eds.), Cambridge University Press, Cambridge, 1995, pp. 13–34.
11. A. Bouchet, "Greedy algorithm and symmetric matroids," *Math. Programming* **38** (1987), 147–159.
12. I.M. Gelfand, M. Goresky, R.D. MacPherson, and V.V. Serganova, "Combinatorial geometries, convex polyhedra, and Schubert cells," *Advances Math.* **63** (1987), 301–316.
13. I.M. Gelfand and V.V. Serganova, "On a general definition of a matroid and a greedoid," *Soviet Math. Dokl.* **35** (1987), 6–10.
14. I.M. Gelfand and V.V. Serganova, "Combinatorial geometries and torus strata on homogeneous compact manifolds," *Russian Math. Surveys* **42** (1987), 133–168; see also I.M. Gelfand, *Collected Papers*, Springer-Verlag, New York, 1989, Vol. 3, pp. 926–958.
15. H. Hiller, *Geometry of Coxeter Groups*, Pitman, Boston, 1982.
16. J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1976.
17. M. Ronan, *Lectures on Buildings*, Academic Press, Boston, 1989.
18. R. Scharlau, "Buildings," preprint 93-016, Universität Bielefeld (SFB 343 Diskrete Structuren in der Mathematik), 1993.
19. V.V. Serganova, A. Vince, and A. Zelevinsky, "Geometric characterization of Coxeter matroids," *Annals of Combinatorics* **1** (1997), 173–181.
20. J. Tits, "A local approach to buildings," *The Geometric Vein (Coxeter Festschrift)*, Springer-Verlag, New York, 1981, pp. 317–322.
21. A. Vince, "The greedy algorithm and Coxeter matroids," *J. Alg. Comb.*, to appear.
22. W. Wenzel, *Geometric Algebra of* Δ*-Matroids and Related Combinatorial Geometries*, Habilitationschrift, Bielefeld, 1991.
23. A.V. Zelevinsky and V.V. Serganova, "Combinatorial optimisation on Weyl groups, greedy algorithms and generalised matroids," preprint, Acad. Sci. USSR, Scientific Council on the Complex Problem "Cybernetics", Moscow, 1989, 24 pp. (in Russian).