Journal of Convex Analysis
Volume 5 (1998), No. 1, 57–62

On a New Result on the Existence of Zeros due to Ricceri*

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Received September 3, 1996
Revised manuscript received June 23, 1997

The purpose of this short paper is to extend a recent result of Ricceri, on the existence of zeros of functions with values in a dual space, to the existence of solutions to inclusions with values in arbitrary topological spaces.

1. Introduction

It is well-known that the existence of a null classical subgradient at a point x of a real function f defined on a real topological vector space is a necessary and sufficient condition for x to be a global minimum point of f. In fact, one of the essential requirements to define a generalized subdifferential of a proper lower semicontinuous function f is that a point x be a local minimum of f whenever the zero functional belongs to the subdifferential at x (see for instance [2]). Hence the detailed study of the existence of zeros of multivalued functions, defined on a real topological vector space with values in the family of the nonempty w*-sets of its topological dual, becomes quite relevant.

In a recent paper Ricceri [9] presented a first result on the existence of zeros of single-valued functions with values in the dual topological space of a real topological vector space in terms of the disconnectedness at a level set of a composition involving the canonical duality. Roughly speaking, Ricceri has shown that "a single-valued continuous function $A\colon X \to E'$ defined on a connected topological space X with values in the topological dual space E' of a real Hausdorff topological vector space E has a zero if the graph of the associated multivalued function

$$\Lambda\colon X \to 2^E, \quad \Lambda(x) = \{e \in E : \langle e, A(x)\rangle_{E,E'} = 1\}$$

is disconnected." This result is applicable in very smooth optimization and it can be extended to the case where the subdifferential is not a singleton. Indeed in [4] (see also Theorem 3.4 below), the authors have pointed out that "a set-valued continuous proper function $A\colon X \to 2^{E'}$ defined on the connected topological space X with w*-compact convex values in the topological dual space E' of a real Hausdorff topological vector space E has a zero i.e., there exists some $\overline{x} \in X$ such that $0_{E'} \in A(\overline{x})$ if the values of the

*The research of the author was partially supported by the PRAXIS and JNICT.

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

associated multivalued function

$$\Lambda\colon X \to 2^E, \quad \Lambda(x) = \{e \in E : 1 \in \langle e, A(x)\rangle_{E,E'}\}$$

are connected and the graph is disconnected." We recall that, unfortunately, the subdifferential of a continuous convex function on a Banach space is not norm-to-weak* continuous Fréchet differentiability implies norm-to-norm continuity (see [6]).

The present paper intends to be a simple contribution to the study of the solvability of abstract equations and inclusions. The main reason for this extension is that the kind of concrete problems which it involves is of interest in domains besides the linear (or convex) applications as will be seen in a forthcoming paper by the author. The main goal is achieved in Theorem 3.1 below.

2. Notations and some results

For the sake of completeness we include here some natural extensions concerning to the single-valued case studied by Ricceri [9].

The elementary facts on multifunctions and on dual systems may be found in [1] and [7], respectively. Some notations are:

(H_0) X is a connected topological space, E is a locally connected topological space and F is a topological space. $u \in F$ is a fixed point.

(H_B) $B\colon E \times F \to \mathbb{R}$ is a single-valued function with the following property: for every $f \in F$, $f \neq u$, $B(\cdot, f)$ is continuous.

(H_ψ) $\psi\colon E \times X \to \mathbb{R}$ is a single-valued function, which is continuous in the first variable: for every $x \in X$, $\psi_x = \psi(\cdot, x)\colon E \to \mathbb{R}$ is continuous. $\psi \equiv 1$ is the most usual case.

($\mathrm{H}_{B,\psi}$) B and ψ are related by the following assumption: for every pair $(x, f) \in X \times F$, $f \neq u$, the set $E_{x,f} = \{e \in E : B(e, f) = \psi(e, x)\}$ is nonempty and does not contain any local extremum of $B(\cdot, f)$.

$G : X \multimap F$ ($G : X \to 2^F$) is a proper multivalued function with connected values, i.e., for every $x \in X$, $G(x)$ is a nonempty connected set. We identify a single-valued function $g\colon X \to F$ with a proper multivalued function with singleton values (it is said that G is single-valued).

We associate with G the multifunction: $\Lambda : X \multimap E$, $x \mapsto \{e \in E : \psi(e, x) \in B(e, G(x))\}$.

Let assume that

(H_G) the set

$$\{e \in E : x \mapsto B(e, G(x)) \text{ is lower semicontinuous}\}$$

is dense in E if G is single-valued this is equivalent to

$$\{e \in E : x \mapsto B(e, g(x)) \text{ is continuous}\};$$

(H_{Λ_1}) for every $x \in X$, $\Lambda(x)$ is connected; and

(H_{Λ_2}) the graph of Λ: $\Gamma = \{(e, x) \in E \times X : \psi(e, x) \in B(e, G(x))\}$ is disconnected.

The most usual ambient is a dual system $(E, F; B)$: E, F being a pair of real Hausdorff locally convex spaces and B a real continuous bilinear separated form on $E \times F$.[1] In the canonical dual system (E, E'), E is a real Hausdorff locally convex space with topological dual $F = E'$, and $B = <\ ,\ >_{E,E'}$ is the duality pairing. B as described above is a type of "generalized duality pairing ". In general, it is $u = 0_F$ $(u = 0_{E'})$.

Remark 2.1. We point out some particular situations.

(i) The condition $(\mathrm{H})_{B,1}$ is satisfied if for every $f \in F \setminus \{u\}$ the two following conditions hold:

$$1 \in B(E, f) \qquad \text{and} \qquad B(\cdot, f) \text{ is an open function.}$$

(ii) If G is single-valued, E is a topological vector space, for every $f \neq u$ and for every x, $B(\cdot, f)$ and $\psi(\cdot, x)$ are linear, every $\Lambda(x)$ is a linear manifold (so (H_{Λ_1}) holds).

(iii) If G is single-valued, under the hypothesis $B(\cdot, u) = 0$, $0 \notin \psi(E, X)$ and $(\mathrm{H}_{B,\psi})$ holds, the solvability of the equation $u = g(x)$ is equivalent to the condition $\Lambda(\overline{x}) = \emptyset$ for some $\overline{x} \in X$.

(iv) If E is a real Hausdorff topological vector space, $B(\cdot, f)$ is linear for every f and $B(\cdot, f) = 0$ implies $f = u$, condition $(\mathrm{H}_{B,\psi})$ is fulfilled whenever ψ is independent of e or $\psi(0, x) = 0$, $x \in X$. This is the case of a dual system.

Theorem 2.2. *Let* E,F, X *and* u *as in* (H_0). *Let* $B : E \times F \to \mathbb{R}$ *and* $\psi \equiv 1$ *be such that* (H_B) *and* $(\mathrm{H}_{B,1})$ *hold. Let* $g : X \to F$ *be such that* (H_{Λ_1}), (H_{Λ_2}) *and* (H_G) *hold (this last hypothesis means that the set*

$$\{e \in E : x \mapsto B\,(e\,, g\,(x)) \text{ is continuous}\}$$

is dense in E*).*

Then, the equation

$$u = g\,(x)$$

is solvable, i.e., there exists some point $\overline{x}$ *in* X *for which* $u = g\,(\overline{x})$.

The proof of this theorem follows the same steps of Theorem 1.1. of [9] (a proof is also therein our Theorem 3.1 below).

In a dual system $(E, F; B)$, this theorem includes two important cases.

Indeed, if X is a connected topological space and $g : X \to F$ is continuous, then

(i) g has a zero whenever $\{(e, x) \in E \times X : 1 = B(e, g(x))\}$ is disconnected.

(ii) if $X \subseteq F$, one and only one of the following statements holds: or g has a fixed point or the set of the solutions of the equation in $E \times X$: $B(e, g(x)) = 1 + B(e, x)$ is connected.

[1] If $(E, F; B)$ is a dual system where E, F are real vector spaces and B is a real separated bilinear form on $E \times F$, it is assumed that the vectorial topologies on E, F are the ones endowed by the form, i.e., $\sigma\,(E, F)$ and $\sigma\,(F, E)$, respectively, unless otherwise expressed.

3. The main result

The main result of this paper is the following abstract theorem. For sake of self-containing and clarity we repeat here some of the essential hypotheses.

Theorem 3.1. *Let* X, E, F *and* u *be as in* (H_0). *Let* $B\colon E \times F \to \mathbb{R}$ *and* $\psi\colon E \times X \to \mathbb{R}$ *fulfill* (H_B), (H_ψ), $(\mathrm{H}_{B,\psi})$ *and* (H_{B_1}): B *is a Darboux function.*

Let $G : X \multimap F$ *be a proper multivalued function with connected values, and let us define the multivalued function* $\Lambda : X \multimap E$, $x \mapsto \{e \in E : \psi(e, x) \in B(e, G(x))\}$. *Assume that the three conditions* (H_G), (H_{Λ_1}) *and* (H_{Λ_2}) *hold.*

Then, the inclusion

$$u \in G(x)$$

is solvable, i.e., there exists some point $\overline{x}$ *in* X *for which* $u \in G(\overline{x})$.

Proof. We argue by contradiction, i.e., let $u \notin G(x)$, $x \in X$. First of all, two remarks must be done:

(α) Λ is proper in virtue of $(\mathrm{H}_{B,\psi})$;

(β) if K is a connected subset of E, $e_1, e_2 \in K$, $g_1, g_2 \in G(x)$ and $\alpha \in \mathbb{R}$ are such that

$$B(e_1, g_1) < \alpha < B(e_2, g_2)$$

then there is a pair $(e_3, g_3) \in K \times G(x)$ verifying $B(e_3, g_3) = \alpha$ (hypothesis (H_{B_1})).[2]

Let Ω_1, Ω_2 be open sets in $E \times X$ such that the family $\{\Omega_1 \cap \Gamma, \Omega_2 \cap \Gamma\}$ is a relatively open nontrivial partition of the graph of Λ: $\Gamma = \{(e, x) \in E \times X : \psi(e, x) \in B(e, G(x))\}$ (see (H_{Λ_2})):

$$\Omega_1 \cap \Gamma \neq \emptyset, \quad \Omega_2 \cap \Gamma \neq \emptyset, \quad \Omega_1 \cap \Omega_2 \cap \Gamma = \emptyset, \quad \Gamma \subseteq \Omega_1 \cup \Omega_2 \, .$$

Let P be the projection of $E \times X$ onto X. We prove that the family $\{P(\Omega_1 \cap \Gamma), P(\Omega_2 \cap \Gamma)\}$ is an open nontrivial partition of X contradicting the connectedness of X.

The two sets of the last family define a disjoint covering of X. Indeed, for every $x \in X$, $\{x\} \times \Lambda(x) = P^{-1}(x) \cap \Gamma$ is a nonempty connected set. Hence, $P(\Gamma) = X$ and $\{x\} \times \Lambda(x)$ is contained in one and only one of the open sets Ω.

It remains to prove the openness of each set of the partition.

Let $x_0 \in P(\Omega_1 \cap \Gamma)$. So $(e_0, x_0) \in \Omega_1 \cap \Gamma$ for some $e_0 \in \Lambda(x_0)$ and (e_0, x_0) belongs to an open rectangle $V_0 \times U_0 \subseteq \Omega_1$ with $V_0 \subseteq E$ connected. Let $f_0 \in G(x_0)$ such that $B(e_0, f_0) = \psi(e_0, x_0)$.

The sets

$$V_1 = \{e \in V_0 : B(e, f_0) < \psi(e_0, x_0)\} \quad \text{and} \quad V_2 = \{e \in V_0 : B(e, f_0) > \psi(e_0, x_0)\}$$

are nonempty open sets because $f_0 \neq u$ and $B(\cdot, f_0)$ has no local extremum on E_{x_0, f_0}.

[2] It is the unique place where the Darboux condition on B is required. It is clear that if G is single-valued it is enough to assume the continuity of B in its first variable (see Theorem 2.2).

By density, it follows that there exist $e_1 \in V_1, e_2 \in V_2$ such that the proper multivalued functions $\phi_i : x \mapsto B(e_i, G(x))$, $i = 1, 2$ are lower semicontinuous.

As $B(e_1, f_0) < \psi(e_0, x_0)$, $B(e_2, f_0) > \psi(e_0, x_0)$, the set

$$U_1 = \{x \in U_0 : \phi_1(x) \cap (-\infty, \psi(e_0, x_0)) \neq \emptyset \quad \text{and} \quad \phi_2(x) \cap (\psi(e_0, x_0), \infty) \neq \emptyset\}$$

is a (open) neighborhood of x_0.

It is $U_1 \subseteq P(\Omega_1 \cap \Gamma)$. Indeed, let $x \in U_1$, $g_1, g_2 \in G(x)$ be such that

$$B(e_1, g_1) < \psi(e_0, x_0) < B(e_2, g_2).$$

As $V_0 \times G(x)$ is connected, from remark (β) above it follows the existence of $(e_3, g_3) \in V_0 \times G(x)$ such that $\psi(e_0, x_0) = B(e_3, g_3) \in B(e_3, G(x))$.

As $\Lambda(x) \neq \emptyset$, $\psi(e_3, x) \in B(e_3, G(x))$, i.e., $(e_3, x) \in \Omega_1 \cap \Gamma$.

In a similar way, it is proved that $P(\Omega_2 \cap \Gamma)$ is a nonempty open set which contradicts X being connected. □

Remark 3.2. As the proof of the theorem shows,the Darboux condition (H_{B_1}) can be weakened. Indeed, it is enough to assume that B satisfies the following condition: "$B(R)$ is an interval for every $R \in \mathcal{R}$ where $\mathcal{R}$ is the family of the nonempty connected rectangles $R = A \times B \subset E \times F$ such that A is open."

Corollary 3.3. *Assume that X, E, F, u, B and $\psi \equiv 1$ satisfy the hypotheses of* Theorem 3.1. *Furthermore, assume that for every $f \in F$, $f \neq u$, the level set $E_f = \{e \in E : B(e, f) = 1\}$ is connected.*
Let $G : X \to F$ be a proper multivalued function with connected values such that the conditions (H_G), (H_{Λ_2}) hold.
Then, there exists some point $\overline{x}$ in X for which $u \in G(\overline{x})$.

Proof. It is only necessary to prove that (H_{Λ_1}) holds. As $\Lambda(x) = H(G(x))$ where H is the multivalued mapping $H(f) = \{e \in E : B(e, f) = 1\}$ defined on $F \setminus \{u\}$, and as $H(f)$ and $G(x)$ are connected, it is enough to prove that H is lower semicontinuous (we recall that the image of a connected set by a lower semicontinuous multifunction with connected values is connected). But the lower semicontinuity of H follows from [8, page 266] (see also [5]). □

The linear case is particularly important and it extends the above mentioned result of [4].

Theorem 3.4. *Let E, X be a real topological Hausdorff vector space with topological dual E' and a connected space, respectively. Let $\psi \colon E \times X \to \mathbb{R}$ be either independent of e or $\psi(0, x) = 0$, $x \in X$ and satisfying (H_ψ). Assume that $G : X \multimap E'$ is a proper multivalued mapping with $\sigma(E', E)$-connected values such that the set*

$$\{e \in E : x \mapsto \langle e, G(x)\rangle_{E,E'} \text{ is lower semicontinuous}\}$$

is dense in E and that $\Lambda : X \multimap E$, $\Lambda(x) = \{e \in E : \psi(e, x) \in \langle e, G(x)\rangle_{E,E'}\}$ has connected values and its graph $\{(e, x) \in E \times X : \psi(e, x) \in \langle e, G(x)\rangle_{E,E'}\}$ is disconnected.

Then, $0 \in G(\overline{x})$ for some point $\overline{x}$ in X.

Proof. $(\mathrm{H}_{B,\psi})$ is fulfilled in virtue of Remark 2.1.iv. Now, we claim that the duality pairing $\langle\ ,\ \rangle_{E,E'}$ satisfies the condition expressed at Remark 3.2.

In fact, if $a_1, a_2 \in E$, the real function

$$\theta\colon \mathbb{R}\times E' \to \mathbb{R}, \quad \theta(\lambda, f) := \lambda\langle a_1, f\rangle_{E,E'} + (1-\lambda)\langle a_2, f\rangle_{E,E'}$$

is $\mathbb{R}\times(E', E)$-continuous. Therefore, the assertion above is clear whenever E is a locally convex space and otherwise because every nonempty connected open set of a topological vector space is arcwise connected (by polygonal arcs). $\square$

Final Remark. In the last days of the final revision of this paper, Cubiotti has kindly sent me a preprint [3]. There a theorem is presented which is very close to our Theorem 3.4 (see also [4]).

Acknowledgements. The author wishes to thank Professor Ricceri for his valuable comments and suggestions. He also acknowledges useful suggestions and criticisms by an anonymous referee.

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