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Unbounded Linear Monotone Operators on Nonreflexive Banach Spaces

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This study of unbounded linear monotone operators $T\colon E \to E^*$ for nonreflexive Banach spaces E was motivated by the (still open) problem of distinguishing between several well-studied classes of maximal monotone set-valued operators (classes which coincide when E is reflexive). It is shown in Theorem 6.7 that in the unbounded linear case these classes are closely related. They may even be identical, as was shown by Bauschke and Borwein to be true for the case of bounded linear monotone operators. (A short new proof of this latter result is given in Section 8.) Earlier sections yield a characterization of maximality, a characterization of maximal monotone unbounded linear symmetric operators (in terms of the convex function $\langle Tx, x\rangle$) and a number of relevant examples. Section 7 contains a proof that, in the linear case, Rockafellar's theorem on the maximality of the sum of two maximal monotone operators is true even in nonreflexive Banach spaces. It also contains a counterexample in Hilbert space showing that the hypotheses cannot be substantially weakened.

1. Introduction

Many of the most useful results concerning maximal monotone set-valued operators are only valid in reflexive Banach spaces. In an effort to extend some of these results to nonreflexive spaces, various authors have introduced certain natural subclasses of maximal monotone operators (subclasses which are identical with the entire class of maximal monotone operators in reflexive spaces). The precise relationships between the various new classes of operators have remained murky, although it is very clear in a particular case: Bauschke and Borwein [2, 3] have shown that all of these notions *coincide* for *bounded linear* monotone operators. In this note, we consider the next simplest case, namely, unbounded linear monotone operators. It may be true that the Bauschke-Borwein characterization remains valid for unbounded monotone operators; Theorem 6.7 below strongly suggests that such is the case. (The simplifications needed to prove these facts about unbounded operators lead to much simpler proofs of the corresponding portions of [3, Theorem 4.1] – see Theorem 8.1.)

After some basics in Sections 2 and 3, in Section 4 we look at adjoint operators and in Section 5 we characterize those linear operators which are subdifferentials of proper lower semicontinuous convex functions. Section 6 is devoted to those operators which are in one of the subclasses mentioned above, that is, those which are of type (D), type

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(NI), or locally maximal monotone. In Section 7 it is shown that, in the linear case, Rockafellar's theorem on the maximality of the sum of two maximal monotone operators is true even in nonreflexive Banach spaces; this is followed by a counterexample showing that the hypotheses cannot be substantially weakened. Section 8 is devoted to the case of continuous linear operators. In the concluding section, Section 9, we give some open problems. The first-named author thanks Jeff Eldridge for numerous helpful conversations on the subject matter of this paper. The second-named author thanks the University of Toulouse, France for its hospitality during some of his work on this paper. Both authors would like to thank Heinz Bauschke for reading a previous version of this paper, and making a number of perceptive comments.

Let E be a real Banach space, $D(T)$ be a linear subspace of E and $T\colon D(T) \to E^*$ a (possibly unbounded) monotone linear operator with range $R(T) \subset E^*$; that is, T is linear and

$$\langle Tx, x\rangle \geq 0 \quad \text{for all} \quad x \in D(T).$$

Let $G(T) \subset E \times E^*$ denote the graph of T, and notice that it is an example of a *monotone subset*; that is, a subset $M \subset E \times E^*$ with the property that, for all $(x, x^*), (y, y^*) \in M$, we have

$$\langle x^* - y^*, x - y\rangle \geq 0. \tag{1.1}$$

We say that T is *maximal monotone* if $G(T)$ is maximal (under inclusion) in the family of all monotone subsets of $E \times E^*$. (As shown in Proposition 3.2(f) below, when $D(T)$ is dense, this is the same as being maximal in the family of graphs of all linear monotone operators.) It is immediate that a monotone subset M is maximal monotone provided $(x, x^*) \in M$ whenever $(x, x^*) \in E \times E^*$ is *monotonically related* to M; that is, inequality (1.1) holds for each $(y, y^*) \in M$.

2. Which monotone linear operators are maximal monotone?

In Theorem 2.5, we will give a criterion for a monotone linear operator to be maximal monotone. Before embarking on the analysis leading up to this, we isolate in Lemma 2.1 a result that we will use a number of times in the sequel.

Lemma 2.1. *Let a, b and $c \in \mathbb{R}$. Then*

$$a\lambda^2 + b\lambda + c \geq 0 \quad \textit{for all} \quad \lambda \in \mathbb{R}$$

if, and only if,

$$a \geq 0, c \geq 0 \quad \textit{and} \quad b^2 \leq 4ac.$$

Definition 2.2. Let $T\colon\ D(T) \to E^*$ be monotone and linear. Write

$$\begin{aligned} H(T) := \{x \in E\colon \quad &\text{there exists} \quad M \geq 0 \ \text{ such that,} \\ &\text{for all} \ \ y \in D(T),\ \langle Ty, x - y\rangle \leq M\|x - y\|\}. \end{aligned} \tag{2.1}$$

$H(T)$ stands for the *halo* of T. It is easy to see that $H(T)$ is closed under multiplication by scalars and that $H(T)$ is an F_σ: take $M = 1,\ 2,\ 3, \ldots$ to get the appropriate closed sets. The relevance of $H(T)$ stems from Lemma 2.3 below. Before starting on that, we recall that if C is a convex subset of E then a function $f\colon\ C \to \mathbb{R} \cup \{\infty\}$ is said to be *convex* if

$$x,\ y \in C \quad \text{and} \quad 0 < \lambda < 1 \quad \Longrightarrow \quad f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y),$$

and a function g on C is said to be *concave* if $-g$ is convex.

Lemma 2.3. *Let $T\colon\ D(T) \to E^*$ be monotone and linear and $x \in E$. Then:*

$$x \in H(T) \iff$$
$$\textit{there exists} \quad x^* \in E^* \quad \textit{such that} \quad (x, x^*) \quad \textit{is monotonically related to} \quad G(T).$$

Proof. ($\Longleftarrow$) If (x, x^*) is monotonically related to $G(T)$ then, for all $y \in D(T)$,

$$\langle Ty, x-y\rangle \le \langle x^*, x-y\rangle \le \|x^*\|\|x-y\|.$$

The required result follows by taking $M := \|x^*\|$.

($\Longrightarrow$) Let $x \in H(T)$. It follows easily from the monotonicity of T that the function $y \to \langle Ty, x-y\rangle$ is concave on D(T) (see the proof in Proposition 5.3(a) that the function $y \to \langle Ty, y\rangle$ is convex). Suppose further that M is as in (2.1), so that,

$$\text{for all} \quad y \in D(T), \quad \langle Ty, x-y\rangle \le M\|x-y\|. \tag{2.2}$$

Let

$$X := \{(y, \lambda) \in E \times \mathbb{R}\colon\ \lambda \ge M\|y-x\|\}$$

and

$$Y := \{(y, \lambda) \in D(T) \times \mathbb{R}\colon\ \lambda \le \langle Ty, x-y\rangle\}.$$

X and Y are nonempty convex subsets of $E \times \mathbb{R}$, and X has nonempty interior. Further, it follows from (2.2) that $\operatorname{int} X \cap Y = \emptyset$. From the Eidelheit separation theorem, there exist $\alpha \in \mathbb{R}$ and $(y^*, \mu) \in E^* \times \mathbb{R}$ such that

$$(y^*, \mu) \ne (0, 0), \tag{2.3}$$

$$y \in E \quad \text{and} \quad \lambda \ge M\|y-x\| \quad \text{imply that} \quad \langle y^*, y\rangle + \lambda\mu \ge \alpha, \tag{2.4}$$

and

$$y \in D(T) \quad \text{and} \quad \lambda \le \langle Ty, x-y\rangle \quad \text{imply that} \quad \langle y^*, y\rangle + \lambda\mu \le \alpha. \tag{2.5}$$

Setting $y = 0$ in (2.5), we obtain that $\lambda\mu \le \alpha$ for all $\lambda \le 0$, so that $\mu \ge 0$. We next show that $\mu > 0$. If we had $\mu = 0$ then, from (2.4),

$$y \in E \quad \text{implies that} \quad \langle y^*, y\rangle \ge \alpha,$$

hence $y^* = 0$, contradicting (2.3). So, indeed, $\mu > 0$. Now set $x^* := y^*/\mu$ and $\beta := \alpha/\mu$. Dividing (2.4) by μ and setting $(y, \lambda) := (x, 0)$, and dividing (2.5) by μ and setting $\lambda := \langle Ty, x-y\rangle$, we obtain that

$$\langle x^*, x\rangle \ge \beta$$

and

$$y \in D(T) \quad \text{implies that} \quad \langle x^*, y\rangle + \langle Ty, x - y\rangle \le \beta.$$

Combining these two inequalities,

$$\text{for all } \ y \in D(T), \quad \langle Ty - x^*, y - x\rangle \ge 0,$$

so (x, x^*) is monotonically related to $G(T)$, as required. This gives the required result. □

Remark 2.4. The authors are grateful to the referee for pointing out that Lemma 2.3 can also be deduced from the Moreau-Rockafellar formula for the subdifferential of the sum of convex functions. Define $g\colon\ E \to \mathbb{R}$ by $g(y) := M\|x - y\|$, and $f\colon\ E \to \mathbb{R} \cup \{\infty\}$ by

$$f(y) := \begin{cases} \langle Ty, y - x\rangle & \text{if } y \in D(T); \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 2.5. *Let $T\colon\ D(T) \to E^*$ be monotone and linear. Then T is maximal monotone if, and only if, $D(T)$ is dense and $H(T) = D(T)$.*

Proof. $(\Longrightarrow)$ Suppose that T is maximal monotone. We will prove first that

$$x^* \in E^* \ \text{ and } \ \langle x^*, y\rangle = 0 \ \text{ for all } \ y \in D(T) \quad \text{imply that} \quad x^* = 0;$$

this will establish that $D(T)$ is dense. Let $x^* \in E^*$ and $\langle x^*, y\rangle = 0$ for all $y \in D(T)$. Then

$$\text{for all } \ y \in D(T), \quad \langle Ty - x^*, y - 0\rangle = \langle Ty - x^*, y\rangle = \langle Ty, y\rangle \ge 0,$$

thus $(0, x^*)$ is monotonically related to $G(T)$. Since T is maximal, $(0, x^*) \in G(T)$ and so $x^* = T0 = 0$. This completes the proof that $D(T)$ is dense. If $x \in D(T)$ then (x, Tx) is monotonically related to $G(T)$ hence, from Lemma 2.3$(\Longleftarrow)$, $x \in H(T)$. If, on the other hand, $x \in H(T)$ then, from Lemma 2.3$(\Longrightarrow)$, there exists $x^* \in E^*$ such that (x, x^*) is monotonically related to $G(T)$. Since T is maximal monotone, $(x, x^*) \in G(T)$, hence $x \in D(T)$. This completes the proof that $H(T) = D(T)$.

$(\Longleftarrow)$ Let $D(T)$ be dense and $H(T) = D(T)$. Suppose that (x, x^*) is monotonically related to $G(T)$. From Lemma 2.3$(\Longleftarrow)$, $x \in H(T)$, hence $x \in D(T)$. Now let $z \in D(T)$, and λ be an arbitrary real number. Then $x + \lambda z \in D(T)$, hence

$$(x + \lambda z, Tx + \lambda Tz) = (x + \lambda z, T(x + \lambda z)) \in G(T).$$

By hypothesis,

$$\langle Tx + \lambda Tz - x^*, x + \lambda z - x\rangle \ge 0.$$

This can be rewritten

$$\lambda^2\langle Tz, z\rangle + \lambda\langle Tx - x^*, z\rangle \ge 0.$$

From Lemma 2.1, $\langle Tx - x^*, z\rangle = 0$. Since this holds for all $z \in D(T)$ and $D(T)$ is dense, $Tx - x^* = 0 \in E^*$, hence $x^* = Tx$. Thus $(x, x^*) = (x, Tx) \in G(T)$. This completes the proof that T is maximal monotone. □

Corollary 2.6 is well known. It can be deduced easily from Theorem 2.5 and the fact that $H(T) \supset D(T)$ when T is monotone.

Corollary 2.6. *Let $T\colon\ E \to E^*$ be monotone and linear. Then T is maximal monotone.*

3. Other preliminary results

We collect some useful elementary observations in Proposition 3.2. First, we introduce two important subclasses of linear operators into the dual space:

Definition 3.1. A linear operator $T\colon D(T) \to E^*$ is said to be *symmetric* (resp. *anti-symmetric*) if for every $x, y \in D(T)$ we have $\langle Tx, y\rangle = \langle Ty, x\rangle$ (resp. $\langle Tx, y\rangle = -\langle Ty, x\rangle$).

Proposition 3.2. *Let $T\colon D(T) \to E^*$ be linear and monotone.*

(a) *An element $(x, x^*) \in E \times E^*$ is monotonically related to $G(T)$ if and only if*

$$\langle x^*, x\rangle \geq 0 \quad \textit{and} \quad [\langle Ty, x\rangle + \langle x^*, y\rangle]^2 \leq 4\langle x^*, x\rangle\langle Ty, y\rangle \qquad \textit{for all} \quad y \in D(T). \tag{3.1}$$

(b) *If T is bounded and maximal, then $D(T) = E$.*
(c) *If $D(T)$ is dense in E and $x \in D(T)$ (or $R(T)$ is weak* dense in E^* and $x^* \in R(T)$) and $(x, x^*) \in E \times E^*$ is monotonically related to $G(T)$, then $(x, x^*) \in G(T)$.*
(d) *If $R(T) = E^*$, then T is maximal. (In fact, we will prove in* Theorem 6.7 *that much more is true.)*
(e) *If T has dense domain $D(T)$ and is symmetric or anti-symmetric, then $G(T)$ is closed in $D(T) \times E^*$ in the topology induced by the weak $\times$ weak* topology.*
(f) *If $D(T)$ is dense in E and T is not maximal monotone, then T has a maximal monotone linear extension.*
(g) *If T is maximal monotone, then it has closed graph in $E \times E^*$ in the product of the norm topologies.*
(h) *If $D(T) = E$, then T is bounded.*
(i) *If T is maximal monotone and one-one, then $R(T)$ is weak* dense in E^*.*
(j) *If T is one-one and $R(T) = E^*$, then $T^{-1}\colon E^* \to E$ is a bounded operator.*

Proof. (a) An element (x, x^*) is monotonically related to $G(T)$ if and only if for all $y \in D(T)$ and $\lambda \in \mathbb{R}$ we have

$$0 \leq \langle T(\lambda y) - x^*, \lambda y - x\rangle = \lambda^2\langle Ty, y\rangle - \lambda[\langle Ty, x\rangle + \langle x^*, y\rangle] + \langle x^*, x\rangle.$$

The result follows from Lemma 2.1.

(b) If T is maximal then, from Theorem 2.5, $D(T)$ is dense in E, hence if T is bounded, it has a unique bounded monotone extension to all of E which, by maximality, must be equal to T; that is, $D(T) = E$.

(c) If $x \in D(T)$ and $(x, x^*) \in E \times E^*$ is monotonically related to $G(T)$, then for all $z \in D(T)$ and $\lambda \in \mathbb{R}$ we have

$$0 \leq \langle T(x + \lambda z) - x^*, (x + \lambda z) - x\rangle = \lambda^2\langle Tz, z\rangle + \lambda\langle Tx - x^*, z\rangle.$$

Lemma 2.1 shows that $\langle Tx - x^*, z\rangle = 0$ for all $z \in D(T)$; by the density hypothesis, $x^* = Tx$. Suppose now that $R(T)$ is weak* dense in E^* and $x^* \in R(T)$, and that (x, x^*) is monotonically related to $G(T)$. There exists $u \in D(T)$ such that $x^* = Tu$ and hence for all $z \in D(T)$ and $\lambda \in \mathbb{R}$ we have

$$0 \leq \langle T(u + \lambda z) - x^*, (u + \lambda z) - x\rangle = \lambda^2\langle Tz, z\rangle + \lambda\langle Tz, u - x\rangle.$$

It follows as before that $\langle Tz, u-x\rangle = 0$ for all $z \in D(T)$ that is, for all $Tz \in R(T)$. Since the latter is weak* dense, we have $x = u$ and therefore $x^* = Tx$.

(d) This is immediate from (c).

(e) Suppose that $x_\alpha \in D(T)$, that $x_\alpha \to x \in D(T)$ in the weak topology and that $Tx_\alpha \to x^* \in E^*$ in the weak* topology. Then, taking the appropriate sign depending on whether T is symmetric or anti-symmetric, for all $z \in D(T)$,

$$\langle Tx, z\rangle = \pm\langle Tz, x\rangle = \pm\lim\langle Tz, x_\alpha\rangle = \lim\langle Tx_\alpha, z\rangle = \langle x^*, z\rangle;$$

since $D(T)$ is dense, we conclude that $x^* = Tx$.

(f) By Zorn's lemma we can always find an extension U of T which is maximal among the linear monotone extensions of T, so it suffices to show that if U is not maximal monotone, then it has a monotone linear proper extension. Suppose that $(x, x^*) \notin G(U)$ is monotonically related to $G(U)$. Note first that by (c) above, $x \notin D(U)$, so the subspace $D(U) + \mathbb{R}x$ properly contains $D(U)$ and one can define $S(y + \alpha x) = Uy + \alpha x^*$ ($\alpha \in \mathbb{R}$, $y \in D(U)$). It follows from (3.1) that the linear extension S is monotone.

(g) Suppose that $(x_n, Tx_n) \in G(T)$ converges to (x, x^*) in the norm topology. If $y \in D(T)$, then

$$\langle x^* - Ty, x - y\rangle = \lim_{n\to\infty}\langle Tx_n - Ty, x_n - y\rangle \geq 0,$$

that is, (x, x^*) is monotonically related to the graph of T, hence $x^* = Tx$, by maximality.

(h) If $D(T) = E$, then by Corollary 2.6, T is maximal monotone, hence by (g), it has closed graph, hence by the closed graph theorem, it is bounded. Another way of seeing this is to use the fact that monotone operators are locally bounded.

(i) If there exists $x \in E$ such that $\langle Ty, x\rangle = 0$ for all $y \in D(T)$, it follows that $(x, 0) \in E \times E^*$ satisfies the inequality (3.1), that is, $(x, 0)$ is monotonically related to $G(T)$. By maximality, $0 = Tx$; since T is one-one, $x = 0$, which proves the weak* density of $R(T)$.

(j) By part (d), T is maximal and by part (g), it has closed graph. Thus, T^{-1} has closed graph, hence is bounded. $\square$

During the course of this paper, we will introduce several monotone linear operators that are not symmetric. Nevertheless, such operators always have a property which trivially holds for symmetric operators; this is brought into evidence by Lemma 3.3 (below). If $F \subset E$, we write

$$F^\perp := \{x^* \in E^*:\ \text{ for all } \ x \in F,\ \langle x^*, x\rangle = 0\}.$$

If $F \subset E^*$, we write

$$F_\perp := \{x \in E:\ \text{ for all } \ x^* \in F,\ \langle x^*, x\rangle = 0\}.$$

Finally, we write

$$N(T) := \{x \in D(T):\ Tx = 0\}.$$

Lemma 3.3. *Let $T\colon D(T) \to E^*$ be linear and monotone and $x \in D(T)$. Then:*

(a) $\langle Ty, x\rangle = 0$ *for all* $y \in D(T) \iff \langle Tx, z\rangle = 0$ *for all* $z \in D(T)$.

(b) $x \in R(T)_\perp \iff Tx \in D(T)^\perp$.
(c) *If $D(T)$ is dense then $D(T) \cap R(T)_\perp = N(T)$.*

Proof. (a) Let $z \in D(T)$ and $\lambda \in \mathbb{R}$. Since $y := x + \lambda z \in D(T)$, $\langle T(x+\lambda z), x\rangle = 0$, hence

$$\lambda^2\langle Tz, z\rangle + \lambda\langle Tx, z\rangle = \langle T(x+\lambda z), \lambda z\rangle = \langle T(x+\lambda z), x+\lambda z\rangle \geq 0.$$

Using Lemma 2.1, it follows that $\langle Tx, z\rangle = 0$. Conversely, let $y \in D(T)$ and $\lambda \in \mathbb{R}$. Since $z := x + \lambda y \in D(T)$, $\langle Tx, x+\lambda y\rangle = 0$, hence

$$\lambda^2\langle Ty, y\rangle + \lambda\langle Ty, x\rangle = \langle T(\lambda y), x+\lambda y\rangle = \langle T(x+\lambda y), x+\lambda y\rangle \geq 0.$$

Using Lemma 2.1 again, it follows that $\langle Ty, x\rangle = 0$. This completes the proof of (a), and (b) is simply a restatement of (a). (c) follows from (b) since $D(T)^\perp = \{0\}$ when $D(T)$ is dense. □

The following lemma ties in $R(T)_\perp$ with the "halo" concept introduced in Section 2.

Lemma 3.4. *Let $T\colon D(T) \to E^*$ be linear and monotone. Then $R(T)_\perp \subset H(T)$.*

Proof. Let $x \in R(T)_\perp$. Then, for all $y \in D(T)$, $\langle Ty, x\rangle = 0$ hence

$$\langle Ty, x-y\rangle = -\langle Ty, y\rangle \leq 0 = 0\|x-y\|.$$

□

We emphasize that the second conclusion of Corollary 3.5 below is that $N(T)$ is closed in E (not merely in $D(T)$).

Corollary 3.5. *Let $T\colon D(T) \to E^*$ be linear and maximal monotone. Then*

$$N(T) = R(T)_\perp \quad \textit{and} \quad N(T) \quad \textit{is closed in} \quad E.$$

Proof. From Theorem 2.5, $D(T)$ is dense, thus Lemma 3.3(c) implies that $D(T) \cap R(T)_\perp = N(T)$. However, from Lemma 3.4 and Theorem 2.5 (again), $R(T)_\perp \subset H(T) = D(T)$, hence $R(T)_\perp = N(T)$, as required. □

4. The adjoint mapping

For a better understanding of the properties of T it is helpful to introduce its adjoint mapping T^*. As usual with unbounded operators, this requires some care.

Definition 4.1. Let $D(T)$ be dense and $T\colon D(T) \to E^*$ be a linear operator. Define $D(T^*) \subset E^{**}$ to be the set of all $x^{**} \in E^{**}$ for which the linear functional

$$D(T) \ni y \to \langle x^{**}, Ty\rangle$$

is continuous. Since $D(T)$ is dense, this functional extends uniquely to all of E and is denoted by $T^*x^{**} \in E^*$; it is characterized by the identity

$$\langle T^*x^{**}, y\rangle = \langle x^{**}, Ty\rangle, \quad y \in D(T).$$

Treating E as a subspace of E^{**}, we define $D(\widetilde{T}) = E \cap D(T^*)$ and $\widetilde{T} = T^*|_E$. More precisely, writing $\widehat{x}$ for the canonical image of x in E^{**},

$$\widetilde{T}x := T^*\widehat{x}, \quad x \in D(T).$$

It is not immediately clear that $D(T^*)$ is nontrivial for an arbitrary unbounded operator T; however, Proposition 4.2 below shows that it is nontrivial for maximal monotone operators. That the inclusions $D(T) \subset D(\widetilde{T})$ or $D(\widetilde{T}) \subset D(T)$ fail in general is shown by Example 4.3 below. The same example shows that it is possible to have $D(T^*) \subset E$. The operator T^* *need not be monotone*, even for bounded anti-symmetric T [2, Example 5.2], but Proposition 4.2 shows that its restriction $\widetilde{T}$ *is* monotone.

Proposition 4.2. *If T is a linear maximal monotone operator, then $D(T^*)$ is weak* dense in E^{**}. Further, $\widetilde{T}$ is monotone.*

Proof. By the Hahn-Banach theorem, to prove the first assertion it suffices to show that if $y^* \in E^*$ and $\langle x^{**}, y^* \rangle = 0$ for all $x^{**} \in D(T^*)$, then $y^* = 0$. But if $y^* \neq 0$, then $(0, y^*) \in E \times E^*$ is not in $G(T)$ which, by Proposition 3.2(g), is a closed linear subspace of $E \times E^*$. The dual of this space is $E^* \times E^{**}$, under the usual pairing

$$\langle (u^*, u^{**}), (v, v^*) \rangle = \langle u^*, v \rangle + \langle u^{**}, v^* \rangle, \quad (v, v^*) \in E \times E^*, \ (u^*, u^{**}) \in E^* \times E^{**}.$$

It follows that there exists an element $(x^*, x^{**}) \in E^* \times E^{**}$ which vanishes on $G(T)$ but not at $(0, y^*)$. The latter statement implies that $\langle x^{**}, y^* \rangle \neq 0$, and therefore we cannot have $x^{**} \in D(T^*)$. But the former statement implies that

$$\langle x^*, y \rangle + \langle x^{**}, Ty \rangle = 0 \quad \text{for all } y \in D(T).$$

This shows that the function $D(T) \ni y \to \langle x^{**}, Ty \rangle = -\langle x^*, y \rangle$ is continuous, so that $x^{**} \in D(T^*)$ (and $-x^* = T^*(x^{**})$), a contradiction.

To prove the second assertion, we will prove that

$$y \in D(\widetilde{T}) \quad \text{implies that} \quad \langle \widetilde{T}y, y \rangle \geq 0. \tag{4.1}$$

If $y \in D(T) \cap D(\widetilde{T})$, then

$$\langle \widetilde{T}y, y \rangle = \langle T^*\widehat{y}, y \rangle = \langle \widehat{y}, Ty \rangle = \langle Ty, y \rangle \geq 0. \tag{4.2}$$

If, on the other hand, $y \in D(\widetilde{T}) \setminus D(T)$, we define $S\colon [D(T) + \mathbb{R}y] \to E^*$ by

$$S(x + \lambda y) = Tx - \lambda T^*\widehat{y}, \quad x \in D(T), \quad \lambda \in \mathbb{R}.$$

This is clearly a proper linear extension of T; consequently it is not monotone. Thus there exist $x \in D(T)$ and $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} 0 > \langle S(x + \lambda y), x + \lambda y \rangle &= \langle Tx - \lambda T^*\widehat{y}, x + \lambda y \rangle \\ &= \langle Tx, x \rangle + \lambda \langle Tx, y \rangle - \lambda \langle T^*\widehat{y}, x \rangle - \lambda^2 \langle T^*\widehat{y}, y \rangle \\ &= \langle Tx, x \rangle + \lambda \langle Tx, y \rangle - \lambda \langle \widehat{y}, Tx \rangle - \lambda^2 \langle \widetilde{T}y, y \rangle \\ &= \langle Tx, x \rangle + \lambda \langle Tx, y \rangle - \lambda \langle Tx, y \rangle - \lambda^2 \langle \widetilde{T}y, y \rangle \\ &= \langle Tx, x \rangle - \lambda^2 \langle \widetilde{T}y, y \rangle. \end{aligned}$$

Consequently $\lambda^2 \langle \widetilde{T}y, y \rangle > \langle Tx, x \rangle \geq 0$, from which $\langle \widetilde{T}y, y \rangle > 0$. If we combine this with (4.2) we obtain (4.1), which completes the proof of Proposition 4.2. $\square$

Example 4.3. In $L^1[0,1]$, let

$$D(T) = \{x \in L^1\colon\ x \ \text{ is absolutely continuous}, \ x(0) = 0 \ \text{ and } \ x' \in L^\infty[0,1]\}.$$

Define $T\colon D(T) \to L^\infty$ by $Tx = x'$. Then $D(T)$ is dense, T is monotone, the convex function $f(x) = \frac{1}{2}\langle Tx, x\rangle$ $(x \in D)$ is not lower semicontinuous, and T is neither symmetric nor anti-symmetric. Moreover, $R(T) = L^\infty$, so T is maximal monotone. Finally, $D(T^*)$ is contained in the canonical image of $L^1[0,1]$ in $(L^\infty)^*$, with

$$D(T^*) = \{\widehat{z}\colon\ z \in L^1,\ z \ \text{ is absolutely continuous}, \ z(1) = 0 \ \text{ and } \ z' \in L^\infty[0,1]\}, \quad (4.3)$$

and T^* is monotone.

Proof. Since $D(T)$ contains, for instance, all the C^1 functions on $[0,1]$ which vanish at 0, it is dense in $L^1[0,1]$. For all $x \in D(T)$ we have $\langle Tx, x\rangle = \int_0^1 x'x = \frac{1}{2}x(1)^2$, so it is clear that T is monotone and that $f(x) = \frac{1}{4}x(1)^2$. Let $x_0(t) = t$ and choose $\{x_n\}$ in $D(T)$ such that $x_n \to x_0$ in L^1 norm but $x_n(1) = 0$ for all n; then $\liminf f(x_n) = 0 < 1 = f(x_0)$, so f is not lower semicontinuous. The fact that T is neither symmetric nor anti-symmetric can easily be verified by comparing $\langle Tx, y\rangle$ with $\langle Ty, x\rangle$ when $x(t) = t$ and $y(t) = t^2$. If $x^* \in L^\infty[0,1]$, then $w(t) := \int_0^t x^*$ defines an element $w \in D(T)$ such that $x^* = Tw$. Thus $R(T) = L^\infty$, and it follows from Proposition 3.2(d) that T is maximal monotone. To prove the last assertion in this example, suppose that $x^{**} \in D(T^*) \subset (L^\infty)^*$; then since $R(T) = L^\infty$, there exists $y \in D(T)$ such that $T^*x^{**} = Ty$. It follows (using integration by parts and the fact that $x(0) = y(0) = 0$) that, for any $x \in D(T)$,

$$\begin{aligned}\langle x^{**}, Tx\rangle &= \langle T^*x^{**}, x\rangle = \langle Ty, x\rangle = \textstyle\int_0^1 y'x \\ &= x(1)y(1) - \textstyle\int_0^1 x'y = \int_0^1 x'z = \langle Tx, z\rangle = \langle \widehat{z}, Tx\rangle,\end{aligned}$$

where z is the element of L^1 given by

$$z(t) = y(1) - y(t) \quad (t \in [0,1]). \qquad (4.4)$$

Using again the fact that $R(T) = L^\infty$, this shows that $x^{**} = \widehat{z}$. Conversely, if for some $y \in D(T)$ we define z as in (4.4) then the functional

$$D(T) \ni x \to \langle \widehat{z}, Tx\rangle = \langle Tx, z\rangle = \textstyle\int_0^1 x'z = x(1)y(1) - \int_0^1 x'y\,dt = \int_0^1 xy'\,dt = \langle Ty, x\rangle$$

is clearly continuous, so $\widehat{z} \in D(T^*)$. This characterization has two interesting consequences: first, it shows that $D(T^*)$ can be identified with a subset of L^1, namely,

$$D(T^*) = \{\widehat{z}\colon\ z \in L^1,\ z = y(1) - y \quad \text{for some} \ \ y \in D(T)\},$$

which gives the representation in (4.3), and that $T^*\widehat{z} = -z'$. The monotonicity of T^* follows since $\langle \widehat{z}, T^*\widehat{z}\rangle = \langle T^*\widehat{z}, z\rangle = -\int_0^1 z'z\,dt = -\frac{1}{2}z(1)^2 + \frac{1}{2}z(0)^2 = \frac{1}{2}z(0)^2 \geq 0$. (In fact, once we know that $R(T) = L^\infty$, it follows from Theorem 6.7 that $D(T)$ is dense, T^* is monotone and that T has a number of other desirable properties.) $\square$

The final lemma of this section shows that we can say more about $D(\widetilde{T})$ and $\widetilde{T}$ if T is symmetric or anti-symmetric. It should be emphasized that *it remains an open question whether $D(\widetilde{T})$ is nontrivial when T is neither symmetric nor anti-symmetric.*

Lemma 4.4. *Suppose that $D(T)$ is dense and the linear operator $T\colon D(T) \to E^*$ is symmetric (resp. anti-symmetric). Then $D(T) \subset D(\widetilde{T})$, $\widetilde{T}|_{D(T)} = T$ (resp. $-T$) and $R(T) \subset R(\widetilde{T})$.*

Proof. Suppose that $x \in D(T)$; then, for all $y \in D(T)$ we have

$$\langle Ty, x\rangle = \langle Tx, y\rangle \quad (\text{resp.} \ -\langle Tx, y\rangle)$$

which is continuous in y, so $x \in D(\widetilde{T})$ and $\widetilde{T}x = Tx$ (resp. $-Tx$). Thus $D(T) \subset D(\widetilde{T})$, and it follows that $R(T) \subset R(\widetilde{T})$. □

5. Linear subdifferentials and convex functions

The next result shows that *symmetric* maximal monotone linear operators are easily and nicely characterized as the subdifferentials of convex lower semicontinuous extended-real-valued functions.

Theorem 5.1. *Suppose that $T\colon D(T) \to E^*$ is a linear monotone mapping. Then T is maximal monotone and symmetric if and only if there exists a proper lower semicontinuous function g on E such that $D(\partial g) = D(T)$, $T = \partial g$ and for all $x \in D(T)$, $g(x) = \frac{1}{2}\langle Tx, x\rangle$.*

Proof. Suppose that T is maximal monotone and symmetric. For any $x \in E$ define

$$g(x) := \sup_{y \in D(T)} \{\langle Ty, x\rangle - \tfrac{1}{2}\langle Ty, y\rangle\}. \tag{5.1}$$

Clearly, g is convex and lower semicontinuous. If $x, y \in D(T)$, then

$$0 \le \tfrac{1}{2}\langle Ty - Tx, y - x\rangle = \tfrac{1}{2}\langle Ty, y\rangle + \tfrac{1}{2}\langle Tx, x\rangle - \langle Ty, x\rangle,$$

which shows that, for $x \in D(T)$, we have $g(x) \le \frac{1}{2}\langle Tx, x\rangle$. On the other hand, taking $y = x$ in the definition of g shows that $g(x) \ge \frac{1}{2}\langle Tx, x\rangle$. Thus, $g(x) = \frac{1}{2}\langle Tx, x\rangle$ for all $x \in D(T)$, in particular, g is proper. Moreover, if $x \in D(T)$, then for all $y \in E$,

$$g(x) + \langle Tx, y - x\rangle = \tfrac{1}{2}\langle Tx, x\rangle + \langle Tx, y - x\rangle = \langle Tx, y\rangle - \tfrac{1}{2}\langle Tx, x\rangle \le g(y),$$

so that $Tx \in \partial g(x)$. Since ∂g is monotone and extends T, the maximality of the latter implies that they are equal.

To prove the converse, suppose that for a lower semicontinuous proper convex function g we have $D(T) = D(\partial g)$ and $T = \partial g$. From Rockafellar's maximal monotonicity theorem, T is maximal monotone. If we further assume that $g(x) = \frac{1}{2}\langle Tx, x\rangle$ whenever $x \in D(T)$, then it follows that T is symmetric. Indeed, by definition, $Tx \in \partial g(x)$ implies that

$$\langle Tx, y\rangle \le \lim_{\lambda \to 0^+} \frac{g(x + \lambda y) - g(x)}{\lambda} \quad \text{for all} \ \ y \in D(T),$$

and the right hand side can be computed directly to be $\frac{1}{2}[\langle Tx, y\rangle + \langle Ty, x\rangle]$, so

$$\langle Tx, y\rangle \le \tfrac{1}{2}[\langle Tx, y\rangle + \langle Ty, x\rangle] \quad \text{for all} \ \ x,\ y \in D(T).$$

By linearity, we actually have equality:

$$\langle Tx, y\rangle = \tfrac{1}{2}[\langle Tx, y\rangle + \langle Ty, x\rangle],$$

hence $\langle Tx, y\rangle = \langle Ty, x\rangle$ for all $x, y \in D(T)$, i.e., T is symmetric. □

It is not hard to see that, *if T is linear, symmetric and maximal monotone and g satisfies the conditions of Theorem 5.1, then* $\mathrm{dom}(g) = E$ *if and only if T is bounded.* ($\mathrm{dom}(g)$ is defined to be $\{x \in E\colon\ g(x) \in \mathbb{R}\}$.) Indeed, if T is bounded then, from Proposition 3.2(b), $D(T) = E$. Since $\mathrm{dom}(g) \supset D(\partial g) = D(T)$, $\mathrm{dom}(g) = E$. Conversely, if $\mathrm{dom}(g) = E$, then (from [10, Proposition 3.3] and the Eidelheit Separation Theorem) $\partial g(x) \neq \emptyset$ for all $x \in E$, therefore $D(T) = D(\partial g) = E$. But ([10, Theorem 2.28]) any monotone operator is locally bounded on the interior of its domain, so T is bounded. If T is not symmetric, then the function g defined by (5.1) can be quite disappointing. This is shown by the following continuation of Example 4.3.

Example 5.2. In Example 4.3, let g be defined by (5.1). If $x \in L^1$ is the function with constant value c then $g(x) = \frac{1}{2}c^2$. If x is not a constant function then $g(x) = \infty$. Consequently, $\mathrm{dom}(g)$ has a trivial intersection both with $D(T)$ and with $D(\widetilde{T})$.

Proof. Suppose first that $x \in L^1$ has constant value c. Then

$$\begin{aligned} g(x) &= \sup_{y \in D(T)} \left[\langle Ty, x\rangle - \tfrac{1}{2}\langle Ty, y\rangle\right] = \sup_{y \in D(T)} \left[\textstyle\int_0^1 y'c - \tfrac{1}{2}y(1)^2\right] \\ &= \sup_{y \in D(T)} \left[cy(1) - \tfrac{1}{2}y(1)^2\right] = \sup_{\lambda \in \mathbb{R}} \left[c\lambda - \tfrac{1}{2}\lambda^2\right] = \tfrac{1}{2}c^2. \end{aligned}$$

If, on the other hand, $x \in L^1$ is not a constant function then there exists $x^* \in L^\infty$ such that $\langle x^*, 1\rangle = 0$ but $\langle x^*, x\rangle \neq 0$. Find $w \in D(T)$ as in Example 4.3 so that $Tw = x^*$. Then

$$w(1) = \textstyle\int_0^1 x^* = \langle x^*, 1\rangle = 0,$$

and consequently $\langle Tw, w\rangle = \frac{1}{2}w(1)^2 = 0$. Thus

$$\begin{aligned} g(x) &\geq \sup_{\lambda \in \mathbb{R}} \left[\langle T(\lambda w), x\rangle - \tfrac{1}{2}\langle T(\lambda w), \lambda w\rangle\right] \\ &= \sup_{\lambda \in \mathbb{R}} \left[\lambda\langle Tw, x\rangle - \tfrac{1}{2}\lambda^2\langle Tw, w\rangle\right] = \sup_{\lambda \in \mathbb{R}} \lambda\langle Tw, x\rangle = \sup_{\lambda \in \mathbb{R}} \lambda\langle x^*, x\rangle = \infty. \end{aligned}$$

□

If T is not necessarily symmetric, we consider a different function, f in Proposition 5.3 below. Example 4.3 shows that f need not be lower semicontinuous, even if T is maximal monotone. On the other hand, the subdifferential of f has a nice characterization; it is $\frac{1}{2}(T + \widetilde{T})$.

Proposition 5.3. *Suppose that $T\colon D(T) \to E^*$ is linear and define*

$$f(x) := \begin{cases} \frac{1}{2}\langle Tx, x\rangle & \text{if } x \in D(T); \\ \infty & \text{otherwise.} \end{cases}$$

(a) *f is convex if and only if T is monotone.*

(b) *Suppose now that $D(T)$ is dense and T is monotone. Then $D(\partial f) = D(T) \cap D(\widetilde{T})$. Moreover, ∂f has the following description: If $x \in D(\partial f)$, then $\partial f(x)$ consists of the single functional $\frac{1}{2}(T + \widetilde{T})x$. Finally,*

$$\{x \in D(T)\colon\ f(x) = 0\} = \{x \in D(\partial f)\colon\ \partial f(x) = \{0\}\}.$$

Proof. (a) is immediate from the following identity: If $0 < \alpha < 1$ and $x, y \in D(T)$, then

$$\alpha f(x) + (1-\alpha) f(y) - f[\alpha x + (1-\alpha) y] = \alpha(1-\alpha)\tfrac{1}{2}\langle T(x-y), (x-y)\rangle.$$

(b) If $x^* \in \partial f(x)$ then $x \in \operatorname{dom}(f) = D(T)$ and, for any $y \in D(T)$ and $\lambda > 0$, we have

$$\langle x^*, y\rangle \le \frac{f(x+\lambda y) - f(x)}{\lambda} = \tfrac{1}{2}\langle Tx, y\rangle + \tfrac{1}{2}\langle Ty, x\rangle + \tfrac{1}{2}\lambda\langle Ty, y\rangle.$$

Letting $\lambda \to 0^+$,

$$\langle x^*, y\rangle \le \tfrac{1}{2}[\langle Tx, y\rangle + \langle Ty, x\rangle], \quad y \in D(T).$$

Since both sides are linear in y, we have

$$\langle x^*, y\rangle = \tfrac{1}{2}[\langle Tx, y\rangle + \langle Ty, x\rangle], \quad y \in D(T).$$

Moreover, the first two of the three terms in this equality are continuous for $y \in D(T)$, hence so is the third; that is, $x \in D(\widetilde{T})$ and $\langle Ty, x\rangle = \langle \widetilde{T}x, y\rangle$ for $y \in D(T)$. Since the latter is dense, we conclude that $x^* = \frac{1}{2}[Tx + \widetilde{T}x]$ and that $D(\partial f) \subset D(T) \cap D(\widetilde{T})$.

To prove the reverse inclusion, suppose that $x \in D(T) \cap D(\widetilde{T})$. We will prove that

$$\text{for all } \; y \in E, \quad f(x+y) \ge f(x) + \langle \tfrac{1}{2}(T + \widetilde{T})x, y\rangle, \tag{5.2}$$

from which $\frac{1}{2}(T + \widetilde{T})x \in \partial f(x)$, hence $x \in D(\partial f)$, as required. If $y \in D(T)$ then, noting that $\langle \widetilde{T}x, y\rangle = \langle Ty, x\rangle$,

$$\tfrac{1}{2}\langle T(x+y), x+y\rangle - \tfrac{1}{2}\langle Tx, x\rangle - \langle \tfrac{1}{2}(T+\widetilde{T})x, y\rangle = \tfrac{1}{2}\langle Ty, y\rangle \ge 0,$$

and (5.2) follows on rearranging the terms. If, on the other hand, $y \in E \setminus D(T)$ then (5.2) follows since $f(x+y) = \infty$. This completes the proof of (5.2).

To prove the last assertion, suppose that $f(x) = 0$; then for all $y \in E$ we have $0 \le f(y) - f(x)$, hence $x \in D(\partial f)$ and $0 \in \partial f(x)$. For the reverse inclusion, if $x \in D(\partial f)$ and $0 \in \partial f(x)$, then $0 = \frac{1}{2}[Tx + \widetilde{T}x]$, hence $\langle Tx, y\rangle + \langle Ty, x\rangle = 0$ for all $y \in D(T)$; taking $y = x$ shows that $0 = \langle Tx, x\rangle$, so $f(x) = 0$. $\square$

Continuing with the assumptions of Proposition 5.3(b) that $D(T)$ is dense and T is monotone, it is useful at this point to introduce the *closure* $\overline{f}$ of f. This is a proper lower semicontinuous convex function which can be defined in several ways: Its epigraph is the closure in $E \times \mathbb{R}$ of the epigraph of f; alternatively, it is the restriction to $E \subset E^{**}$ of the second Fenchel dual f^{**} of f. For our purpose, the following "local" definition will be most convenient:

$$\overline{f}(x) = \lim_{\epsilon \to 0^+} \inf f[B(x;\epsilon) \cap D(T)], \quad x \in E.$$

It follows that if $x \in \operatorname{dom}(\overline{f})$ then there exists $\{x_n\} \subset \operatorname{dom}(f) = D(T)$ such that $x_n \to x$ and $f(x_n) \to \overline{f}(x)$ and, (almost) conversely, if there exists $\{x_n\} \subset \operatorname{dom}(f) = D(T)$ such that $x_n \to x$ and $f(x_n) \to \lambda \in \mathbb{R}$ then $x \in \operatorname{dom}(\overline{f})$ and $\overline{f}(x) \le \lambda$. If $x \in D(T)$, then $\overline{f}(x) \le f(x)$, while $\overline{f}(x) = f(x)$ if and only if f is lower semicontinuous at x. It follows easily from the definitions that if $x \in D(\partial f)$, then $\partial f(x) = \partial\overline{f}(x)$, so $D(\partial f) \subset D(\partial\overline{f})$. It is clear that $f(\lambda x) = \lambda^2 f(x)$ for all $x \in D(T)$, $\lambda \in \mathbb{R}$, and it then follows from the definitions that $\overline{f}(\lambda x) = \lambda^2\overline{f}(x)$ for all $x \in \mathrm{dom}(\overline{f})$, $\lambda \in \mathbb{R}$. This implies that $\mathrm{dom}(\overline{f})$ is closed under multiplication by scalars so, since it is convex, it is a linear subspace of E.

If, in addition, T is anti-symmetric then $f \equiv 0$ and hence $\mathrm{dom}(\overline{f}) = E$ and $\overline{f} \equiv 0$. It is routine to verify that in Example 4.3 (where T is not anti-symmetric), it is still true that $\mathrm{dom}(\overline{f}) = E$ ($\equiv L^1$) and $\overline{f} \equiv 0$.

Proposition 5.4. *Let $D(T)$ be dense, T be linear and monotone, f and $\overline{f}$ be as above and $x \in \mathrm{dom}(\overline{f})$. Then $x \in D(\partial\overline{f})$ if and only if there exists a unique element $x^* \in E^*$ such that*

(i) $\overline{f}(x) = \frac{1}{2}\langle x^*, x\rangle$ *and*

(ii) *there exists $\{x_n\} \subset D(T)$ such that $x_n \to x$, $f(x_n) \to \overline{f}(x)$ and*

$$\langle x^*, z\rangle = \tfrac{1}{2}\lim_{n\to\infty}\langle Tx_n, z\rangle + \tfrac{1}{2}\langle Tz, x\rangle \quad \textit{for all} \quad z \in D(T). \tag{5.3}$$

Proof. Suppose that $x \in D(\partial\overline{f})$ and let x^* be any element of $\partial\overline{f}(x)$. Since $x \in \mathrm{dom}(\overline{f})$, there exists a sequence $\{x_n\} \subset D(T)$ such that $x_n \to x$ and $f(x_n) \to \overline{f}(x)$. Fix $z \in D(T)$. Given $\beta \in \mathbb{R}$, let $y_n = x_n + \beta z$, so that $y_n \in D(T)$. We have

$$\begin{aligned}\beta\langle x^*, z\rangle + \langle x^*, x_n - x\rangle &= \langle x^*, y_n - x\rangle \le \overline{f}(y_n) - \overline{f}(x) \le f(y_n) - \overline{f}(x)\\ &= \tfrac{1}{2}\langle T(x_n + \beta z), x_n + \beta z\rangle - \overline{f}(x)\\ &= \tfrac{1}{2}\beta[\langle Tx_n, z\rangle + \langle Tz, x_n\rangle] + \tfrac{1}{2}\beta^2\langle Tz, z\rangle + f(x_n) - \overline{f}(x).\end{aligned}$$

Letting $n \to \infty$, we see that

$$\beta\langle x^*, z\rangle \le \tfrac{1}{2}\liminf_{n\to\infty}\beta\langle Tx_n, z\rangle + \tfrac{1}{2}\beta\langle Tz, x\rangle + \tfrac{1}{2}\beta^2\langle Tz, z\rangle.$$

If $\beta > 0$, we can divide by β and let $\beta \to 0^+$ to conclude that

$$\langle x^*, z\rangle \le \tfrac{1}{2}\liminf_{n\to\infty}\langle Tx_n, z\rangle + \tfrac{1}{2}\langle Tz, x\rangle.$$

Taking $\beta < 0$ and dividing (hence reversing the inequality) leads similarly to

$$\langle x^*, z\rangle \ge \tfrac{1}{2}\limsup_{n\to\infty}\langle Tx_n, z\rangle + \tfrac{1}{2}\langle Tz, x\rangle.$$

It follows that $\lim_{n\to\infty}\langle Tx_n, z\rangle$ exists and

$$\langle x^*, z\rangle = \tfrac{1}{2}\lim_{n\to\infty}\langle Tx_n, z\rangle + \tfrac{1}{2}\langle Tz, x\rangle.$$

Since this holds for any $x^* \in \partial\overline{f}(x)$ and z in the dense set $D(T)$, the set $\partial\overline{f}(x)$ is a singleton. Finally, for all $\lambda \in \mathbb{R}$,

$$\lambda^2\overline{f}(x) - \lambda\langle x^*, x\rangle + [\langle x^*, x\rangle - \overline{f}(x)] = \overline{f}(\lambda x) - \overline{f}(x) - \langle x^*, \lambda x - x\rangle \ge 0,$$

since $x^* \in \partial\overline{f}(x)$. From Lemma 2.1, $\langle x^*, x\rangle \le 4\overline{f}(x)[\langle x^*, x\rangle - \overline{f}(x)]$, which can be rewritten $[2\overline{f}(x) - \langle x^*, x\rangle]^2 \le 0$. Thus $\overline{f}(x) = \frac{1}{2}\langle x^*, x\rangle$, as required.

To prove the converse, suppose that $x \in \mathrm{dom}(\overline{f})$ and that there exists $x^* \in E^*$ satisfying (i) and (ii). Suppose also that $z \in \mathrm{dom}(\overline{f})$. Then there exists a sequence $\{z_m\} \subset D(T)$ such that $z_m \to z$ and $f(z_m) \to \overline{f}(z)$. From the monotonicity of T, for all m and n,

$$\begin{aligned} 2f(z_m) - \langle Tx_n, z_m\rangle - \langle Tz_m, x_n\rangle + 2f(x_n) \\ &= \langle Tz_m, z_m\rangle - \langle Tx_n, z_m\rangle - \langle Tz_m, x_n\rangle + \langle Tx_n, x_n\rangle \\ &= \langle Tz_m - Tx_n, z_m - x_n\rangle \geq 0. \end{aligned}$$

Letting $n \to \infty$, using (ii) and dividing by 2,

$$f(z_m) - \langle x^*, z_m\rangle + \overline{f}(x) \geq 0.$$

Letting $m \to \infty$,

$$\overline{f}(z) - \langle x^*, z\rangle + \overline{f}(x) \geq 0.$$

Thus, using (i),

$$\overline{f}(z) - \overline{f}(x) \geq \langle x^*, z\rangle - 2\overline{f}(x) = \langle x^*, z - x\rangle.$$

Consequently $x^* \in \partial\overline{f}(x)$, and so $x^* \in D(\partial\overline{f})$. □

It is natural to ask whether $\partial\overline{f}$ is an extension of T. This needn't be so: If it were, then for all $x \in D(T)$, we would have $x \in D(\partial\overline{f})$ and $Tx \in \partial\overline{f}(x)$. By Proposition 5.4, $\partial\overline{f}(x) = \{Tx\}$ and $\overline{f}(x) = \frac{1}{2}\langle Tx, x\rangle = f(x)$. This implies that f would be lower semicontinuous at each point of $D(T)$, which – by Example 4.3 above – is not generally true. It *is* true that $\partial\overline{f}$ is an extension of $\frac{1}{2}(T + \widetilde{T})$, and it might be a linear extension (see below). Indeed, the domain $D(T) \cap D(\widetilde{T})$ of the latter is, by Proposition 5.3, the same as $D(\partial f) \subset D(\partial\overline{f})$. Since f is lower semicontinuous at each point of $D(\partial f)$, it is equal to $\overline{f}$ on this set and therefore (Proposition 5.3 again) if $x \in D(T) \cap D(\widetilde{T})$, then $\partial\overline{f}(x) = \partial f(x) = \frac{1}{2}(T + \widetilde{T})(x)$. Thus, the maximal monotone operator $\partial\overline{f}$ extends $\frac{1}{2}(T+\widetilde{T})$. If $(x, x^*) \in G(\partial\overline{f})$ and $z \in D(T) \cap D(\widetilde{T})$ then (with $\{x_n\}$ as in Proposition 5.3)

$$\langle x^*, z\rangle = \tfrac{1}{2}\lim_{n\to\infty}\langle Tx_n, z\rangle + \tfrac{1}{2}\langle Tz, x\rangle = \tfrac{1}{2}\lim_{n\to\infty}\langle \widetilde{T}z, x_n\rangle + \tfrac{1}{2}\langle Tz, x\rangle = \tfrac{1}{2}\langle(\widetilde{T} + T)z, x\rangle.$$

To check linearity, let (x, x^*), $(y, y^*) \in G(\partial\overline{f})$, λ, $\mu \in \mathbb{R}$, and $(\lambda x + \mu y, w^*) \in G(\partial\overline{f})$. Then, from the above, for all $z \in D(T) \cap D(\widetilde{T})$, we have $\langle x^*, z\rangle = \frac{1}{2}\langle(\widetilde{T} + T)z, x\rangle$, $\langle y^*, z\rangle = \frac{1}{2}\langle(\widetilde{T} + T)z, y\rangle$ and $\langle w^*, z\rangle = \frac{1}{2}\langle(\widetilde{T} + T)z, \lambda x + \mu y\rangle$. Consequently, for all $z \in D(T) \cap D(\widetilde{T})$,

$$\langle w^*, z\rangle = \tfrac{1}{2}\lambda\langle(\widetilde{T} + T)z, x\rangle + \tfrac{1}{2}\mu\langle(\widetilde{T} + T)z, y\rangle = \lambda\langle x^*, z\rangle + \mu\langle y^*, z\rangle = \langle \lambda x^* + \mu y^*, z\rangle.$$

It follows that *$\partial\overline{f}$ is a linear extension of $\frac{1}{2}(T + \widetilde{T})$ if $D(T) \cap D(\widetilde{T})$ is dense*: for in this case the equalities above imply that that $w^* = \lambda x^* + \mu y^*$.

The authors are grateful to Heinz Bauschke for pointing out that if $D(T)$ is dense then so is $D(\partial\overline{f})$. Indeed, $\mathrm{dom}(\overline{f}) \supset \mathrm{dom}(f) = D(T)$, so $\mathrm{dom}(\overline{f})$ is dense. However, from the Brøndsted-Rockafellar theorem (see [10, Theorem 3.17]), $D(\partial\overline{f})$ is dense in $\mathrm{dom}(\overline{f})$.

6. Type (D), type (NI) and locally maximal monotone linear operators

We first consider linear monotone operators of type (D). Recall the definition for general monotone operators [8, 7]. (If $A \subset E$ (resp. $A \subset E \times E^*$), we denote by $\widehat{A}$ its natural embedding in E^{**} (resp. $E^{**} \times E^*$).)

Definition 6.1. A monotone operator $T\colon D(T) \to 2^{E^*}$ is said to be of *type* (D) provided the following holds: Whenever $(x^{**}, x^*) \in E^{**} \times E^*$ is monotonically related to $\widehat{G(T)}$, there exists a bounded net $(x_\alpha, x_\alpha^*) \subset G(T) \subset E \times E^*$ such that $\widehat{x}_\alpha \to x^{**}$ in the weak* topology of E^{**} and $\|x_\alpha^* - x^*\| \to 0$.

Remark 6.2. It was proved in [8] and [12] that *the subdifferential of a proper convex lower semicontinuous function is always maximal monotone of type* (D) *and locally maximal monotone.* (The definition of "locally maximal monotone" will be given after Example 6.4.) Such subdifferentials have many other properties, which are collected together in [14].

In what follows, we will be considering a subspace F of E with the property that $\widehat{F} = F^{\perp\perp}$. Note that this is equivalent to saying that $\widehat{F}$ is weak* closed in E^{**}, hence is satisfied by any finite dimensional subspace of E. The special case when F is one-dimensional turns out to be very useful in Example 6.4 below.

Lemma 6.3. *Let $D(T)$ be dense, $T\colon D(T) \to E^*$ be a monotone linear operator and suppose that there exists a subspace F of $D(T)$ such that $\widehat{F} = F^{\perp\perp}$ and $R(T) = F^{\perp}$. Then:*

(a) *$F \subset N(T)$.*
(b) *$R(T^*) \subset R(T)$.*
(c) *$N(T^*) = \widehat{F}$.*
(d) *$\widehat{G(T)}$ is a maximal monotone subset of $E^{**} \times E^*$, hence T is of type* (D).
(e) *If, in addition, T is symmetric or anti-symmetric then*

$$R(T^*) = R(T) \quad \textit{and} \quad D(T^*) = \widehat{D(T)}.$$

Proof. (a) Clearly $F \subset F^{\perp}{}_{\perp} = R(T)_{\perp}$ thus, from Lemma 3.3(c), $F \subset N(T)$.

(b) Let $x^* \in R(T^*)$ and $y \in F$. We choose $x^{**} \in D(T^*)$ such that $T^* x^{**} = x^*$. From (a),

$$\langle x^*, y\rangle = \langle T^* x^{**}, y\rangle = \langle x^{**}, Ty\rangle = \langle x^{**}, 0\rangle = 0.$$

Thus we have proved that $x^* \in F^{\perp} = R(T)$.

(c) $N(T^*) = R(T)^{\perp} = (F^{\perp})^{\perp} = F^{\perp\perp} = \widehat{F}$. The authors are grateful to Heinz Bauschke for supplying them with this proof.

(d) Let $(x^{**}, x^*) \in E^{**} \times E^*$ be monotonically related to $\widehat{G(T)}$, and let $y \in F$. From (a), $y \in N(T)$, that is to say, $(\widehat{y}, 0) \in \widehat{G(T)}$. Consequently, for all $\lambda \in \mathbb{R}$,

$$0 \le \langle x^{**} - \lambda\widehat{y}, x^* - 0\rangle = \langle x^{**}, x^*\rangle - \lambda\langle x^*, y\rangle.$$

This implies that $\langle x^*, y\rangle = 0$. Thus we have proved that $x^* \in F^\perp = R(T)$, and so we can find $x \in D(T)$ such that $Tx = x^*$. Then, for every $z \in D(T)$ and $\lambda \in \mathbb{R}$ we have

$$0 \le \langle x^{**} - (\widehat{x} + \lambda\widehat{z}), Tx - T(x + \lambda z)\rangle = \lambda^2\langle Tz, z\rangle - \lambda\langle x^{**} - \widehat{x}, Tz\rangle.$$

From Lemma 2.1, $\langle x^{**} - \widehat{x}, Tz\rangle = 0$ for all $z \in D(T)$ and so $x^{**} - \widehat{x} \in R(T)^\perp = F^{\perp\perp}$. Since $F^{\perp\perp} = \widehat{F}$, there exists $y \in F$ such that $x^{**} - \widehat{x} = \widehat{y}$. Thus $x^{**} = \widehat{x} + \widehat{y} = \widehat{x + y}$. On the other hand, $x^* = Tx = Tx + 0 = T(x + y)$, so $(x^{**}, x^*) = (\widehat{x + y}, T(x + y)) \in \widehat{G(T)}$.

(e) By (b) and Lemma 4.4, $R(T^*) \subset R(T)$ and $R(T) \subset R(\widetilde{T}) \subset R(T^*)$, from which $R(T^*) = R(T)$, as required. It is clear from Lemma 4.4 that $\widehat{D(T)} \subset D(T^*)$, so it only remains to prove that $D(T^*) \subset \widehat{D(T)}$. Let $x^{**} \in D(T^*)$. Then $T^*(x^{**}) \in R(T^*)$ and, from (b), there exists $x \in D(T)$ such that $Tx = T^*x^{**}$. Using Lemma 4.4, we have $Tx = \pm\widetilde{T}x = \pm T^*\widehat{x}$, thus $T^*x^{**} = \pm T^*\widehat{x}$. Consequently, $T^*(x^{**} \mp \widehat{x}) = 0$, that is to say, $x^{**} \mp \widehat{x} \in N(T^*)$. From (c), there exists $y \in F$ such that $\widehat{y} = x^{**} \mp \widehat{x}$. But then $x^{**} = \widehat{y} \pm \widehat{x} = \widehat{y \pm x} \in \widehat{D(T)}$. □

Example 6.4. In $L^1[0,1]$ let

$$D(T) = \{x \in L^1\colon\ x \text{ is absolutely continuous },\ x' \in L^\infty[0,1] \text{ and } x(0) = x(1)\}.$$

Define $T\colon D(T) \to L^\infty$ by $Tx = x'$. Then:

(a) $D(T)$ is dense in E; T is linear and anti-symmetric (hence monotone).

(b) $R(T^*) = R(T)$ and $D(T^*) = \widehat{D(T)}$.

(c) $\widehat{G(T)}$ is a maximal monotone subset of $E^{**} \times E^*$, hence T is of type (D).

Proof. (a) It is easily verified that $D(T)$ is dense. The first assertion about T is obvious and the second follows from integration by parts.

(b,c) First, note that $x^* \in L^\infty$ is in $R(T)$ if and only if $\int_0^1 x^* = 0$; indeed, if $x^* \in R(T)$, then $x^* = x'$ for some $x \in D(T)$ and hence $\int_0^1 x^* = x(1) - x(0) = 0$. On the other hand, if $\int_0^1 x^* = 0$, then $x^* = Tx$, where $x(t) = \int_0^t x^*$ $(t \in [0,1])$ defines a member of $D(T)$. To sum up: $R(T) = \{x^* \in L^\infty\colon\ \langle x^*, 1\rangle = 0\}$. Thus we can apply Lemma 6.3 with F the one-dimensional subspace of L^1 consisting of the constant functions. (b) now follows from Lemma 6.3(e), and (c) from Lemma 6.3(d). □

We now introduce the two other classes of operators that we will consider in this section, the operators of type (NI) (see [13]) and the locally maximal monotone operators (see [6]). For multivalued operators, their definitions read as follows:

Definition 6.5. Let $T : E \to 2^{E^*}$ be monotone. We say that T is *of type* (NI) if

$$\text{for all } (x^{**}, x^*) \in E^{**} \times E^*, \quad \inf_{(y,y^*) \in G(T)} \langle \widehat{y} - x^{**}, y^* - x^*\rangle \le 0.$$

We say that T is *locally maximal monotone* provided the following holds: For any open convex subset U of E^* such that $U \cap R(T) \ne \emptyset$, if $(x, x^*) \in E \times U$ is such that

$$y \in D(T) \text{ and } y^* \in Ty \cap U \quad \text{imply that} \quad \langle x^* - y^*, x - y\rangle \ge 0$$

then $x^* \in Tx$.

Our main result on these three classes of operators is going to be Theorem 6.7. As a prelude to this, we introduce in Lemma 6.6 a technical result in which we assume relatively few continuity conditions; precisely, T is not assumed to be continuous and g is not assumed to be lower semicontinuous. Lemma 6.6 replaces the Fenchel Duality Theorem used in [2] and [3]. We do not know whether the decomposition technique of [2] and [3] can be used in this case. This decomposition technique uses the symmetric operator $P = \frac{1}{2}(T + \widetilde{T})$ and the anti-symmetric operator $S = \frac{1}{2}(T - \widetilde{T})$. When $D(T) = E$ then $D(\widetilde{T}) = E$ also, so $D(P) = D(S) = E$. However, in the situation that we are considering, $D(P) = D(S) = D(T) \cap D(\widetilde{T})$, and there does not seem to be any *prima facie* reason why this set should contain anything other than 0 (though we do not have an example in which $D(T) \cap D(\widetilde{T})$ is not dense in E).

Lemma 6.6. *Let* $T\colon\ D(T) \to E^*$ *be linear,* $g\colon\ E \to \mathbb{R} \cup \{\infty\}$ *be convex and* U *be a convex open subset of* E^* *such that* $(\mathrm{dom}(g) \times U) \cap G(T) \neq \emptyset$. *Let* C *be the nonempty convex subset* $D(T) \cap \mathrm{dom}(g)$ *of* E. *Then*

$$\max_{x^{**} \in E^{**}} \inf_{x \in C,\ x^* \in U} [g(x) + \langle x^{**}, Tx - x^* \rangle] = \inf_{x \in C,\ Tx \in U} g(x).$$

Proof. Let $m := \inf_{x \in C,\ Tx \in U} g(x)$. If $y \in C$ and $Ty \in U$ then, for all $x^{**} \in E^{**}$ (setting $x = y$ and $x^* = Tx$),

$$g(y) = g(y) + \langle x^{**}, Ty - Ty \rangle \geq \inf_{x \in C,\ x^* \in U} [g(x) + \langle x^{**}, Tx - x^* \rangle].$$

Taking the infimum over y, $m \geq \inf_{x \in C,\ x^* \in U} [g(x) + \langle x^{**}, Tx - x^* \rangle]$. Thus it remains to prove that

$$\text{there exists } x^{**} \in E^{**} \text{ such that } \inf_{x \in C,\ x^* \in U} [g(x) + \langle x^{**}, Tx - x^* \rangle] \geq m. \tag{6.1}$$

This is obvious if $m = -\infty$, so we can and will assume that $m \in \mathbb{R}$. Let

$$X := \{(Tx, \lambda)\colon\ x \in C,\ \lambda \in \mathbb{R},\ g(x) \leq \lambda\} \quad \text{and} \quad Y := U \times (-\infty, m].$$

X and Y are nonempty convex subsets of $E^* \times \mathbb{R}$, and Y has nonempty interior. Further, we can derive from the definition of m that $X \cap \operatorname{int} Y = \emptyset$. From the Eidelheit separation theorem, there exist $\alpha \in \mathbb{R}$ and $(y^{**}, \mu) \in E^{**} \times \mathbb{R}$ such that

$$(y^{**}, \mu) \neq (0, 0), \tag{6.2}$$

$$x \in C \text{ and } \lambda \geq g(x) \text{ imply that } \langle y^{**}, Tx \rangle + \lambda\mu \geq \alpha, \tag{6.3}$$

and

$$x^* \in U \text{ and } \lambda \leq m \text{ imply that } \langle y^{**}, x^* \rangle + \lambda\mu \leq \alpha. \tag{6.4}$$

Fix $(z, z^*) \in (\mathrm{dom}(g) \times U) \cap G(T)$. From (6.4),

$$\lambda \leq m \text{ implies that } \lambda\mu \leq \alpha - \langle y^{**}, z^* \rangle,$$

and so $\mu \geq 0$. We next show that $\mu > 0$. If we had $\mu = 0$ then, from (6.3) and (6.4),

$$(x \in C \ \text{ implies that } \ \langle y^{**}, Tx\rangle \geq \alpha) \quad \text{and} \quad (x^* \in U \ \text{ implies that } \ \langle y^{**}, x^*\rangle \leq \alpha).$$

Now $z \in C$ and $z^* = Tz$ thus, combining the above two implications,

$$x^* \in U \quad \text{implies that} \quad \langle y^{**}, x^*\rangle \leq \langle y^{**}, z^*\rangle.$$

Since $z^* \in U$ and U is open, this would imply that $y^{**} = 0$, contradicting (6.2). So, indeed, $\mu > 0$. Now set $x^{**} := y^{**}/\mu$ and $\beta := \alpha/\mu$. Dividing (6.3) by μ and setting $\lambda = g(x)$, and dividing (6.4) by μ and setting $\lambda = m$, we obtain

$$x \in C \quad \text{implies that} \quad \langle x^{**}, Tx\rangle + g(x) \geq \beta$$

and

$$x^* \in U \quad \text{implies that} \quad \langle x^{**}, x^*\rangle + m \leq \beta.$$

Consequently,

$$x \in C \ \text{ and } \ x^* \in U \quad \text{imply that} \quad \langle x^{**}, Tx\rangle + g(x) \geq \langle x^{**}, x^*\rangle + m,$$

that is to say,

$$x \in C \ \text{ and } \ x^* \in U \quad \text{imply that} \quad g(x) + \langle x^{**}, Tx - x^*\rangle \geq m.$$

This gives (6.1), and completes the proof of Lemma 6.6. □

Theorem 6.7. *Let $T : D(T) \to E^*$ be linear and monotone. Then* (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longleftrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (f) *:*

(a) $R(T) = E^*$.

(b) $\widehat{G(T)}$ *is a maximal monotone subset of* $E^{**} \times E^*$.

(c) T *is maximal monotone of type* (D).

(d) T *is maximal monotone of type* (NI).

(e) T *is locally maximal monotone.*

(f) T^* *is monotone.*

Proof. ((a) $\Longrightarrow$ (b)) Let $(x^{**}, x^*) \in E^{**} \times E^*$ be monotonically related to $\widehat{G(T)}$. From (a), we can find $x \in D(T)$ such that $Tx = x^*$. Then, for every $z \in D(T)$ and $\lambda \in \mathbb{R}$ we have

$$0 \leq \langle x^{**} - (\widehat{x} + \lambda\widehat{z}), Tx - T(x + \lambda z)\rangle = \lambda^2\langle Tz, z\rangle - \lambda\langle x^{**} - \widehat{x}, Tz\rangle.$$

From Lemma 2.1, $\langle x^{**} - \widehat{x}, Tz\rangle = 0$ for all $z \in D(T)$. Since $R(T) = E^*$, this implies that $x^{**} = \widehat{x}$. Thus $(x^{**}, x^*) = (\widehat{x}, Tx) \in \widehat{G(T)}$, giving (b). (This implication can also be obtained from Lemma 6.3 with $F := \{0\}$ since, from Proposition 3.2 and Theorem 2.5, $D(T)$ is dense.)

It is obvious that (b)$\Longrightarrow$(c), and it was proved in [13, Lemma 15] that (c) $\Longrightarrow$ (d) (even in the general set-valued case).

((d) $\Longrightarrow$ (c)) Let $(x^{**}, x^*) \in E^{**} \times E^*$ be monotonically related to $\widehat{G(T)}$. We first prove that,

$$\text{for all } \ y^{**} \in D(T^*), \quad \langle y^{**}, x^* \rangle = \langle x^{**}, T^* y^{**} \rangle. \tag{6.5}$$

To this end, let $y^{**} \in D(T^*)$. Let λ be an arbitrary real number and write

$$\alpha := \lambda^2 \langle y^{**}, T^* y^{**} \rangle + \lambda [\langle x^{**}, T^* y^{**} \rangle - \langle y^{**}, x^* \rangle].$$

One can verify by direct computation that, for all $x \in D(T)$,

$$\alpha = \langle \widehat{x} - x^{**}, Tx - x^* \rangle - \langle \widehat{x} - (x^{**} + \lambda y^{**}), Tx - (x^* - \lambda T^* y^{**}) \rangle.$$

It follows that

$$\begin{aligned} \alpha &= \sup_{x \in D(T)} [\langle \widehat{x} - x^{**}, Tx - x^* \rangle - \langle \widehat{x} - (x^{**} + \lambda y^{**}), Tx - (x^* - \lambda T^* y^{**}) \rangle] \\ &\geq \inf_{x \in D(T)} \langle \widehat{x} - x^{**}, Tx - x^* \rangle - \inf_{x \in D(T)} \langle \widehat{x} - (x^{**} + \lambda y^{**}), Tx - (x^* - \lambda T^* y^{**}) \rangle. \end{aligned}$$

The first infimum is ≥ 0 since (x^{**}, x^*) is monotonically related to $\widehat{G(T)}$, and the second is ≤ 0 since T is of type (NI). Consequently, $\alpha \geq 0 - 0 = 0$. Thus we have proved that, for all $\lambda \in \mathbb{R}$,

$$\lambda^2 \langle y^{**}, T^* y^{**} \rangle + \lambda [\langle x^{**}, T^* y^{**} \rangle - \langle y^{**}, x^* \rangle] \geq 0,$$

and (6.5) now follows from Lemma 2.1. Now write $M := \|x^{**}\|$ and let (y^*, y^{**}) be an arbitrary element of $E^* \times D(T^*)$. From (6.5),

$$\begin{aligned} \langle x^{**}, y^* \rangle + \langle y^{**}, x^* \rangle &= \langle x^{**}, y^* + T^* y^{**} \rangle \\ &\leq M \|y^* + T^* y^{**}\| \\ &= \sup_{x \in D(T), \ \|x\| \leq M} \langle y^* + T^* y^{**}, x \rangle, \end{aligned}$$

the last equality following from the density of $D(T)$. Thus, for all $(y^*, y^{**}) \in E^* \times D(T^*)$,

$$\langle x^{**}, y^* \rangle + \langle y^{**}, x^* \rangle \leq \sup_{x \in D(T), \ \|x\| \leq M} [\langle \widehat{x}, y^* \rangle + \langle y^{**}, Tx \rangle]. \tag{6.6}$$

In fact, (6.6) is true for all $(y^*, y^{**}) \in E^* \times E^{**}$, for if $y^{**} \in E^{**} \setminus D(T^*)$ and $M > 0$ then

$$\sup_{x \in D(T), \ \|x\| \leq M} [\langle \widehat{x}, y^* \rangle + \langle y^{**}, Tx \rangle] \geq \sup_{x \in D(T), \ \|x\| \leq M} [\langle y^{**}, Tx \rangle - M \|y^*\|] = \infty.$$

Since $E^* \times E^{**}$ is the dual of $E^{**} \times E^*$ with respect to $w(E^{**}, E^*) \times \| \ \|$, we obtain from this and the bipolar theorem that (x^{**}, x^*) is in the $w(E^{**}, E^*) \times \| \ \|$-closure of the convex set $\{(\widehat{x}, Tx)\colon \ x \in D(T), \ \|x\| \leq M\}$. Thus we can find the net specified in the definition of "type (D)", which completes the proof of (c). (The net $\{Tx_\alpha\}$ can be taken to be bounded since it can be made to converge to x^* in norm.)

((d) $\Longrightarrow$ (e)) Let U be a convex open subset of E^* such that $U \cap R(T) \neq \emptyset$. Suppose also that $(y, y^*) \in E \times U$ and

$$\inf_{x \in D(T), \ Tx \in U} \langle Tx - y^*, x - y \rangle \geq 0.$$

Our aim is to prove that

$$(y, y^*) \in G(T). \tag{6.7}$$

From Lemma 6.6, with

$$g(x) := \begin{cases} \langle Tx - y^*, x - y \rangle & \text{if } x \in D(T); \\ \infty & \text{otherwise,} \end{cases}$$

(so $C = D(T)$) there exists $x^{**} \in E^{**}$ such that

$$\inf_{x \in D(T),\ x^* \in U} [\langle Tx - y^*, x - y \rangle + \langle x^{**}, Tx - x^* \rangle] \geq 0,$$

that is to say,

$$\inf_{x \in D(T),\ x^* \in U} [\langle \widehat{x} - \widehat{y} + x^{**}, Tx - y^* \rangle - \langle x^{**}, x^* - y^* \rangle] \geq 0.$$

This can be rewritten:

$$\inf_{x \in D(T)} \langle \widehat{x} - \widehat{y} + x^{**}, Tx - y^* \rangle \geq \sup_{x^* \in U} \langle x^{**}, x^* - y^* \rangle. \tag{6.8}$$

Since T is of type (NI),

$$\inf_{x \in D(T)} [\langle \widehat{x} - (\widehat{y} - x^{**}), Tx - y^* \rangle] \leq 0,$$

that is to say,

$$\inf_{x \in D(T)} [\langle \widehat{x} - \widehat{y} + x^{**}, Tx - y^* \rangle] \leq 0.$$

Consequently, from (6.8), $\sup_{x^* \in U} \langle x^{**}, x^* - y^* \rangle \leq 0$. Since $y^* \in U$ and U is open, we derive from this that $x^{**} = 0$. Substituting this back in (6.8), we obtain

$$\inf_{x \in D(T)} [\langle \widehat{x} - \widehat{y}, Tx - y^* \rangle] \geq 0.$$

that is to say,

$$\inf_{x \in D(T)} [\langle Tx - y^*, x - y \rangle] \geq 0.$$

(6.7) now follows from the maximal monotonicity of T.

((e) $\Longrightarrow$ (f)) Suppose that (f) fails, so there exists $x^{**} \in D(T^*)$ such that

$$\langle x^{**}, T^* x^{**} \rangle < 0. \tag{6.9}$$

This implies that

$$T^* x^{**} \neq 0. \tag{6.10}$$

Since $D(T)$ is dense, there exists $y \in D(T)$ such that $\langle T^* x^{**}, y \rangle < 0$, hence

$$\langle x^{**}, Ty \rangle < 0. \tag{6.11}$$

Now write U for the convex open set $\{x^* \in E^*\colon\ \langle x^{**}, x^* \rangle < 0\}$. It follows from (6.9) and (6.11) that $T^*x^{**} \in U$ and $Ty \in U$. In particular, $U \cap R(T) \neq \emptyset$. Now

$$\begin{aligned}\inf_{x \in D(T),\ Tx \in U} \langle Tx - T^*x^{**}, x - 0 \rangle &= \inf_{x \in D(T),\ Tx \in U} \langle Tx - T^*x^{**}, x \rangle \\ &= \inf_{x \in D(T),\ Tx \in U} [\langle Tx, x \rangle - \langle x^{**}, Tx \rangle] \geq 0,\end{aligned}$$

since $\langle Tx, x \rangle \geq 0$ for $x \in D(T)$ and $\langle x^{**}, Tx \rangle < 0$ for $Tx \in U$. It now follows from the locally maximal monotonicity of T that $(0, T^*x^{**}) \in G(T)$, hence $T^*x^{**} = 0$. This contradiction of (6.10) establishes (f), and completes the proof of Theorem 6.7. $\square$

Remark 6.8. Gossez has noted in [9] that the bipolar theorem and a technique of Brézis can be used to establish the implication (c) $\Longrightarrow$ (f) of Theorem 6.7.

We also point out that *if T is monotone, one-one and $R(T) = E^*$, then $G(T^*)$ is a maximal monotone subset of $E^{**} \times E^*$.* We know already from the result above that $G(T^*)$ is a monotone subset of $E^{**} \times E^*$. From Proposition 3.2(j), there exists a bounded linear operator S from E^* into E such that, for all $y \in D(T)$, $S(Ty) = y$. Then, for all $x^* \in E^*$,

$$\langle S^*x^*, Ty \rangle = \langle x^*, S(Ty) \rangle = \langle x^*, y \rangle$$

and so $S^*x^* \in D(T^*)$ and $T^*(S^*x^*) = x^*$. This shows that $R(T^*) = E^*$. The maximality then follows from an argument parallel to that in Proposition 3.2(d).

In Example 6.9 below, we give an example in which 6.7(f) is satisfied but 6.7(b) is not. Since T is continuous in this case, it will follow from Theorem 8.1 that in fact 6.7(c) is satisfied (but 6.7(b) is not).

Example 6.9. Define $T\colon \ell^1 \to \ell^\infty$ by

$$T(x_1, x_2, x_3, x_4, \dots) = (-x_2, x_1, -x_4, x_3, \dots).$$

T is bounded, linear and anti-symmetric, hence maximal monotone. Further, T^* is anti-symmetric but $\widehat{G(T)}$ is not a maximal monotone subset of $(\ell^\infty)^* \times \ell^\infty$.

Proof. Write φ for the canonical map from c_0 into ℓ^∞. We first show that

$$T^* = -T\varphi^*. \tag{6.12}$$

In order to do this, let x and x^{**} be arbitrary elements of $E = \ell^1$ and $E^{**} = (\ell^\infty)^*$, respectively. Since $R(T) = \ell^1 \subset \varphi(c_0) \subset \ell^\infty$, there exists $y \in c_0$ such that $Tx = \varphi y$. But then

$$\langle T^*x^{**}, x \rangle = \langle x^{**}, Tx \rangle = \langle x^{**}, \varphi y \rangle = \langle \varphi^* x^{**}, y \rangle = \langle \varphi y, \varphi^* x^{**} \rangle = \langle Tx, \varphi^* x^{**} \rangle.$$

However, from Lemma 4.4, $Tx = -T^*\widehat{x}$, thus

$$\langle T^*x^{**}, x \rangle = -\langle T^*\widehat{x}, \varphi^*x^{**} \rangle = -\langle \widehat{x}, T\varphi^*x^{**} \rangle = -\langle T\varphi^*x^{**}, x \rangle,$$

which establishes the validity of (6.12). It follows from two applications of (6.12) that

$$\begin{aligned}\langle x^{**}, T^*x^{**} \rangle &= -\langle x^{**}, T\varphi^*x^{**} \rangle = -\langle T^*x^{**}, \varphi^*x^{**} \rangle \\ &= \langle T\varphi^*x^{**}, \varphi^*x^{**} \rangle = \langle T(\varphi^*x^{**}), \varphi^*x^{**} \rangle = 0.\end{aligned}$$

Thus T^* is anti-symmetric, as required. Now let $x^{**} \in R(T)^\perp \setminus \{0\} \subset (\ell^\infty)^*$ (for instance, we could take x^{**} to be a "Banach limit"). Since T is anti-symmetric, for all $x \in \ell^1$,

$$\langle \widehat{x} - x^{**}, Tx - 0\rangle = \langle \widehat{x}, Tx\rangle - \langle x^{**}, Tx\rangle = \langle Tx, x\rangle - \langle x^{**}, Tx\rangle = 0 - 0 = 0.$$

So $(x^{**}, 0)$ is monotonically related to $\widehat{G(T)}$. If we had $x^{**} \in \widehat{\ell^1}$ then there would exist $x \in \ell^1 \subset \ell^\infty$ such that $x^{**} = \widehat{x}$; so $\langle x^{**}, x\rangle = \langle \widehat{x}, x\rangle > 0$. Since $x \in R(T)$, this would contradict $x^{**} \in R(T)^\perp$. Consequently, $x^{**} \notin \widehat{\ell^1}$, from which $(x^{**}, 0) \notin \widehat{G(T)}$. Thus $\widehat{G(T)}$ is not a maximal monotone subset of $(\ell^\infty)^* \times \ell^\infty$. □

Remark 6.10. Example 6.9 and Proposition 4.2 might lead one to suspect that if φ is the canonical map from c_0 into ℓ^∞, $T\colon\ \ell^1 \to \ell^\infty$ is monotone and $R(T) \subset \varphi(c_0)$ then T^* is monotone. The following example shows that this suspicion is false. Define T by $(Tx)_n = \sum_{k \geq n} x_k$. We leave it to the reader to check that T is monotone and $R(T) \subset \varphi(c_0)$. Let $e := (1, 1, 1, \dots) \in \ell^\infty$ and $e^{(1)} := (1, 0, 0, \dots) \in \ell^1$, and fix $x^{**} \in \varphi(c_0)^\perp$ so that $\langle x^{**}, e\rangle = -2$. Since x^{**} vanishes on $\varphi(c_0)$, $\varphi^* x^{**} = 0$. Then, exactly as in Example 6.9, for all $x \in E$, there exists $y \in c_0$ such that

$$\langle T^* x^{**}, x\rangle = \langle x^{**}, Tx\rangle = \langle x^{**}, \varphi y\rangle = \langle \varphi^* x^{**}, y\rangle = \langle \varphi y, \varphi^* x^{**}\rangle = \langle Tx, \varphi^* x^{**}\rangle.$$

so $T^* x^{**} = 0$. Further, by direct computation, $T^* \widehat{e^{(1)}} = e$. Thus

$$\langle \widehat{e^{(1)}} + x^{**}, T^*(\widehat{e^{(1)}} + x^{**})\rangle = \langle \widehat{e^{(1)}} + x^{**}, e\rangle = 1 - 2 < 0,$$

and so T^* is not monotone.

7. The sum problem

A fundamental and pervasive problem concerning maximal monotone operators is to determine precisely when the sum $S + T$ of two such operators (which is trivially monotone) is maximal. (We recall that the domain of the sum is by definition the intersection $D(S) \cap D(T)$.) Early work on this question culminated with Rockafellar's 1970 theorem [11] that in a reflexive Banach space, $S+T$ is maximal under the "constraint qualification"

$$\operatorname{int} D(S) \cap D(T) \neq \emptyset. \tag{7.1}$$

A different proof of this was given by Brezis, Crandall and Pazy [4]. Recently, Attouch, Riahi and Thera [1] and Chu [5] have weakened this condition further. It follows from the results in [1] that we need only assume that

$$\bigcup_{\lambda > 0} \lambda[D(S) - D(T)] = E. \tag{7.2}$$

In fact, it is shown in [15] that the constraint qualifications of [1] and [5] are equivalent. It still remains an open problem whether, even with (7.1), the sum theorem is valid for nonlinear operators in nonreflexive Banach spaces.

We will show in Theorem 7.2 that, even in nonreflexive Banach spaces, $S + T$ is maximal monotone in the linear case under condition (7.1). We do not know if the same is true under condition (7.2). However, we will show in Example 7.4 that, even for linear operators in the space ℓ_2, we cannot weaken (7.2) further to the condition

$$D(T) - D(S) \text{ is dense.}$$

We start off with a simple preliminary lemma.

Lemma 7.1. *Let $T\colon\ D(T) \to E^*$ be maximal monotone and linear and $(x, u^*) \in E \times E^*$. Then*

$$\inf_{y \in D(T)} [\langle Ty - u^*, y - x\rangle] \le 0.$$

Proof. If $(x, u^*) \in G(T)$ the result is immediate by taking $y := x$. If $(x, u^*) \notin G(T)$, it follows (with strict inequality) from the definition of maximal monotonicity. $\square$

We point out that if (7.1) is satisfied then $\operatorname{int} D(S) \ne \emptyset$ hence, since $D(S)$ is a subspace of E, $D(S) = E$. Consequently, Theorem 7.2 implies that if S and T are maximal monotone and linear and (7.1) is satisfied then $S + T$ is maximal monotone, as advertised above.

Theorem 7.2. *Let $S\colon\ E \to E^*$ be monotone and linear and $T\colon\ D(T) \to E^*$ be maximal monotone and linear. Then $S + T\colon\ D(T) \to E^*$ is maximal monotone.*

Proof. Let (x, z^*) be monotonically related to $G(S + T)$. Then

$$\text{for all } \ y \in D(T), \quad \langle Sy - z^*, y - x\rangle - \langle Ty, x - y\rangle = \langle (S + T)y - z^*, y - x\rangle \ge 0. \tag{7.3}$$

We now proceed as in Lemma 2.3, replacing the continuous convex function $y \to M\|x - y\|$ by the continuous convex function $y \to \langle Sy - z^*, y - x\rangle$, and obtain $x^* \in E^*$ such that

$$y \in E \ \text{ and } \ \lambda \ge \langle Sy - z^*, y - x\rangle \quad \text{imply that} \quad \langle x^*, y\rangle + \lambda \ge \beta, \tag{7.4}$$

and

$$y \in D(T) \ \text{ and } \ \lambda \le \langle Ty, x - y\rangle \quad \text{imply that} \quad \langle x^*, y\rangle + \lambda \le \beta. \tag{7.5}$$

Consequently

$$y \in E \quad \text{implies that} \quad \langle x^*, y\rangle + \langle Sy - z^*, y - x\rangle \ge \beta$$

and

$$y \in D(T) \quad \text{implies that} \quad \langle x^*, y\rangle + \langle Ty, x - y\rangle \le \beta.$$

Subtracting $\langle x^*, x\rangle$ from these inequalities and setting $\gamma := \beta - \langle x^*, x\rangle$, we derive that:

$$\inf_{y \in E} \langle Sy - (z^* - x^*), y - x\rangle \ge \gamma \quad \text{and} \quad \inf_{y \in D(T)} \langle Ty - x^*, y - x\rangle \ge -\gamma. \tag{7.6}$$

Now T is given to be maximal monotone, and it follows from Corollary 2.6 that S is maximal monotone. Thus, from two applications of Lemma 7.1, $\gamma \le 0$ and $-\gamma \le 0$, from which $\gamma = 0$. Substituting back in (7.6),

$$\inf_{y \in E} \langle Sy - (z^* - x^*), y - x\rangle \ge 0 \quad \text{and} \quad \inf_{y \in D(T)} \langle Ty - x^*, y - x\rangle \ge 0.$$

Since S and T are maximal monotone, $(x, z^* - x^*) \in G(S)$ and $(x, x^*) \in G(T)$. Consequently, $(x, z^*) = (x, (z^* - x^*) + x^*) \in G(S + T)$. This completes the proof that $S + T$ is maximal monotone. $\square$

Remark 7.3. The authors are grateful to the referee for pointing out that Theorem 7.2 can also be deduced from the Moreau-Rockafellar formula for the subdifferential of the sum of convex functions. (See the remark following Lemma 2.1.)

We now define continuous linear operators $V,\ W\colon \ell_2 \to \ell_2$ by

$$Vx := (x_1, x_2 - x_1, x_3 - x_2, \dots) \quad \text{and} \quad Wx := (x_1 - x_2, x_2 - x_3, x_3 - x_4, \dots)$$

for $x = (x_n) \in \ell_2$. Both V and W are injective, so we can define unbounded linear operators S and T by $S := V^{-1}$ (with $D(S) = R(V)$) and $T := W^{-1}$ (with $D(T) = R(W)$).

Example 7.4. The operators S and T are maximal monotone of type (D) and $D(S) \cap D(T)$ is dense in ℓ_2 (a necessary condition for maximality of their sum) but $S + T$ is not maximal, even though $D(S) - D(T)$ is dense in ℓ_2.

Proof. For all $x \in \ell_2$,

$$\langle Vx, x\rangle = \langle Wx, x\rangle = \tfrac{1}{2}\|Vx\|^2 \geq 0,$$

hence V and W are monotone. It follows from this and Theorem 6.7 that S and T are both maximal monotone of type (D). Consider the set D of all finitely non-zero sequences $x = (x_n)$ in ℓ_2 such that $\sum x_k = 0$. If $x \in D$ then $x = Vy$ and $x = Wz$, where $(y_n),\ (z_n) \in \ell_2$ are given by

$$y_n = \sum_{k \leq n} x_k \quad \text{and} \quad z_n = \sum_{k \geq n} x_k, \quad n = 1, 2, 3, \dots.$$

So $D \subset R(V) \cap R(W) = D(S) \cap D(T)$. On the other hand, D is dense in ℓ_2 – to see this, it is sufficient to show that it is dense in the set of all finitely non-zero sequences. If $y = (y_1, y_2, \dots, y_m, 0, 0, \dots)$ is one of the latter, with $s = \sum y_k$, then for any $n \geq 1$ consider

$$(y_1, y_2, \dots, y_m, -s/n, -s/n, \dots, -s/n, 0, 0, \dots) \quad (n \ \text{ terms of the form } \ -s/n).$$

Clearly this is in D and of distance $|s|/\sqrt{n}$ from y. So D is dense in ℓ_2, and consequently so also is the larger set $D(S) \cap D(T)$. If $x \in D(S) \cap D(T)$, then $\sum_{k=1}^{\infty} x_k = 0$, so

$$(S + T)(x)_n = \sum_{k=1}^{\infty} x_k + x_n = x_n.$$

Thus $S + T$ has a proper monotone extension from $D(S) \cap D(T) \subset \ell_2$ to the identity map on ℓ_2. Consequently, $S + T$ is not maximal monotone. □

Remark 7.5. Our use of the operators S and T was motivated by [2, Remark 4.4], who used the corresponding (bounded) operators from ℓ_1 to ℓ_∞.

8. The continuous case

We now consider the case when $D(T) = E$ and T is continuous, and obtain a new proof of the equivalence of (i), (ii), (iv) and (v) in [3, Theorem 4.1].

Theorem 8.1. *Let T be continuous and linear. Then the conditions* (a), (b), (c) *and* (d) *are equivalent.*

(a) *T is (maximal) monotone of type* (D).
(b) *T is (maximal) monotone of type* (NI).
(c) *T is locally maximal monotone.*
(d) *T^* is monotone.*

Proof. From Theorem 6.7, we only have to prove that (d) $\Longrightarrow$ (b). Let $(x^{**}, x^*) \in E^{**} \times E^*$. From (d), T is monotone and, further, for all $x \in E$, $\langle \widehat{x} - x^{**}, T^*(\widehat{x} - x^{**}) \rangle \geq 0$. Consequently, for all $x \in E$,

$$\begin{aligned}\langle \widehat{x} - x^{**}, Tx - x^* \rangle &\leq \langle \widehat{x} - x^{**}, Tx - x^* \rangle + \langle \widehat{x} - x^{**}, T^*(\widehat{x} - x^{**}) \rangle \\ &= \langle \widehat{x} - x^{**}, Tx + T^*\widehat{x} - (x^* + T^*x^{**}) \rangle.\end{aligned}$$

However, the map $x \to Tx + T^*\widehat{x}$ is the subdifferential of the continuous convex function $x \to \langle Tx, x \rangle$. From [8] and [13, Lemma 15] it is of type (NI). It follows that

$$\inf_{x \in E} \langle \widehat{x} - x^{**}, Tx + T^*\widehat{x} - (x^* + T^*x^{**}) \rangle \leq 0,$$

and consequently

$$\inf_{x \in E} [\langle \widehat{x} - x^{**}, Tx - x^* \rangle \leq 0.$$

This completes the proof of (b). □

9. Open Problems

- As we have noted earlier: it remains an open question whether $D(\widetilde{T})$ is nontrivial when T is neither symmetric nor anti-symmetric. If this is the case, does it necessarily follow that $D(T) \cap D(\widetilde{T})$ is nontrivial? If so, must it be dense in E? If Proposition 3.2(g) could be strengthened to show that $G(T)$ is closed in the product of the norm and weak* topologies, then a proof similar to that of Proposition 4.2 would show that $D(\widetilde{T})$ is dense in E.
- The analysis of Section 8 brings to mind the obvious question whether there are any converse results in Theorem 6.7. For instance, if T^* is monotone, is T of type (D)?
- It follows from Theorem 2.5 that if T is maximal monotone then $D(T)$ is a dense F_σ. Is it the domain of some naturally defined proper convex lower semicontinuous function?
- Our proof of Theorem 8.1 appeals to [8] and [13]. Now [8] uses some quite sophisticated functional analysis. Is there a more elementary proof of Theorem 8.1?
- If $S\colon\ D(S) \to E^*$ and $T\colon\ D(T) \to E^*$ are linear and maximal monotone and

  $$D(S) - D(T) = E$$

  (i.e., (7.2) is satisfied) is $S + T$ necessarily maximal monotone?

References

[1] H. Attouch, H. Riahi, M. Théra: Somme ponctuelle d'opérateurs maximaux monotones, Serdica Math. J. 22 (1996) 267–292.

[2] H. H. Bauschke, J. M. Borwein: Continuous linear monotone operators on Banach spaces, Pacific J. Math. (to appear).

[3] H. H. Bauschke, J. M. Borwein: Maximal monotonicity of dense type, local maximal monotonicity, and monotonicity of the conjugate are all the same for continuous linear operators (preprint).

[4] H. Brézis, M. G. Crandall, A. Pazy: Perturbations of nonlinear maximal monotone sets in Banach space, Comm. Pure Appl. Math. 23 (1970) 123–144.

[5] L.-J. Chu: On the sum of monotone operators, Mich. Math. J. 43 (1996) 273–289.

[6] S. P. Fitzpatrick, R. R. Phelps: Bounded approximants to monotone operators on Banach spaces, Ann. Inst. Henri Poincaré, Analyse Non Linéaire 9 (1992) 573–595.

[7] S. P. Fitzpatrick, R. R. Phelps: Some properties of maximal monotone operators on nonreflexive Banach spaces, Set-Valued Analysis 3 (1995) 51–69.

[8] J.-P. Gossez: Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs, J. Math. Anal. Appl. 34 (1971) 371–395.

[9] J.-P. Gossez: On the range of a coercive maximal monotone operator in a nonreflexive Banach space, Proc. Amer. Math. Soc. 35 (1972) 88–92.

[10] R. R. Phelps: Convex Functions, Monotone Operators and Differentiability, Lecture Notes Math. 1364, Springer-Verlag, 1989, 2nd Edition, 1993.

[11] R. T. Rockafellar: On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75–88.

[12] S. Simons: Subdifferentials are locally maximal monotone, Bull. Australian Math. Soc. 47 (1993) 465–471.

[13] S. Simons: The range of a monotone operator, J. Math. Anal. Appl. 199 (1996) 176–201.

[14] S. Simons: Subdifferentials of convex functions, in: Recent Developments in Optimization Theory and Nonlinear Analysis, Y. Censor, S. Reich (eds.), American Mathematical Society, Providence, Rhode Island, Contemporary Mathematics 204 (1997) 217–246.

[15] S. Simons: Pairs of monotone operators, Trans. Amer. Math. Soc. 350 (1998) 2913–2980.

[16] E. Zeidler: Nonlinear Functional Analysis and its Applications, Vol II/A, Linear Monotone Operators, Springer-Verlag, 1985.