Journ@l Electronique d'Histoire des Probabilités et de la Statistique

Electronic Journ@l for History of Probability and Statistics

Vol 9; Décembre/December 2013

www.jehps.net

Marc Barbut's research on the measure of inequalities

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ABSTRACT

Marc Barbut's last publication in Spain: "The income distribution. A theory of Fréchet" (2010) let us review some of his key-research points related to the modeling and the measurement of Pareto inequalities. Marc Barbut [2003a] is interested in the extensions of Pareto's income distribution model I.From P. Lévy, Barbut notes two properties of the stable laws related to Pareto's model. Barbut [2010] generalized an income distribution model proposed by Fréchet [1939] which is formed by a potential function for low and medium incomes and by Pareto's model for the high incomes. Barbut [1998] used those conditional means to characterize Pareto's model II and Barbut [2003b] proposes a reformulation of the discussion between Pareto and Sorel.

KEYWORDS – Marc Barbut, Income distribution, Measure of inequalities.

RÉSUMÉ

La dernière publication de Marc Barbut en Espagne "The income distribution. A theory of Fréchet (2010) nous a permis de revisiter quelques moments clé des recherches de Marc Barbut à propos de la modélisation et la mesure des inégalités de Pareto. Marc Barbut [2003a] discute de deux propriétés des lois stables établies par Lévy et liées au modèle de Pareto. Marc Barbut (2010) a généralisé le modèle de distribution du revenu proposé par Fréchet [1939], composé d'une fonction potentielle pour les revenus faibles et moyens, et du modèle de Pareto pour les revenus élevés. Marc Barbut [1998] a utilisé ces moyennes conditionnelles pour caractériser le modèle II de Pareto et Marc Barbut [2003b] propose une reformulation de la discussion entre Pareto et Sorel.

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Marc Barbut's articles on the measurement of inequality and its reflections on Pareto, Sorel, Levy and Fréchet served as inspiration for various researches on Pareto and Gini by Basulto et al [2010 and 2011] of Seville University, which we extend in the current paper.

1.On the interpretation of the parameter α in the Pareto’s model I

Marc Barbut spent much of his research on inequality with discussions on the interpretation of parameter α in the Pareto’s Model I. The article published in the journal Empiria “Idología, matemáticas y Ciencias y social V. Pareto, G. Sorel Sorel y la ambigüedad en la comparación de las desigualdades" is perhaps a good example to summarize the debate.

G. Sorel proposed, in the pages of his article "Le Devenir Social" [1897], the following interpretation of the parameter α :

> *J’appelle nombre de personnes ayant un revenu donné le nombre des gens ayant un revenu compris dans un intervalle relativement constant autour du chiffre considéré : ainsi, je prendrai, par exemple, les nombres des revenus compris entre 4900 et 5100, francs – entre 9800 et 10200 – puis entre 14700 et 15300, etc..., chaque tranche a une épaisseur relative constante. D’après la loi de Bernoulli ces écarts se traduisent, à peu près, par des estimations égales. Je compare ensuite les nombres de revenus compris entre deux tranches successives : je suppose qu’il y ait, par exemple, 8000 personnes ayant des revenus compris entre 4900 et 5100 et 6000 ayant des revenus dans la tranche suivante : je dirai que sur 8000 candidats à l’ascension 6000 seulement arrivent à atteindre l’échelon supérieur, que 2000 ont échoué et que par suite la difficulté de l’ascension est mesurée par le rapport de 2000 à 8000.*
>
> *Si on traite la question par le calcul infinitésimal, on trouve que la loi de M. Pareto revient à cette interprétation : « La difficulté pour l’ascension est constante et mesurée par la quantité α »* [Sorel, 1897, p. 587].

Let’s see an explanation of this text:

For the Pareto Model I

$$1-F_X(x)=\bar{F}_X(x)=\left(\frac{h}{x}\right)^{\alpha}, \quad \text{pour} \quad x\geq h>0 \text{ and } \alpha>0,$$

where X is the income variable. Let’s consider the transformation $Y=Ln(X)$, i.e. Y is the natural logarithm of X. Sorel, after Daniel Bernoulli, interpreted the values of Y as income’s utilities[3]. The distribution function of the new random variable of Y is

$$\bar{F}_Y(y)=\bar{F}_X(e^y)=\frac{h^{\alpha}}{e^{y\alpha}}.$$

Since $\bar{F}_Y(y_h)=\bar{F}_X(h)=1$, where $y_h=Ln(h)$ is the minimum of the variable *Y*, the distribution function of Y will be equal to

[3] <<The utility is a measure of well-being or satisfaction obtained from consumption, or at least setting a good or service>>, (Daniel Bernoulli, Specimen Theoriae Novae de Mensura Sortis, 1738).

$$\bar{F}_Y(y)=\bar{F}_X(e^y)=\frac{e^{y_h\alpha}}{e^{y\alpha}}=e^{-(y-y_h)\alpha}.$$

So we see that Y follows an exponential model with parameters (y_h,α).

If we select three successive values of the random variable Y, $y_i < y_{i+1} < y_{i+2}$, such that $y_{i+1}-y_i=y_{i+2}-y_{i+1}=\lambda$, i.e. the difference between these values is constant and equal to $\lambda>0$, then incomes corresponding to utilities $y_i < y_{i+1} < y_{i+2}$, are, respectively, $e^{y_i}<e^{y_{i+1}}<e^{y_{i+2}}$, or $\frac{e^{y_{i+1}}}{e^{y_i}}=\frac{e^{y_{i+2}}}{e^{y_{i+1}}}=e^{\lambda}$. Thus, we see that pairs of values of Y with equal differences correspond to pairs of values of X with equal quotients.

We now calculate what Sorel called "the difficulty of being reached", corresponding to the following expression

$$q_{i+1|i}=\left[1-\frac{\bar{F}_Y(y_{i+1})-\bar{F}_Y(y_{i+2})}{\bar{F}_Y(y_i)-\bar{F}_y(y_{i+1})}\right]. \quad (1)$$

In (1), the difference $\bar{F}_Y(y_{i+k})-\bar{F}_y(y_{i+1+k})$ is the proportion of individuals with income's utilities in the interval $[y_{i+k},y_{i+1+k})$, for $k=0,1$.

Formula (1) is the conditional probability that an individual who has his income's utility included in the interval $[y_i,y_{i+1})$ does not reach the interval $[y_{i+1},y_{i+2})$. Formula (1) expresses the individuals' reduction from the interval $[y_i,y_{i+1})$ to interval $[y_{i+1},y_{i+2})$, compared to individuals in the interval $[y_i,y_{i+1})$. If, for example, the first interval contains 8000 individuals and 6000 individuals the second, the expression (1) is $\frac{8000-6000}{8000}=0.25$, i.e. the number of individuals in the second interval decreased by 25% compared to individuals of the first interval. Sorel interprets the formula (1) with these words:

> *Il faut bien observer que la notion d'ascension n'est pas donnée pour la statistique ; elle est ajoutée pour l'interprétation ; on assimile les revenus à des corps mobiles ayant un mouvement déterminé et éprouvant plus ou moins de difficultés à se mouvoir: comme on n'a pas de moyens pour calculer les forces, on ne sait rien de la direction et de la nature de ce mouvement. Dans nos sociétés, il existe certainement une classe qui prospère et absorbe la plus grande partie des revenus ; c'est toujours d'elle qu'on parle quand on raisonne sur la richesse nationale: dans son sein il y a une ascension et celle-ci peut se concilier avec la déchéance du plus grand nombre de manière à produire une apparente ascension nationale. C'est de ce mouvement fictif d'enrichissement que traitent les économistes dans leurs apologies de la société capitaliste* [Sorel, 1897, p. 588].

A dynamic model of social change that leads to Pareto's Model I was proposed by Champernowne [1953].

If we divide the formula (1) by λ, we have

$$\left[1-\frac{\bar{F}_Y(y_{i+1})-\bar{F}_Y(y_{i+2})}{\bar{F}_Y(y_i)-\bar{F}_y(y_{i+1})}\right]\left(\frac{1}{\lambda}\right), \quad (2)$$

which is the conditional probability density.

Operating in (2), we get

$$\frac{\dfrac{\bar{F}_Y(y_i)-2\bar{F}_Y(y_{i+1})+\bar{F}_Y(y_{i+2})}{\lambda^2}}{\dfrac{\bar{F}_Y(y_i)-\bar{F}_y(y_{i+1})}{\lambda}}. \quad (3)$$

If the expression (3) we change the utilities $y_i < y_{i+1} < y_{i+2}$ by $y < y+\lambda < y+2\lambda$ and also we take its limit when $\lambda \to 0$, using the definitions of the first and the second derivatives of a function, then we have

$$-\frac{F_Y^{"}(y)}{F_Y^{'}(y)}. \quad (4)$$

If we now demand that (4) is a constant equal to parameter α, then we have a differential equation whose general solution is the exponential model $\bar{F}_Y(y)=e^{-(u-u_h)\alpha}$ [J. S. Chipman,1976]. We see that the condition $-\frac{F_Y^{"}(y)}{F_Y^{'}(y)}=\alpha$ characterized the Pareto Model I.

If in (3) we substitute $\bar{F}_Y(y)=e^{-(y-y_h)\alpha}$, i.e. we assume that income is distributed according to Pareto's model I, then we get the following expression

$$\frac{1-2e^{-\alpha\lambda}+e^{-2\alpha\lambda}}{\left[1-e^{-\alpha\lambda}\right]\lambda}, \ (5)$$

and taking the limit in (5) when $\lambda \to 0$, we have

$$\lim_{\lambda\to 0}\frac{1-2e^{-\alpha\lambda}+e^{-2\alpha\lambda}}{\left[1-e^{-\alpha\lambda}\right]\lambda}=\alpha. \ (6)$$

Expression (6) is used to interpret the parameter α, because if λ is enough small then we have the approximation $\frac{q_{i+1|i}}{\lambda}=\alpha$. When the conditional probability $q_{i+1|i}$ increases (decreases), then the parameter α increases (decreases). Calling $p_{i+1|i}=1-q_{i+1|i}$, the result of Sorel provides an interpretation of the parameter α, showing that when the parameter α decreases (increases), then increases (decreases) $p_{i+1|i}$, i.e., increases (decreases) the number of individuals with income utilities in an interval, when passing to next interval of greater utility.

2. The stable laws of Paul L□vy and the Pareto model

In his article "Homme moyen ou homme extrême: de Vilfredo Pareto (1896) à Paul Lévy (1936) en passant par Maurice Fréchet et quelques autres [2003]", Barbut collected certain relationships among some results of Lévy, Pareto and Fréchet.

In this section we have collected two results related to Paul Lévy stable laws and Pareto's Model I.

Stable laws- also called α -stable, stable Paretian or Lévy stable- were introduced by Lévy [1925] during his investigations of the behavior of sums of independent random variables. They are the only limiting laws of normalized sums of independent, identically distributed random variables.

The α -stable distribution requires four parameters for complete descriptions: an index of stability $\alpha \in (0,2]$, a skewness parameter $\beta \in [-1,1]$, a scale parameter and a location parameter.

According to W. Feller [Chapter 6, vol. II, 1966]: a distribution R is stable if the variables $X, X_1, X_2, ..., X_n$, independent identically distributed random variables, with the same distribution R, verify, for all n, that the sum $S_n = X_1 + X_2 + ... + X_n$ has the same distribution as the variable $c_n X + \gamma_n$, where $c_n > 0$. The normal model is an example of stable distribution.

A first result is the relationship between the first law of Pareto and the limit theorems proposed by Paul Lévy for a positive stable laws with parameters $0 < \alpha < 1$ and $\beta = 1$.

(1) We have considered the distribution function of a stable model with $\alpha = 0.7$ and $\beta = 1$, with a location parameter equal to zero and a scale parameter $\sigma = 3.646$. In order to calculate the distribution function of the stable model considered, we have simulated 500 random samples of size n = 20 from the stable model, using the procedure of Chamber, Mallows and Stuck [1976].

We have also calculated other approximation to this stable model from the following limit theorem proposed by P. Lévy:

(2) We have selected 500 independent samples from Pareto Model I with $\alpha = 0.7$ and a minimum income $h = 1$, with size n=20. For each sample we have calculated the expression $\left(\sum_{k=1}^{n} x_k\right) n^{-\frac{1}{\alpha}}$, which approximates to the stable distribution function. These calculations are based on the following result of Paul Lévy:

Paul Lévit showed that the expression $\left(\sum_{k=1}^{n} x_k\right) n^{-\frac{1}{\alpha}}$ converges in distribution to a stable law with parameters $\alpha = 0.7$ and $\beta = 1$, when $n \to \infty$ [P. Lévy, 1937, p. 203].

We have collected in Figure 1 the two distribution functions calculated in paragraphs (1) and (2) above.

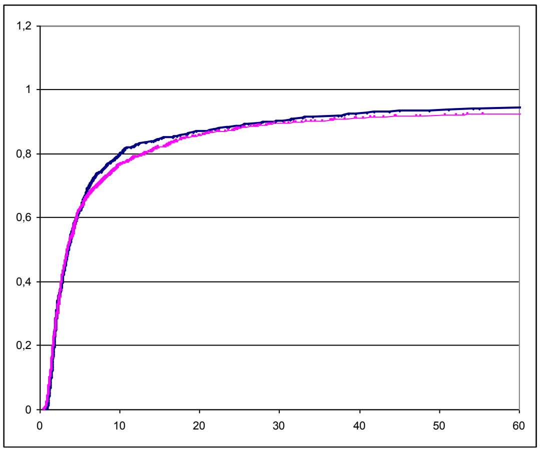

Figure 1

Figure 1 shows that the approach of Paul Lévy (the curve below) approximates well the distribution function of stable model considered.

A second result is the relationship between the Pareto Model I and the stable laws: when $\alpha < 2$, the variance is infinite and the tails are asymptotically equivalent to the Pareto law i.e. they exhibit a power-law behavior. More precisely, using a central limit theorem type argument it can be shown that [Janicki and Weron, 1994]:

$$\lim_{x\to\infty} x^{\alpha} P\left(X > x\right) = C_{\alpha}\left(1+\beta\right)\sigma^{\alpha},$$

where $C_{\alpha} = \frac{1}{\pi}\Gamma\left(\alpha\right)\sin\frac{\pi\alpha}{2}$.

In our example, when the variable x increases then the distribution function of the stable model approximates to Pareto Model I.

Figure 2 shows the cumulative probabilities above 0.49.

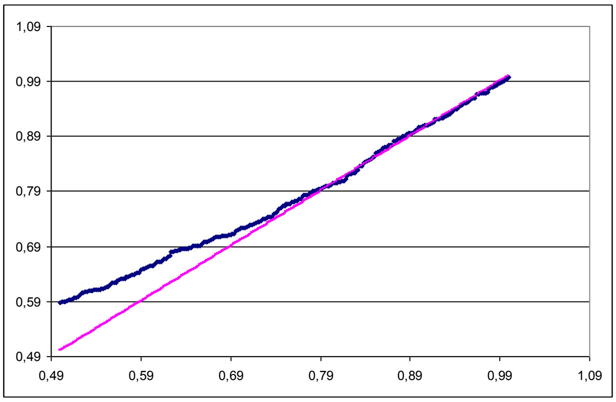

Figure 2

In the Figure 2 we see that the approximation is valid from an accumulated probability of 0.79.

3. The income distribution: a theory of Fr□chet [2010]

In section 1 of this article, Barbut noted the existence of a large file in the Paris Academy of Sciences (Box F-10) containing papers and manuscripts of Maurice Fréchet conducted between 1930 and 1960. Several of these manuscripts are related to Fréchet conducted research on the distribution of income and some writings about improving Pareto Law. We‘ll be interested in one of the paper cited in the literature by Fréchet, the one entitled: "Sur les formules of répartition des revenus" [1939], published in the Revue de L'Institut International de Statistique.
In this paper, 1939, Fréchet proposed a model of income distribution that could be applied to the entire range of incomes. Thereby completing the Pareto’s law, because this law is valid only for large incomes. We will see that Barbut proposes a model of income distribution that covers the entire range of income and also is more general than the Fréchet’s model.

Before presenting the main result of Fréchet, we will link the model exponential $\bar{F}_Y(y) = e^{-y}$, $y \geq 0$, with the model of income distribution of Pareto.

Let the transformation $y = \alpha Ln(x+a) + \beta x Ln(10) + b$, where $\alpha > 0$, $\beta \geq 0$, $-\infty < a < +\infty$, and Ln(.) is the natural logarithmic function. Then the random variable X has the distribution function $F_X(x)$, such that $\bar{F}_X(x) = \bar{F}_Y(y)$, where $\bar{F}_X(x) = 1 - F_X(x)$. There is also a minimum income $h > 0$, such that $\alpha Ln(h+a) + \beta x Ln(10) + b = 0$, and thus $b = -\alpha Ln(h+a) - \beta h Ln(10)$ and $h > -a$ for $a < 0$. Thus $\bar{F}_Y(0) = \bar{F}_X(h) = 1$.

Hence the distribution function of X is

$$1 - F_X(x) = \bar{F}_X(x) = \bar{F}_Y(y) = e^{-y} = e^{-Ln(x-h)^{\alpha} - \beta x Ln(10) - b},$$

that is

$$\bar{F}_X(x) = \frac{e^{-b} 10^{-\beta x}}{(x+a)^{\alpha}} = \frac{10^{h\beta}(h+a)^{\alpha}}{(x+a)^{\alpha}} 10^{-\beta x}. \quad (7)$$

Taking $\beta^* = \dfrac{\beta}{\log(e)}$, then (7) is

$$\bar{F}_X(x) = \frac{e^{h\beta^*}(h+a)^{\alpha}}{(x+a)^{\alpha}} e^{-\beta^* x}. \quad (7')$$

We see that (7) is the third model of income distribution de Pareto. For $\beta = 0$, we obtain the Pareto Model II and if, in addition, $a = 0$, then we obtain the Pareto Model I. For $\beta = 0$, the model (7) was collected by D'Addario [1949].

The transformation $y = \alpha Ln(x+a) + \beta x Ln(10) + b$ is an example of the "method of translation" proposed by Edgeworth [1898]. This method searches a distribution function from a law of errors. For example, the first or the second law of Laplace errors, using a suitable transformation of variables.

Considering now the work of Fréchet. He went from the first law of Laplace errors with density function, i.e.

$$f_Y(y) = \frac{e^{-|y|}}{2},\ -\infty < y < +\infty,\ (8)$$

and making the transformation $y = \alpha Ln(x-h) + b,\ X \geq h > 0$.

Knowing that if $y > 0$, then $\int_0^y e^{-|t|} = \int_0^y e^{-t} dt = 1 - e^{-y}$, and if $y < 0$, then $\int_0^y e^{-|t|} = \int_0^y e^t dt = e^y - 1$. Let's calculate the distribution function of the variable X.

The function $y = \alpha Ln(x-h) + b$ intersects to the abscissa axis at point x_M, such that $\alpha Ln(x_M - h) + b = 0$, where $e^{-b} = (x_M - h)^\alpha$.

For $x \geq x_M$, $\bar{F}_X(x) = \bar{F}_Y(y) = \frac{1}{2} - \frac{1}{2}\left(1 - e^{-y}\right) = \frac{1}{2} e^{-y} = \frac{e^{-b}}{2(x-h)^\alpha} = \frac{A}{(x-h)^\alpha}$, with

$A = \frac{e^{-b}}{2} = \frac{(x_M - h)^\alpha}{2}$. Also, as $\bar{F}_X(x_M) = \bar{F}_Y(0) = \frac{1}{2}$, then x_M is the median of $F_X(x)$.

For $h < x \leq x_M$, $F_X(x) = \frac{1}{2}\int_{-\infty}^{y} e^t dt = \frac{e^y}{2} = \frac{e^{Ln(x-h)^\alpha + b}}{2} = \frac{e^b}{2}(x-h)^\alpha = B(x-h)^\alpha$, where

$B = \frac{e^b}{2} = \frac{1}{2(x_M - h)^\alpha}$.

Hence the distribution function of X is

$$F_X(x) = \begin{cases} B(x-h)^\alpha, h \leq x \leq x_M \\ 1 - \frac{A}{(x-h)^\alpha}, x \geq x_M \end{cases}.\ (9)$$

We see that when $x \geq x_M$ then we obtain the Pareto Model II. If we further assume that, $h=0$, then we have the Pareto Model I, where x_M is the income minimum.

Considering now the density function of the distribution function $F_X(x)$.

When $x \geq x_M$, the density function of X is $f_X(x)=\frac{A\alpha}{(x-h)^{\alpha+1}}$. When $h<x\leq x_M$, then $f_X(x)=B\alpha(x-h)^{\alpha-1}$. For $x=x_M$, we obtain $\frac{A\alpha}{(x_M-h)^{\alpha+1}}=B\alpha(x_M-h)^{\alpha-1}$.

This density function has a mode equal to x_M, since their derivatives are:

-For $x \geq x_M$, $\frac{df_X(x)}{dx}=-A\alpha\frac{(\alpha+1)(x-h)\alpha}{(x-h)^{2(\alpha+1)}}$, which is negative and it is decreasing. The derivative on the right at the median is equal to $-\frac{\alpha(\alpha+1)}{2(x_M-h)^{\alpha}}$.

-For $h<x\leq x_M$, $\frac{df_X(x)}{dx}=B\alpha(\alpha-1)(x-h)^{\alpha-2}$, which is positive if $\alpha>1$, so, $f_X(x)$ is a function increasing. The derivative on the left at the median is equal to $\frac{\alpha(\alpha+1)}{2(x_M-h)^{\alpha}}$.

Hence the median is equal to the mode. In conclusion, Fréchet proposed a model of income distribution, which had a mode equal to the median, covering the whole range of income and being the right tail a Pareto model.

Let us consider now the section 2 the article of Barbut, where he has proposed the following distribution function:

$$F_X(x)=\begin{cases} B(x-h)^{\beta}, h\leq x\leq x_M \\ 1-\frac{A}{(x+a)^{\alpha}}, x\geq x_M \end{cases}, (10)$$

where $B=\frac{1}{2(x_M-h)^{\beta}}$ and $A=\frac{(x_M+a)^{\alpha}}{2}$ are positives.

If $F_X(x_M)=\frac{1}{2}$, then x_M is the median. Also if $a<0$, then $x\geq x_M>-a$.

The density function of this model is

$$f_X(x)=\begin{cases} B\beta(x-h)^{\beta-1}, h\leq x\leq x_M \\ \frac{A\alpha}{(x+a)^{\alpha+1}}, x\geq x_M \end{cases}. (11)$$

If we demand that $B\beta\left(x_M-h\right)^{\beta-1}=\frac{A\alpha}{\left(x_M+a\right)^{\alpha+1}}$, then $\frac{\alpha}{\beta}=\frac{x_M+a}{x_M-h}$ (Formula 2.11 of Barbut).
The first derivative of (11) is:

$$\frac{df_X(x)}{dx}=\begin{cases}B\beta(\beta-1)(x-h)^{\beta-2}, h\le x\le x_M\\ -\frac{A\alpha(\alpha+1)}{(x+a)^{\alpha+2}}, x\ge x_M\end{cases}$$

We see that if we want that the density function to increases when $h\le x\le x_M$, then. $\beta>1$. The other branch of the density function, when $x\ge x_M$, is decreasing. Also the median is equal to mode.

The model (10) proposed by Barbut can be generated, as we have seen with Fréchet, from the exponential the model (8) and the following transformation

$$y=\begin{cases}\beta Ln(x-h)+b_2, h\le x\le x_M\\ \alpha Ln(x+a)+b_1, x\ge x_M\end{cases},$$

where x_M verifies the conditions: $\beta Ln\left(x_M-h\right)+b_2=0$ and $\alpha Ln\left(x_M+a\right)+b_1=0$. Being $A=\frac{e^{-b_1}}{2}$ y $B=\frac{e^{b_2}}{2}$. Also $\frac{b_1}{b_2}=\frac{\alpha}{\beta}\frac{Ln\left(x_M+a\right)}{Ln\left(x_M-h\right)}$, and from formula 2.11 of Barbut, we obtain the expression $\frac{b_1}{b_2}=\frac{Ln\left(x_M+a\right)^{\left(x_M+a\right)}}{Ln\left(x_M-h\right)^{\left(x_M-h\right)}}$ that does not include the parameters α and β.

Fréchet used the same transformation for each of the tails of his model and Barbut used different transformations for each of the two parties which divide the income values.

An example: if in (11), $\alpha=3.42$, $\beta=3$, $a=3$ y $h=10$ then result the Figure 3 next.

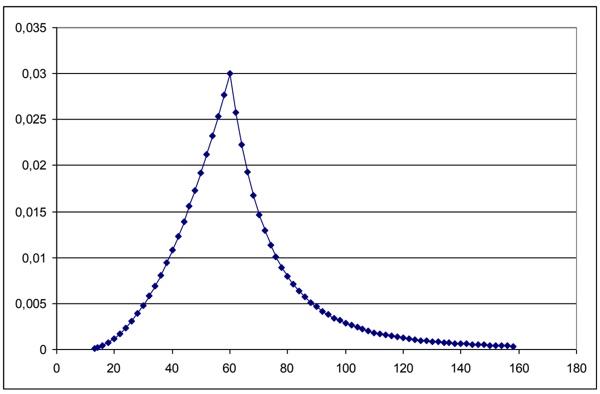

Figure 3

Where the median, $x_M = 60$, is equal to the mode. This Figure 3 is similar to Figure 51 of Pareto [Cours d'Economie, volume II, page 315].

4. Conditional means

An example of conditional means is the formula $M(x) = \frac{x\alpha}{\alpha - 1}$ that Bowley (1914) introduced from the Pareto Model I, where $M(x)$ is the average income of the incomes greater than x, $x \geq h > 0$ and $\alpha > 1$.

For the Pareto Model I, Bresciani-Turroni introduced in 1910 the following formula:

$$R(x) - R(y) = \delta\left[xN(x) - yN(y)\right],$$

where, $R(x)$ and $N(x)$ are total income and individuals, respectively, with incomes above x, being $\delta = \frac{\alpha}{\alpha - 1}$.

An equivalent expression of the latter formula is

$$R(x) - \delta xN(x) = R(y) - \delta yN(y).$$

When $x = h$ and being $\mu = \frac{R(h)}{N(h)} = h\delta$, then $R(y) - \delta yN(y) = 0$. Hence $M(x) = \frac{x\alpha}{\alpha - 1}$. Bresciani-Turroni could have got the above formula of Bowley.
A direct calculation to get the formula $M(x) = \frac{x\alpha}{\alpha - 1}$, is: from the expression $R(x) = \frac{\alpha N(h) h^\alpha}{(\alpha - 1) x^{\alpha - 1}}$ which we divide by $xN(x) = \frac{xN(h) h^\alpha}{x^\alpha}$, resulting $\frac{M(x)}{x} = \delta$.

The condition $\frac{M(x)}{x} = \delta$, for all $x \geq h > 0$ and $\delta > 1$, characterized the Pareto Model I, with $\alpha > 1$. Let us see this result:

If we assume $\frac{M(x)}{x} = \delta$, that is $\frac{R(x)}{xN(x)} = \delta$, then $\int_x^\infty tf(x)dx = \delta x \int_x^\infty f(x)dx$, for all $x \geq h > 0$.Taking the derivative at each term the above identity, we obtain $-xF'(x) = \delta N(x) - \delta xF'(x)$, where $F(x)$ is the distribution function of $f(x)$ and $N(x) = 1 - F(x)$. As $\frac{dF(x)}{dx} = -\frac{d}{dx}(1 - F(x)) = -\frac{dN(x)}{dx}$, we obtain the differential equation $xN'(x) = \delta N(x) + \delta xN'(x)$, that is $\frac{N'(x)}{N(x)} = -\frac{\delta}{x(\delta - 1)}$. The solutions of this difference equation are $Ln(N(x)) = -\frac{\delta}{\delta-1} Ln(x) + Ln(A)$. Then $N(x) = \frac{A}{x^{\frac{\delta}{\delta-1}}}$, and adding the condition that $N(h) = \frac{A}{h^{\frac{\delta}{\delta-1}}}$, we get $A = N(h) h^{\frac{\delta}{\delta-1}}$, which is the Pareto Model I with parameter $\alpha = \frac{\delta}{\delta-1} > 1$.

In the article "Una familia de distribuciones: de las Paretianas a las Contra-Paretianas. Aplicación al estudio de la concentración urbana y su evolución", Barbut started with the hypothesis $M(x) = \beta x + h$, with the necessary conditions $\beta \geq 0$ y $h \geq 0$, and he got:

-For $\beta > 1$, the Pareto Model II.
-For $0 \leq \beta \leq 1$, distributions called "counter-Pareto".
-For $\beta = 1$, an exponential model with two parameters.

A functional approach to inequality is $M(x(p)) = \frac{\int_{x(p)}^{\infty} tf(t)dt}{\int_{x(p)}^{\infty} f(t)dt}$, where $x(p)$ is the quantile for $0 < p < 1$, that is $x(p)$ is an income that is below the proportion p of individuals, and $M(x(p))$ is the average income for incomes above $x(p)$. For the Pareto Model I with $\alpha > 1$, we have seen that $M(x(p)) = x(p)\frac{\alpha}{\alpha-1}$, which may be written $M(x(p)) = \frac{x(p)}{h}\frac{h\alpha}{\alpha-1} = \frac{x(p)}{h}\mu$, where $\mu = \frac{h\alpha}{\alpha-1}$ and $\frac{x(p)}{h} = (1-p)^{-\frac{1}{\alpha}}$.

Let us see how this functional criterion behaves when we compare two different models that follow the Pareto Model I with $\alpha > 1$.
Case I: $\alpha_2 < \alpha_1, h_2 > h_1$, then $M_2(x_2(p)) > M_1(x_1(p))$ for all $0 < p < 1$.

Proof: $\alpha_2 < \alpha_1$, then $\frac{\alpha_2}{\alpha_2-1} > \frac{\alpha_1}{\alpha_1-1}$ and also $(1-p)^{-\frac{1}{\alpha_2}} > (1-p)^{-\frac{1}{\alpha_1}}$, and from $h_2 > h_1$, we have $\mu_2 > \mu_1$ and also $\frac{x_2(p)}{h_2} > \frac{x_1(p)}{h_1}$, hence $M_2(x_2(p)) > M_1(x_1(p))$, $0 < p < 1$. If $\alpha_2 = \alpha_1$ and $h_2 > h_1$ or if $\alpha_2 < \alpha_1$ and $h_2 = h_1$, then we have also the same result. An example: $\alpha_1 = 1.8$, $h_1 = 24$, $\alpha_2 = 1.6$ and $h_2 = 34$.

Case II: $\alpha_2 > \alpha_1$ and $\mu_2 = \mu_1$, then $h_2 > h_1$ and
$M_2(x_2(p)) < M_1(x_1(p))$ for all $0 < p < 1$.

Proof: as $h = \mu \frac{\alpha - 1}{\alpha}$, and $\alpha_2 > \alpha_1$, then $\frac{\alpha_2 - 1}{\alpha_2} > \frac{\alpha_1 - 1}{\alpha_1}$, and so $h_2 > h_1$.

Also, if $\alpha_2 > \alpha_1$, we have $\frac{x_2(p)}{h_2} < \frac{x_1(p)}{h_1}$, then $M_2(x_2(p)) < M_1(x_1(p))$ for all $0 < p < 1$. An example: $\alpha_1 = \frac{4}{3}, h_1 = 20$, $\alpha_2 = 2$ and $h_2 = 40$

Case III: $\alpha_2 > \alpha_1$ and $\mu_2 > \mu_1$, then $h_2 > h_1$ and the functions $M_2(x_2(p))$ and $M_1(x_1(p))$ intersect in a point.

Proof: $\alpha_2 > \alpha_1$, then $\frac{\alpha_2 - 1}{\alpha_2} > \frac{\alpha_1 - 1}{\alpha_1}$ and also $(1-p)^{-\frac{1}{\alpha_2}} < (1-p)^{-\frac{1}{\alpha_1}}$; and as $\mu_2 > \mu_1$, then $h_2 > h_1$. Also $\alpha_2 > \alpha_1$, then $\frac{x_2(p)}{h_2} < \frac{x_1(p)}{h_1}$, as $M(x(p)) = \frac{x(p)}{h}\mu$, where $\mu_2 > \mu_1$ y $\frac{x_2(p)}{h_2} < \frac{x_1(p)}{h_1}$, then $M_2(x_2(p))$ y $M_1(x_1(p))$ intersect in a point, because $M(x(p)) = x(p)\frac{\alpha}{\alpha - 1}$ is a linear function of $x(p)$, for all $0 < p < 1$. An example: $\alpha_1 = 1.3, h_1 = 20$, $\alpha_2 = 1.5$ and $h_2 = 39$.

Case IV: $\alpha_2 > \alpha_1$ and $\mu_2 < \mu_1$, then $M_2(x_2(p)) < M_1(x_1(p))$ for all $0 < p < 1$.

Proof: if $\alpha_2 > \alpha_1$ then $(1-p)^{-\frac{1}{\alpha_2}} < (1-p)^{-\frac{1}{\alpha_1}}$, and as $M(x(p)) = \frac{x(p)}{h}\mu$, where $\frac{x(p)}{h} = (1-p)^{-\frac{1}{\alpha}}$, then $M_2(x_2(p)) < M_1(x_1(p))$ for all $0 < p < 1$. Examples: (a) $\alpha_1 = 1.3, h_1 = 20$, $\alpha_2 = 1{,}5$ and $h_2 = 24$, then $h_2 > h_1$; (b) with $h_2 = 19$, then $h_2 < h_1$; and (c) with $h_2 = 20$, then $h_2 = h_1$.

An application collected by G. Sorel is the ratio between the quantile $x(p) = h(1-p)^{-\frac{1}{\alpha}}$ and the income average μ, for $p = 0.95$. This ratio $\frac{x(0.95)}{\mu} = \frac{h(1-0.95)^{-\frac{1}{\alpha}}}{h\frac{\alpha}{\alpha-1}} = \frac{0.05^{-\frac{1}{\alpha}}}{\frac{\alpha}{\alpha-1}}$, which is a function that depends only of $\alpha > 1$.

The Sorel Table (page 605) is as follows:

α	2	1.70	1.50	1.40	1.30	1.20
$\frac{x(0.95)}{\mu}$	2.236	2.398	2.456	2.427	2.311	2.023

The Sorel's Table shows that when α increases ($\alpha > 1$), then the ratio $\frac{x(0.95)}{\mu}$ increases first to a maximum and then decreases. The maximum of the ratio $\frac{x(0.95)}{\mu}$ is equal to 1.5010 [Barbut, 2007]. Also, Barbut says "Il n'est reste pas moins que dès $\alpha > 1.5010$, <<seuil de richesse>> et revenu moyen tendent à se rapprocher; en ce sens, l'inégalité diminue quand α augmente".

We show that $\frac{x(0.95)}{\mu} = \frac{M(x(0.95))}{M(\mu)}$, i.e. $\frac{x(0.95)}{\mu}$ is the ratio between the average income of 5% of the richest and the income average above the average income μ.

The functions $\frac{M(\mu)}{h} = \left[\frac{\alpha}{\alpha-1}\right]^2$ and $\frac{M(x(0.95))}{h} = \frac{\alpha}{\alpha-1}(0.05)^{-\frac{1}{\alpha}}$ are plotted in Figure 4.

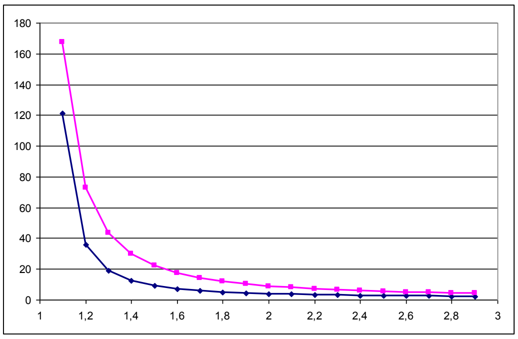

Figure 4

In the Figure 4, $\frac{M(x(0.95))}{h}$ is the upper curve and $\frac{M(\mu)}{h}$ is the lower curve.

We have seen that the ratio $\frac{x(0.95)}{\mu}$ has a maximum when $\alpha = 1.5010$ (the Sorel table takes $\alpha_{\max} = 1.5$), where the maximum value of this function is $\alpha_{\max} = \frac{Ln(1-p)}{1-Ln(1-p)}$. The equation $\frac{x(p)}{\mu} = 1$, i.e. $\frac{(1-p)^{-\frac{1}{\alpha}}}{\frac{\alpha}{\alpha-1}} = 1$, has as solution to the value $\tilde{\alpha}$ such that $\left(\frac{\tilde{\alpha}-1}{\tilde{\alpha}}\right)^{\tilde{\alpha}} = 1-p$, and for $1-p = 0.05$ we obtain the following approximation $\tilde{\alpha} = 1.06361$. We see that when $\alpha < \tilde{\alpha}$ then $M(\mu) > M(x(0.95))$, i.e. the average above μ is higher than the average over the quantile $x(0.95)$, and for $\alpha = \tilde{\alpha}$ coincides, i.e. $M(\mu) = M(x(0.95))$, and for $\alpha > \tilde{\alpha}$ then $M(\mu) < M(x(0.95))$. If $\alpha \to \infty$, then

$\frac{M(x(0.95))}{M(\mu)} \to 1$. The function $\frac{M(x(0.95))}{M(\mu)}$, which is a function of α, with $\alpha > 1$, is plotted in Figure 5.

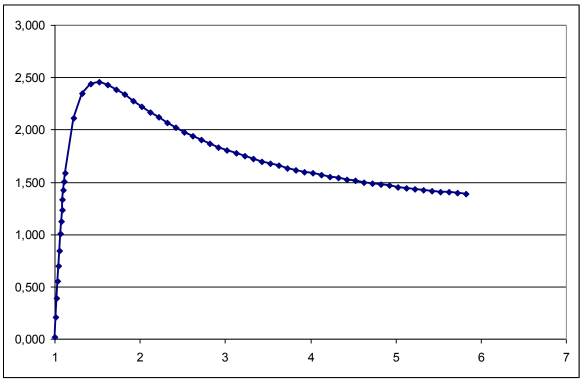

Figure 5

Figure 5 shows the values $\tilde{\alpha} = 1.06361$ and the maximum $\alpha_{\max} = 1.5010$. If $\alpha \to 0$, then $\frac{M(x(0.95))}{M(\mu)} \to 0$. If $\alpha \to \infty$, then $\frac{M(x(0.95))}{M(\mu)} \to 1$.

Now, we can give an interpretation of the Sorel Table. If we look at Figure 5, we see that the ratio $\frac{x(0.95)}{\mu}$ increases from $\alpha = 1$ to the value $\alpha_{\max} = 1.5010$, and gradually decreased from this maximum value to the value unity. The Sorel table does not include that :(1) for $\alpha < \tilde{\alpha}$, with $\alpha > 1$, then $\frac{x(0.95)}{\mu} < 1$, i.e. the mean μ is greater than the 0.95 quantile; (2) when $\alpha = \tilde{\alpha}$, then μ is equal to the 0.95 quantile and finally (3) for $\alpha > \tilde{\alpha}$, the 0.95 quantile is greater than the mean.

We see that for values of α around $\tilde{\alpha} = 1.06361$ or large, then the 0.95 quantile is close to the mean μ.

Let us see some Figures.

Example I

If $\alpha = \tilde{\alpha}$ and $h = 100$, then $\mu = x(0.95) = 1671.9105$, also, $M(\mu) = M(p(0.95)) = 27952.8481$, being $\frac{x(0.95)}{\mu} = 1$. The distribution function, $1 - F(x)$, the Pareto Model I, is

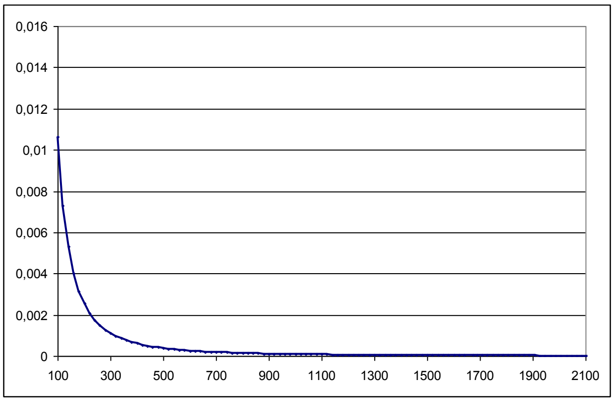

Figure 6

Figure 6 shows the income in the range $[100, 2100]$, with a percentage of 3.923% for individuals with incomes above 2100.

Example II

For $\alpha = 1.05$, $h = 100$, we get: $\mu = 2100$, $x(0.95) = 1734.1081$, $M(\mu) = 44100$ and $M(x(0.95)) = 36416.2717$, being $\dfrac{x(0.95)}{\mu} = 0.8257$. Then, the distribution function, $1 - F(x)$ is

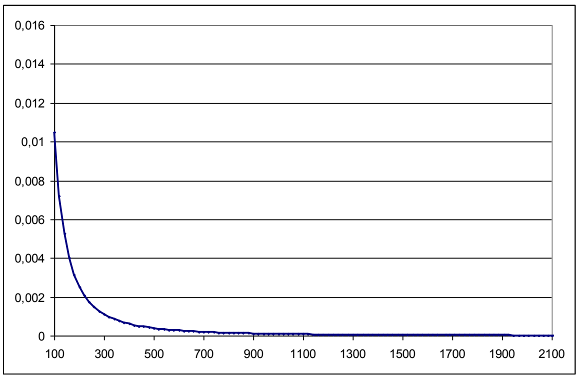

Figure 7

Figure 7 shows the income in the range $[100, 2100]$, where the percentage of individuals with incomes above 2100 is 4.089%. Comparing this Example II with I, we see that when α decreases and μ increases, then $M(x(0,95))$ increases, as shown in Case IV.

Example III

Finally, for $\alpha = 1.5, h = 100$, we get: $\mu = 300$, $x(0.95) = 736.8063$, $M(\mu) = 900$ and $M(x(0.95)) = 2210.4188$, being $\dfrac{x(0.95)}{\mu} = 2.4560$. Then, the distribution function, $1 - F(x)$ is

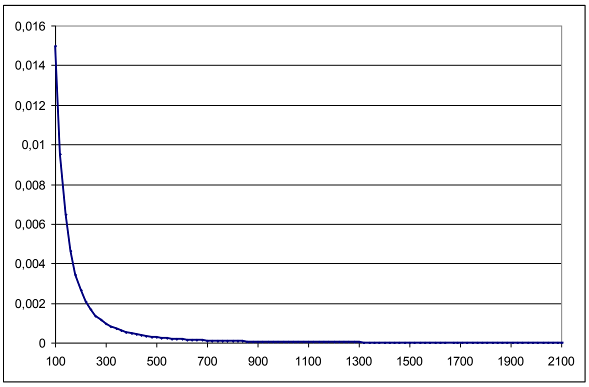

Figure 8

Figure 8 shows the income in the range $[100, 2100]$, where the percentage of individuals with incomes above 2100 is 1.039%. Comparing this Example III with I, we see that when α increases and μ decreases, then $M\left(x(0,95)\right)$ decreases, as shown in Case IV.

These comparative results do not change when the minimum income $h = 100$ changes. The same is applied to $\frac{x(0.95)}{\mu}$.

As Barbut observed, when $\alpha > 1$ increases, then inequality decreases, because the quantile $x(0.95)$ is close to μ. Now, we have seen that $\frac{x(0.95)}{\mu} = 1$ in Example I, the above inequality concept cannot be applied because when $\alpha \to \tilde{\alpha}$, with $\alpha > \tilde{\alpha}$ and $\alpha < \alpha_{\max}$, inequality decreases, and when $\alpha \to \infty$ and $\alpha < \alpha_{\max}$, inequality decreases too.

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