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Theories of the General Interest and the Logical Problem of Aggregation

English translation of "Les théories de l'intérêt général et le problème logique de l'agrégation" *Economie appliquée,* volume V nº 4, Octobre-Décembre 1952, pp.501-551.

Theories of the General Interest, and the Logical Problem of Aggregation[1]

G. Th. Guilbaud[2]

"Le peuple a les opinions très saines; par exemple: (…) travailler pour l'incertain; aller sur la mer (…)." "Or, quand on travaille pour demain et pour l'incertain, on agit avec raison car on doit travailler pour l'incertain, par la *règle des partis* qui est démontrée. Saint Augustin a vu qu'on travaille pour l'incertain sur mer, en bataille etc, mais il n'a pas vu la *règle des parti*s qui démontre qu'on le doit."[3] (The opinions of the people are very sound; for example: (…) they work for uncertain things, they go out to sea (…). Now, when one works for tomorrow and for something uncertain, one behaves reasonably, because one should work for uncertain things according to the rule of *departing the stakes* which has been proved. Saint Augustin had seen that people worked for uncertain things, on the sea, in battles, etc., but he had not seen this rule which proved that one should do this.)

We know that this rule is the point of departure, not only for the technique that ultimately came to be known as *probability calculus*, but also and mainly for the use of mathematics in the study of human actions.

"Opinions du people saines" (Sound opinions of the people) notes Pascal; a century and a half later. On the same topic, Laplace reiterates this idea in the conclusion of his *Essai philosphique sur les probabilités* : "La théorie des probabilités n'est, au fond, que le bon sens réduit au calcul : elle fait apprécier avec exactitude ce que les esprits justes sentent par une sorte d'instinct, sans qu'ils puissant souvent s'en rendre compte[4]." (Fundamentally, probability theory is just common sense condensed in computations: it assesses in an exact manner what sound minds feel by a kind of instinct, often without being aware of it.) But then why should we bother with tiresome computations if these only confirm common sense? Laplace explains himself through the example of mathematical theory applied to the moral sciences: "On a encore soumis au calcul la probabilité des témoignages, les votes et les décisions des assemblées électorales et délibérantes et les jugements des tribunaux. Tant des passions, d'intérêts divers et de circonstances compliquent les questions relatives à ces objets, qu'elles sont presque toujours insolubles. Mais la solution de problèmes plus simples et qui ont avec elles beaucoup d'analogies, peut souvent répandre sur ces questions difficiles et importantes, de grandes lumières que la sûreté du calcul rend toujours préférables aux raisonnements les plus spécieux."[5] (Once more we have been able to compute the probabilities of testimonies, votes and decisions of electoral and deliberating assemblies and court rulings. Strong feelings, diverse interests and circumstances complicate the questions related to these situations to the point where they are almost always impossible to solve. But the solutions to simpler problems provide us with many analogies that may shed much light on these difficult and important questions in a way that is, due to

1 English translation of "Les théories de l'intérêt général et le problème logique de l'agrégation" *Economie appliquée,* volume V n° 4, Octobre-Décembre 1952, pp.501-551.

2 1911-2007. See biography and publications in *Mathématiques et Sciences humaines* (184, http://www.ehess.fr/revue-msh/recherche_gb.php?par=numero).

3 Pascal, *Pensées*, n°s 324 and 234, in the Brunschwig edition, n°s 101 and 577 of Lafuma.

4 *Oeuvres de Laplace*, édition nationale Paris, 1847, v. VII, p.169.

5 Laplace, *Essai philosophique sur les probabilités* (1812); *Introduction à la théorie analytique des probabilités* , *Œuvres*, Paris, 1847, v. VII, p.164.

the certainty of computation, always preferable to the most specious reasoning). What is important is "d'apprendre à se garantir des illusions qui souvent nous égarent" (to learn how to guard ourselves against the delusions that often lead us astray).

The analysis of mechanisms of collective choice provides a beautiful example of this use of mathematics: it is easy to draw a parallel between the recent scholarly works on *Welfare* and the intuitions of Hobbes and Rousseau concerning the general will — and if one is so inclined, to marvel at the considerable efforts required to show what many very bright minds held for obvious. However, Laplace was not mistaken, the computations were not made in vain — to the extent that they dispersed some delusions and brought about a revival of a very ancient topic of study.

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Throughout the XVIII century, it is under the heading of probability calculus that we find almost all attempts to mathematically analyze human problems. This is mainly due to the fact that the initial influence from Pascal had persisted: one attempts to elucidate not only the conduct of players properly speaking but also that of man struggling against uncertainty. But it is also because the last thing that one finds when making a work is the one that should be placed first. For a student today, the theory of choice appears to be much simpler than that of games: first one reasons, following Pareto, by completely eliminating risk and uncertainty to attain pure choice. But the first researchers did not attack the problems that we consider today to be the simplest ones. Here, as in many other domains whether or not they are mathematical, thinking advances in two directions: by combining it explores consequences but it also traces its way back towards the principles ; beginnings can never be really *elementary.*

And thus, choice theory, in its early days, emerged from the reflections on two types of problems considered today to be complicated: firstly, choice under uncertainty and secondly the choice of a collectivity. The first theme, that of uncertainty prompted Daniel Bernoulli and Buffon to sketch the first drafts of a theory of total or marginal utility, given the name of a theory of "moral values". The second theme, that of collective decisions and conflicting wills is no less important: the non comparability of utilities and the major difficulties of expressing a collective will were analyzed by Condorcet, Laplace and some others[6]. But if we take Buffon's *Arithmétique morale* or Condorcet's *Mathématique sociale* as our point of departure then there is no continuous tradition that traces a path all the way to the *New Welfare Economics*: there is a deep divide within the history: Cournot and Poisson were the last representatives of a style that would be ignored or mocked at for a century, whereas Walras or Pareto, to mention only eponymous heroes, hardly considered themselves as the heirs of the social mathematicians of earlier times.

When speaking of the history of mathematical economics or econometrics, it used to be common practice to present Cournot as its founder or precursor.[7] Precursor, one could say that, in the sense that it is *today* that it becomes possible to grasp the significance of his work (by adding to his works in economics strictly speaking his book on *The theory of chance,* and just as importantly his philosophy of history) but we should not imagine that there were a straight path between him and us — rather there were long detours and much narrow-mindedness.

6 Concerning Condorcet, one should read: *La mathématique sociale du marquis de Condorcet (The social mathematics of the marquis Condorcet)*, by G.G. Granger, Paris, P.U.F., 1956.
7 But cf. Tinbergen, *Econometrics*, Philadelphia, 1951, p. 9.

With the superficial sarcasms that he directed at Cournot and Laplace, Joseph Bertrand attacked what was already an old tradition. In an ill-tempered and rather unintelligent manner he was persuaded that everything was almost over, that enlightenment would definitely condemn a temporary straying from the right path and that probability calculus would reject its impure origins to concern itself only with the material world of physics or biology, leaving the "moral" universe to rely on "common sense" alone to avoid "delusions". It is true that the ambitions of certain scholars had been immoderate and that their foolhardiness or the inflexibility of preconceived mathematical tools would discredit them in the eyes of an unprepared opinion. How could we not be worried by the title of Poisson's work[8]: *Recherches sur la probabilité des jugements en matière criminelle et en matière civile, précédées des règles générales du calcul des probabilities*? (*Research on the probability of judgments in criminal and civil cases, preceded by general rules of probability calculus*) But "ceux que nous appelons anciens étaient véritablement nouveaux en toutes choses et formaient l'enfance des hommes proprement"[9] (those who we call old were actually new in every way and really formed the childhood of later men…). Altogether, there is perhaps nothing to regret. Progress is not linear and some books must wait a long time before they are truly and actively read.

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The profound studies of K.J. Arrow brought back attention to some logical difficulties related to the construction of a collective will. I hope that I will not diminish the value and originality of this beautiful research if I mention Rousseau or Condorcet in relation to it: an intellectual tradition in which mathematical and logical work had been suspended was recovered and this without neither destroying nor repeating. We should congratulate those who did the work and forge ahead. A preliminary question must be raised: what is the real relationship between the research that is our point of departure and probability calculus. For the modern reader, risk theory is only a chapter of choice theory, whereas, from Pascal to Cournot, the theory of chance seems to have been essential in any theory of decisions and human judgments. It would be lengthy and perhaps premature to attempt to undertake here an in-depth analysis. We will content ourselves with a few cursory remarks.

Neither Pascal nor Fermat was the first to compute chance; they knew or sensed this, nevertheless they were aware that they were innovating, because they had seen the *use* that could be made of such computations. Calculating probabilities is just a form of combinatorics among others — what matters is its use: to guide human behavior. That was indeed the program of Jacob Bernoulli's posthumous book: "Pour ce qui est sûr et hors de doute, nous parlons de connaissance ou de compréhension; pour tout le reste nous disons seulement *conjecture* or *opinion*. *Conjecturer* quelque chose, c'est mesurer son degré de probabilité: ainsi le savoir conjecturer ou stochastique se définit pour nous comme savoir mesurer, le plus exactement possible, les degrés de probabilités, *afin que*, dans nos décisions et nos actions, nous puissions toujours choisir ou accepter ce qui nous aura paru meilleur, plus satisfaisant, plus sûr, plus prudent : seul objet à quoi s'applique toute la sagesse du philosophe, toute la prévoyance du politique."[10] (for that which is beyond doubt, we will speak of knowledge or understanding; for all the rest we will just say *conjecture* or *opinion*. To *conjecture* something is to measure how probable it is: thus the ability to conjecture or stochastique is defined for us as the ability to measure as exactly as possible the degree of likelihood, *in order* to ensure that our decisions or actions choose or accept what seems to be the best, the most satisfactory, the most sure, the most

8 Published in Paris a few months before Cournot's *Recherches sur les principes mathématiques de la théorie des richesses* and at about the same time as an article by Cournot in the *Journal de Liouville*, on the probability of judgments (1838).
9 Pascal, fragment from an (uncompleted) treatise about emptiness.
10 Jacobi Bernoulli…*Ars conjectandi…Basilae*…,MDCCXIII, Pars. IV, Cap. II, p.213.

prudent alternatives: these are the sole issues that fully preoccupy the sharp minds of philosophers and political planners.)

It is very regrettable not to have retained in its original sense the word introduced by Bernoulli, the *Stochastique,* that is, the ability to conjecture whose Greek origins do not at all evoke chance but rather the decision[11]. The main question is how to lay a foundation for methods of rational behavior and with respect to this objective, probability calculus in itself is only an auxiliary tool. This is what was forgotten, about a century ago when probability calculus believed that it was becoming autonomous when it detached itself from the study of human decisions to be concerned only with "natural philosophy", that is, physics.

However, today this is being brought into question: J. Neyman and A. Wald made efforts to constitute statistics as a science of how to conduct actions; the syntheses of B. de Finetti and of L. Savage establish profound connections between pure probability theory and game theory; but is was still too early to extract a valid philosophy from this. Let us just bear in mind that it is often a good discipline to postpone the introduction of randomness in a theory, if only to notice how deeply rooted the need for models of incomplete knowledge is.

Once a scientific stochastic had been constituted, it was natural to include in it methods for criticizing judgments and testimonies and consequently also the examination of collective decisions. This was done early [12] and one of the important steps on the way was the *Essai sur l'application de l'analyse de probabilité des décisions rendues à la pluralité des voix*, (*Essay on use of probability analysis in decisions taken by majority*) published by Condorcet in 1785. This work does not have a good reputation: it infuriated J. Stuart Mill; Todhunter deemed it obscure and full of contradictions; Bertrand; illegible, useless and ridiculous. It cannot be denied that a considerable part of Condorcet's theories are rather reckless: moreover it is precisely the random aspects of the theory that are suspect. As soon as the analysis of games of chance had become established, it became tempting to use it as a "model": the ineluctable errors that each man commits in his judgments were quite naturally compared to random draws. The Bernoulli *urn* filled with black and white balls became the typical and soon the only random instrument. But the Bernoulli model naturally assumes independence between successive draws: a rather unrealistic hypothesis — this is one of the central problems — when it comes to studying how the individual errors of the members of the same jury are aggregated. We know that it took a long time before one was able to dispose of this excessively restrictive hypothesis[13]. First Cournot and then Bertrand were very uneasy, one after the other they could not refrain from alluding to the meeting between Pantagruel and the judge Bridoye "lequel sententioit les procez au sort des dèz" (who determined the trials by a throw of the dice).

But we also find in Condorcet a distinction that would be further developed by his immediate successors and which clearly isolates the random domain. Daunou, for instance, notes[14] that the intrinsic value of a decision taken by the plurality of votes obviously depends on the

11 *Stochazomai* means to aim (in darts or javelin) and also to conjecture, without particularly implying a random draw. We find the science of *Stochastics* in Plato from whom Bernoulli borrowed it.

12 Pascal probably thought of this but we only have the hearsays of Fileau and La Chaise which we should consider with circumspection. A Scottish mathematician, Craig, published towards 1700 an unfortunate essay of *Mathematical Theology* where the Pascalian bet is particularly abused. Some historians (Lubbeck, Todhunter) even ascribe to this same Craig the oldest known writings on the probability of testimonies — an anonymous dissertation published in the *Transactions philosophiques* in London in 1969, entitled *Calculation of the credibility of human testimony*; this dissertation may have been a source of inspiration for the article Probabilité in Diderot's *encyclopédie*.

13 Not so long ago, Schumpeter contested the modern extensions of the notion of a random variable. Cf. *Business cycles,* p. 194, note and cf. *Economie appliquée*, 1954, p. 264-265.

14 Dissertation from the year IX in the class of the Moral and Political Sciences at the Institut de France: *Sur les élections.*

insightfulness and impartiality of the judges or voters; let us add on their error probability. But instead of asking whether the decisions that result from a given procedure is good or bad, one can ask to what extent this choice is a good representation of the wish of the collectivity. This new and to some degree preliminary partial problem is no less difficult: "en regardant une élection, ou (ce qui est la même chose) une décision à la pluralité des voix entre plusieurs propositions, comme un moyen de terminer les débats en adhérant au voeu du plus grand nombre, il reste encore beaucoup de difficulté à déterminer quel est réellement ce voeu" (If we consider an election, or (it is the same thing) a vote based decision between several propositions as a way to end debates by adhering to the wish of the greater number, it remains very difficult to determine what this wish really is), declared Lacroix[15] in a comment on Condorcet.

Let us thus initially leave aside the theory of errors and examine some of the difficulties that had been identified by our authors, Condorcet, Laplace and Lacroix[16] between 1780 and 1820. We recognize familiar themes from contemporary theories of the general interest and collective decisions.

The Condorcet paradox

In an election where there are only two candidates, a ballot will immediately establish who is for *A* and against *B* and who is of the opposite opinion. Having counted the votes, we will be able to tell what fraction of the electoral body *wants A* to be chosen and does at the same time not want *B* to be chosen. If it is urgent to reach a decision, one can quite naturally follow the opinion of the party that outnumbers the other one. The majority rule applies just as well when one must choose between two mutually exclusive proposals instead of between two candidates. Of course, we will not here enter into considerations of whether it is a good thing to follow the opinion of the majority: we merely note that it is possible to ascribe a precise meaning to the expression "the opinion of the majority".

However, says Condorcet, things are not at all the same when there are at least three candidates, or three mutually exclusive proposals. It is not impossible to use the majority rule in the conventional way but the significance of this law changes profoundly. One knows that every voter voted for the candidate that he esteems the most but one ignores the preference ranking that he would give to the other candidates if he had to choose between them. It is thus perfectly possible that electing the candidate with the greatest support would lead to greater discontent than electing the candidate who came second. One can sense this and experience confirms it; but can this question be analyzed rigorously?

In the case of a single alternative " *A or B* ", each voter expresses *at the same time*:

(1) His wish that *A*, for instance be elected; and (2) that *B* should not. If *A* receives 60% of the votes, those who are discontent represent 40% of the electoral body. Maximal satisfaction coincides with minimal discontent. In the case of three options, *A*, *B* and *C* each voter may, while voting for *A*, fear the election of *B* more than that of *C* or conversely. Now, if we count the votes, we cannot distinguish these nuances of the opinions: simple majority rules take no account of negative wishes. Some people will see this as an advantage, as they do not wish to encourage a "conspiratorial mentality", others see in it a lack of fairness. But prior to these moral questions, we are faced with a precluding technical problem which we did not have in the case of 2 options and which obviously remains present for any greater number of options: if there are more than two candidates each voter also has several ways of being dissatisfied, we can no longer identify maximal satisfaction with minimal discontent.

15 S.F. Lacroix, *Traité élémentaire des probabilités*, Paris, 1816, p. 248.
16 In the following section, I follow very closely the original texts without always giving a detailed reference.

If one wants the election to follow the wishes of the electoral body as closely as possible, the votes must first of all express all the nuances of the individual opinions and cannot be limited to the designation of *the* preferred candidate (or preferred option). It has sometimes been proposed that each elector give, on his vote bulletin, a *number* representing his ranking of each candidate and that the votes be counted by adding the points that each candidate obtained: He who has the smallest sum will be elected. This is the system proposed by Borda in 1781[17], discussed at the convention in 1793[18] and then adopted in Geneva and elsewhere; one still encounters it. But what meaning can one give to such an addition of ordinal numbers? One has spoken of the "merit" of the candidates but can this merit be measured? And if it could for each elector, could the notion of average merit be justified? Moreover, one could imagine that the opinion of each voter is not always suitably conveyed by 3 equidistant numbers: 1, 2 and 3. Can they be allowed to choose other scales? But if one takes this direction, the difficulties will be considerable: the question is how to give the individual opinions a mathematical *form* which allows them to be aggregated. If this form is imposed, like Borda proposed, it will probably betray the real wishes of the voter — if it is more flexible, on one hand, the voter will have difficulties expressing himself in this exceedingly complicated numerical language, and on the other hand, comparability may be compromised.

Thus all our authors recognize the great difficulty, not to say impossibility of translating the voters' appreciation of the candidates or the opinions into a number[19] — and they decide to return to the only thing that is directly observable, that is the preference order according to which voters can rank their options. The general outline of this argumentation will invariably remind the modern reader of the economists' quarrels about comparing — and it is the same problems that resurface, probably in an independent manner a century later. Indeed the demand for a strictly *ordinal* indicator appears in these lines of Lacroix (1816) "Les numéros affectés à chaque candidat" (the numbers ascribed to each candidate) can be used for comparisons "pourvu qu'on ne leur assigne d'autre fonction que d'exprimer le rang dans lequel ils sont placés par les électeurs"[20] (as long as they are not assigned any other function than that of expressing the ranking that the latter were given by the voters) — as well as in Pareto (1911) : "Pour déterminer l'équilibre économique nous n'avons nullement besoin de connaître la mesure du plaisir, un indice du plaisir nous suffit."[21] (To determine the economic equilibrium, there is no need to know the measure of pleasure, an index of pleasure is sufficient)

Having arrived at this point it seems that Condorcet hesitated between several directions of research without being able to finally resign himself to make a choice. Indeed, there are two distinct points of view whose analogues reappear in utility theory[22] and which we will now successively examine.

According to the first one, we admit in theory that utility may be measured but that the measuring cannot be carried out directly and that it is accessible only through implicit estimations: the decisions of economic agents; in the same way that we could acknowledge that the "merit" of candidates or the value of a suggested option could in principle be represented by a number but that voters are incapable of doing this and can only give us qualitative approximations, and that we must, based on these imperfect expressions, attempt

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17 A dissertation on elections by ballot, *Histoire de l'Académie des Sciences for 1781*, Paris, 1784, p. 657 and on.

18 Lhuilier, *Examen du mode d'élection proposé en février 1793 à la Convention nationale,* Paris, in-8^{o}, 1794.

19 It took a long time before the juries of University examinations and selection procedures accepted the use of numbers for grading. Today this has become a standard practice that is rarely criticized.

20 Loc. cit., p.251.

21 *Encyclopédie des sciences mathématiques,* t. I, vol. IV, fascicule 4, p. 609, Paris-Leipzig, 1911.

22 Cf. Hicks, *Value and Capital*, Oxford, 1946, 2nd ed., p. 11-19; Samuelson, *Foundations of Economic Analysis* , Harvard University Press, 1948, p. 90-92; Arrow, *Social Choice and Individual Values,* New York, 1951, p.9-11.

to reconstruct as best we can the unknown measure. This first point of view, leads to a problem of statistical inference which naturally involves probability calculus: the question is how to determine the most likely measure. We find the clearest formulation of this approach in Laplace who on several occasions and without ever mentioning Condorcet restated the problem in the same terms and indicated several types of solutions, among which the Borda system[23].

According to the second point of view, all measures are strictly forbidden and the purely ordinal character of value judgments is affirmed. If one believes that numbers must be used, it is with the precaution that Lacroix takes in the text quoted above: the number is nothing but a *ranking*; it is an *index*, as Pareto would say. But at the same time — and this is another hypotheses — one excludes all objectivity: one could indeed imagine that the different rankings that each voter makes is only a deformed image of an ideal ordering of the options according to their *real* values. Then we are faced once again with a problem of statistical inference; but a qualitative rather than a quantitative one[24]. The question is how to return to the common source by identifying the various individual equations. However, for Condorcet[25] as for most of his successors, the rejection of the cardinal number and of the measure was associated with a rejection of trans-individual objectivity: this attitude could be compared with one that we also find among economists: often measurable utility and objective utility are simultaneously rejected. This does not follow from any logical necessity but only perhaps from a spontaneous association between that which is subjective and that which is qualitative or between that which is mathematical and that which is numerical[26].

Let us accept, however, the double hypothesis: all that really exists is the *individual orders* of preference. Can we infer from the expression of this order as it is given by a ballot an expression for the "wish of the majority"? To answer this question, Condorcet introduces a method, of fundamental importance[27] for analyzing the individual opinions. He does not introduce it only for analytical convenience, he also makes some comments regarding it that are inspired by the practice of deliberations "dans le cas des décisions compliquées, il faut faire en sorte que le *système des propositions simples qui les forment*[28] soit rigoureusement développé, que chaque avis possible soit bien exposé, que la voix de chaque votant soit prise sur chacune des propositions qui forment cet avis et non sur le résultat seul"[29] (when dealing with decisions in complicated questions, it must be ensured that the *system of elementary propositions from which these are formed* is rigorously developed, and that each voter makes a choice about each of the propositions that form an opinion and not on the result alone).

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23 Laplace's study appeared first in the *Journal de l'Ecole Polytechnique*, vol. VII-VIII, p. 169 sq., then in *Théorie analytique des probabilités* and in *L'essai philosophique*. See *Oeuvres de Laplace*, édition nationale, Paris, 1847, t. VII, p. 101-103, 296-299.

24 Cf. M.G. Kendall, *Rank correlation methods*, London, Griffin, 1948, 460 p.

25 On one hand, Condorcet's main concern was to distinguish between the objective and the intersubjective. In his *Tableau général de la science qui a pour objet l'application du calcul aux sciences politiques et morales* (posthumous publication, 1795), he sums up his theses in the following way: "Dans les élections on distingue celles qui expriment un *voeu* de préférence de la majorité et celles qui n'expriment qu'un *jugement* en faveur de la capacité absolue des sujets préférés." (We distinguish between the elections that express a preferred wish of the majority and those that only express a *judgment* that favors the absolute capacity of the preferred citizens.) On the other hand, he always sought to reason in terms of ordinals.

26 Il ne faut pas croire, avertit Cournot, que partout où l'on voit des nombres ou des mesures, on voit des mathématiques, ni que le règne des chiffres soit le règne des mathématiques…(One must not believe, Cournot warns us, that wherever we see numbers or measures we also see mathematics, nor that the reign of numbers is the reign of mathematics…) (*Considérations sur la marche des idées*…, Livre V, chap. I).

27 This method is universally used. In the area we are dealing with here, it appears constantly. Cf. *The Postulate of relevancy* of E.U. Huntington, the *condition of independence* of K. J. Arrow (*Social Choice and Individual Values*, p.26-27); the postulates of M. Flemming (*Quarterly Journal of Economics*, LXVI (1952), n°3 (Aug.) pp.370-375).

28 My italics.

29 The quoted *Essai,* Preliminary introduction, p.69.

The principle of Condorcet motivates us to reduce every opinion to its *simplest components*, and thus to always consider binary votes which present less difficulties, as we have seen previously. But to know "the will of the majority" it is not necessary to have multiple ballots: if we ask each voter 10 rank the candidates (or the options) in order of preference, we have all the elements necessary for a complete exposition. In establishing a certain order, for instance:

First ……… A
Second……. B
Third……… C

the voter affirms (implicitly) the following judgments:

A is better than B
B " " " C
A " " " C

which we will express by

$$A > B, B > C, A > C^{30}.$$

Thus each ballot will be decomposed into a system of simple judgments.

Imagine an electoral body of sixty voters divided in the following manner[31]:

23 have given the order $A > C > B$
19 " " " " $B > C > A$
16 " " " " $C > B > A$
2 " " " " $C > A > B$

We analyze the voting as we have said:

In comparing A with B we have

23 + 2 = 25 votes for $A > B$
19 + 16 = 35 votes for $B > A$

Condorcet proposed that the opinion of the majority be

" B is better than A "

In the same way, by comparing A and C, we have

23 votes for $A > C$
19 + 1 6 + 2 = 37 votes for $C > A$,

from which we conclude that the majority prefers C to A. Finally, let us compare B and C:

19 votes for $B > C$
23 + 16 + 2 = 41 votes for $C > B$,

so the majority prefers C to B.

This analysis leads us to present as “the will of the majority” the three judgments:

$C > B$, $B > A$ and $C > A$

that is, the order $C > B > A$. If it were necessary to choose only one candidate, we would choose C[32].

30 'This is the notation used by Condorcet and later by Lacroix: we find in these authors a clear idea of ordinal structure.

31 Condorcet, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* Discours préliminaire, p. 58. Cf. the eleventh hypothesis of the first part, page 119. The same numerical examples are used in later works, in particular in the *Essai sur la constitution et les fonctions des assemblées provinciales*.

Condorcet himself showed the limitations of his method. The two problems which are not equally serious are the following; "Il n'existe qu'une seule méthode rigoureuse de connaître le voeu de la pluralité dans une élection. Elle consiste à prendre ce voeu sur le mérite respectif de tous les concurrents comparés deux à deux…mais 1º Cette méthode est très longue…; 2º Il peut arriver qu'aucun concurrent ne soit déclaré supérieur à tous les autres par la pluralité[33]." (There is only one rigorous way of knowing the wish of the plurality in an election. This consists of basing this wish on the respective merits of all candidates compared two by two ... but ... first, this method is very long ... : second, it is possible that none of the candidates will be claimed superior to all the others by the plurality.)

The first problem, length and difficulty of practical application, is less interesting for us than the second, which indicates the possibility of a situation where there is no valid solution. Let us look at the second numerical example constructed by Condorcet, with the numbers slightly changed.

$$\text{Vote:}\quad \left[\begin{array}{ll} 23: & A > B > C \\ 17: & B > C > A \\ 2: & B > A > C \\ 10: & C > A > B \\ 8: & C > B > A \end{array}\right.$$

$$\text{Analysis:}\quad \begin{array}{lll} B > C: 23 + I7 + 2 = 42\ ; & & C > B: 10 + 8 = 18 \\ C > A: 17 + 10 + 8 = 35\ ; & & A > C: 13 + 2 = 25 \\ A > B: 23 + 10 = 33\ ; & & B > A: 17 + 2 + 8 = 27 \end{array}$$

By the law of the majority, we should hold three judgments,

$$B > C\ ,\ C > A, \text{ and } A > B,$$

but they are incompatible!

The reason for this failure is easy to detect: the three elementary judgments obtained from an individual opinion are interdependent; if we state $A > B$ and $B > C$ we must have $A > C$. But if we apply the rule of the majority to each of them, nothing can ensure that the three results will be consistent. In the preceding example there is really a majority for $B > C$, a majority for $C > A$, and a majority for $A > B$; but these three majorities are distinct, none of the voters belonging to all three.

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32 Condorcet notices that an ordinary election where each voter puts only one name in the urn would have given 23 votes for *A*, 19 for *B*, and 18 for *C* and that this vote gives a wrong idea of the real preferences of the collectivity. It is a first paradox, commented in these terms in the *Essai sur la constitution et les fonctions des assemblées provinciales* (Oeuvres de Condorcet, Arago edition, Paris, 1847), Vol. VIII, p. 193: "Qu'entend-on par être élu? N'est-ce pas être jugé préférable à ses concurrents? Pourquoi fait-on dépendre ce jugement de l'opinion de la majorité? C'est parce qu'on regarde comme plus probable une proposition déclarée vraie, par 15 personnes par exemple, que sa contradictoire déclarée vraie par 10 seulement. Ainsi, celui qui obtient véritablement le voeu de la pluralité dans une élection doit être celui dont la supériorité sur ses concurrents est la plus probable, et conséquemment celui qui a été jugé par la pluralité supérieur à chacun des deux autres. Or, il est possible, s'il y a seulement 3 candidats, qu'un d'entre eux ait plus de voix qu'aucun des deux autres ; et que cependant l'un de ces derniers, celui même qui a eu le moins de voix, soit réellement regardé par la pluralité comme supérieur à chacun de ses concurrents. *Cette assertion paraît paradoxale*, mais on *sentira* qu'elle peut être vraie si on fait réflexion que celui qui vote en faveur d'un des candidats prononce bien qu'il le croit supérieur à chacun des deux autres, mais ne prononce point son opinion sur leur mérite respectif, que dès lors son jugement est incomplet…on a deviné cet inconvénient par une sorte d'instinct longtemps avant qu'on en eût prouvé la réalité…" (What do we mean by being elected ? Is it not to be judged preferable to the competition? Why do we make this judgment depend on the opinion of he plurality? It is because we think that a statement claimed as true by 15 persons is more likely to be true than the contradictory statement claimed as true by only 10 persons. So the one who really obtains the preference of the plurality in an election must be the one who seems superior to all. But it is possible, if there are only 3 candidates, that one of them will have more votes than the others but that one of the latter, perhaps the one who had the smallest number of votes, will be looked upon y the plurality as superior to each of the others. This assertion seems paradoxical, but one feels that it could be true, if one recalls that when we vote for one candidate, we claim that we believe he is superior to the others but we do not give our opinion inconvenience by a kind of instinct a long time before we proved its reality...) Notice that Condorcet uses the word *plurality* when we use *majority* (which, in French, was an Anglicism of the eighteenth century).

33 *Sur la forme des élections* (Arago édition, Paris. 1847), Volume IX. p.305 : Brunswick edition (1804), Volume V. p.29.

The existence of this phenomenon makes the twofold task of the legislator more difficult: that is, "connaître le vrai jugement de la pluralité toutes les fois qu'il existe, et lors même que ce jugement n'existe pas" (to know the true judgment of the plurality whenever it exists, and even when such a judgment does not exist) — for example, in the case where the different elementary judgments are not compatible — and " indiquer le choix qui doit être adopté pour avoir un moindre risque de tomber dans l'erreur"[34]. (to indicate the choice which should be adopted in order that the risk of error be as small as possible.) This is why after having studied the paradox, Condorcet could not resign himself to conclude that it is impossible to attribute any coherent opinion to the electoral body (it would be necessary to claim a kind of "standoff"), He looks for a lesser evil, that is to say, among all *coherent* opinions, the one which is supported by the largest possible number of votes. But, as we will see later, the difficulty is more profound and every attempt at a solution is more or less arbitrary. Condorcet does not ignore this problem and returns to it several times. Daunou, without using mathematics, " présente contre cette manière de combiner les propositions des difficultés très fondées, et pense que, dans le cas douteux, il ne saurait y avoir de majorité bien prouvée" (presents some very well-founded difficulties with this method of combining statements, and he thinks that in the dubious case there would not be a well·defined majority); this means that the paradox brought to light by Condorcet is, in a certain sense, impossible to eliminate. It is precisely this ineluctable aspect which K. J. Arrow[35]studied in great detail. Let us spend some time on it.

The Condorcet Effect

Numerical examples cannot hope to make us understand the mechanism of the paradox; a general analysis is necessary. Consider an assembly which has to choose between three options, *A*, *B*, and *C*, and suppose that each member of this assembly has for the three options a *consistent* set of preferences — that is, he arranges the three options in a given *order*. Some, for instance, will order in this way:

$$A > B > C \qquad (1)$$

This we will call *opinion* (1). Five other opinions of this type are possible *a priori*

$$A > C > B \qquad (2)$$
$$C > A > B \qquad (3)$$
$$C > B > A \qquad (4)$$
$$B > C > A \qquad (5)$$
$$B > A > C \qquad (6)$$

Thus the members of this assembly are supposed to be divided into six opinion categories or *parties*, which we will designate by the numbers above[36].

34 *Essai sur les assemblées provinciales* , the first work on ways to evaluate the opinion of an electoral body, Œuvres de Condorcet, Arago edition, Vol. VIII, p. 573.

35 K. J. Arrow mentions the "well-known paradox of voting" without referring to its previous origins. He mentions the *Report on Methods of Election* by E. J. Nanson and attributes to the latter the discovery of this paradox (*Social Choice* and *Individual Values* p. 3, n. 3 and p. 95). In fact Nanson published in *Transactions of the Royal Society of Victoria* (Vol.19, Melbourne. 1883) a study which draws its inspiration explicitly from Condorcet and Borda — known without doubt, through Todhunter, *A History of the mathematical Theory of Probability*, 1865 — and which does not add anything new to the meaning of the paradox ("These results are well known"). Nanson's aim was to find an election rule of the kind proposed by Condorcet: "In this case (inconsistency) there is no real majority and we cannot arrive at any result without abandoning some one of the three propositions. It seems reasonable that the one which is affirmed by the smallest minority should be abandoned" (loc. cit., p. 213).

36 Notice that the enumeration of categories has not been done at random: between two contiguous categories there is less difference than between two more distant ones. If one considers that the chain closes on itself and that (6) and (1) are contiguous: it is possible to represent the categories as placed around a circle.

Let N be the total number of voters in the assembly and N_i (i = 1,2, ... ,6) the number of voters in each of the six categories. The search for a collective opinion should be made, following Condorcet's principle, by decomposition: first examine[37] the attitude of the assembly concerning A and B. The six categories are regrouped into two classes:

$$(A > B) = (1) \text{ and } (2) \text{ and } (3)$$
$$(A < B) = (4) \text{ and } (5) \text{ and } (6)$$

and similarly for the other alternatives:

$$(A > C) = (6) \text{ and } (1) \text{ and } (2)$$
$$(A < C) = (3) \text{ and } (4) \text{ and } (5)$$

and

$$(B > C) = (5) \text{ and } (6) \text{ and } (1)$$
$$(B < C) = (2) \text{ and } (3) \text{ and } (4)$$

The rule of the majority is used to compare the number of voters in two opposed classes.

For instance, if we have[38]

$$N_1 + N_2 + N_3 > N_4 + N_5 + N_6$$

we will consider ($A > B$) the collective judgment.

The problem is to find out whether the three collective judgments considered can be inconsistent and in which circumstances this could happen. Inconsistency will occur if we have

$$(A > B), (B > C), \text{ and } (C > A)$$

or, inversely, if

$$(A < B), (B < C), \text{ and } (C < A).$$

The first case occurs when we have simultaneously

$$(i) \qquad \begin{aligned} N_1 + N_2 + N_3 &> N_4 + N_5 + N_6 \\ N_5 + N_6 + N_1 &> N_2 + N_3 + N_4 \\ N_3 + N_4 + N_5 &> N_6 + N_1 + N_2 \end{aligned}$$

Conditions (i') for the second case can be written by inverting the three inequalities.

Conditions (i) are perfectly possible for properly chosen numerical values of the N_k (k = 1,2, ..., 6); we knew already, from Condorcet's example, that application of the decomposition principle and the majority principle could sometimes lead to inconsistency. We can construct as many new examples as we please: for instance — this example is mentioned frequently[39] because it is a simple one — by choosing:

$$N_1 = N_3 = N_5 = 1$$
$$N_2 = N_4 = N_6 = 0$$

37 Either by a simple vote or by the results of a preliminary preferential vote.

38 The > is used here with its arithmetical meaning "greater than" where in preceding lines and in the following line, it means "preferred to". But both meanings have the same logical structure — ambiguities are impossible.

39 We shall see, a little further on, the significance of this example.

we satisfy conditions *(i)*.This is an assembly of three persons who hold, respectively, opinions (1), (3) and (5). It would also be sufficient to choose:

$$N_1 = N_3 = N_5 > N_2 = N_4 = N_6.$$

Some systems of numbers N_k, that is, some distributions of individuals among the six possible opinions, lead to the situation envisaged by Condorcet. It will be convenient, in the rest of this paper, following the custom of the physical and natural sciences, to call this special situation the "Condorcet effect". It is natural to wonder if these distributions — defined by the systems of inequalities (i) and (i') — possess some other properties more intuitive than their definition, and we wonder whether the Condorcet effect is more or less exceptional. This is not a matter, for the time being, of empirical or historical research: assemblies meeting to decide on the best of three proposed solutions; in the present state of the subject, psycho-social phenomena more complicated than those we envisage can certainly arise — and, besides, attempts at documentation can only furnish isolated examples[40] with no valid indication of frequency.

It is, however, possible to use the traditional technique of tabulating *a priori* all possible cases — the results of which can be given in the language of probabi1ity theory. In saying that for a dice the probability of an ace is equal to 1/6, we are only comparing the ace to the set of faces, that is, to the set of all possible events. In the same way, we can consider *all* possible distributions of a given number of voters into categories and compute the proportion of those which yield the Condorcet effect. The result can be stated in terms of a probability without necessarily referring to the corresponding game of dice[41].

Let us begin with a simple case, an assembly of three persons. We can count possibilities: first, all three have the same opinion; second, all three have different opinions; third, two against one. In the first case (unanimity) there is evidently nothing to say, as the ballot problem does not even arise; in the last case a majority obviously exists — no Condorcet effect; but in the second case there are three different opinions, and it is proper to distinguish the several types.

If, for example, the three opinions are

$$A > B > C \qquad (1)$$
$$A > C > B \qquad (2)$$
$$C > A > B \qquad (3)$$

we see that ($A > B$) is accepted unanimously and ($A > C$) and ($C > B$) by a majority. These three opinions are consistent and lead us to take opinion (2) for the opinion of the majority[42].

With (1), (2), (4) [cf. p. 10] we have the same conclusion: (2) is adopted as the opinion of the majority on each of the three individual ballots. But if the three opinions are (1), (3), (5), the three conclusions are inconsistent[43], as they are when the opinions are (2), (4), (6). It is possible to verify that these configurations are the only ones which lead to the Condorcet effect.

40 K. J. Arrow mentions the U. S. Congress deliberations on school appropriations (*Social Choice and Individual Values*, p. 3, n. 3).

41 This game would consist of throwing a dice a certain number of times, N, letting the face obtained represent one voter's opinion, and at the end, finding the result of this ballot by the Condorcet method. The habit among statisticians of mentioning this kind of game, a specious subordination of statistics to probability, disconcerts the uninitiated: they see obscure intentions in this language and consequently are, with good reason, shocked.

42 Intermediary in circular ordering (see footnote 36)

43 This is exactly the example already mentioned.

It would be possible in this case (only three voters) to be satisfied with a summary table of the different possibilities, but we can easily guess that such enumeration will become laborious when the number of voters increases. This is the reason why it is preferable, when the diversity begins to tire the imagination, to replace purely qualitative enumerations with strictly quantitative ones.

To reach this quantitative expression, it is necessary to standardize in some way: we will consider, for instance, that the first voter can adopt any one of the six opinions, and likewise for the second and third, so that the result is:

$$6 \times 6 \times 6 = 216$$

possibilities. Among these 216 possibilities there are twelve[44] which give rise to the Condorcet effect — a little less than 6 percent. A computation of the same type for any number above three would be easy to carry out by the usual methods of combinatorial analysis. We can see that the computed proportion increases slightly with the number of voters:

3 voters ………………	5.6 %
5 voters ………………	7.0 %
9 voters ………………	7.8 %
25 voters ………………	8.4 %

For a very big number one computes by the usual means the limiting value, which is just under 9 percent[45].

In this hypothesis, it appears that, without being rare, the Condorcet effect represents only a small fraction of the possibilities (between 6 and 9 percent): if it occurs much more often than one time out of ten in a long enough run of observations, we can conclude that it is not foolish to look for a specific *cause* which, in the midst of the observed collectivities, orients the antagonistic individual opinions in a way that makes it more difficult for a true majority opinion to emerge (see p.55).

*
* *

The key to the whole system is breaking down each individual opinion into simple judgments. Everything is done as if we were determining the opinion of everyone by a battery of three questions, according to the following table, in which the signs + and – mean yes and no, the possible answers to the three questions x, y, and z. Question x, for example, is "Do you prefer *B* to *C*?"

44 These twelve possibilities are (1)(3)(5); (1)(5)(3); (3)(1)(5); (3)(5)(1); (5)(1)(3); (5)(3)(1); and the six analogous arrangements of (2), (4), and (6).

45 Value given by computing:

$$1 - \frac{3}{\pi} \text{ arc cos} \left(\frac{1}{\sqrt{3}} \right) = 0.0877\ldots$$

Table 1

	Questions			
	$x = (B > C?)$	$y = (C > A?)$	$z = (A > B?)$	opinions
(1)	+	–	+	$A > B > C$
(2)	–	–	+	$A > C > B$
(3)	–	+	+	$C > A > B$
(4)	–	+	–	$C > B > A$
(5)	+	+	–	$B > C > A$
(6)	+	–	–	$B > A > C$

This table does not mention all possible combinations of the signs + or –; two cases are obviously missing,

$$(+++) \text{ and } (---),$$

because these two systems of answers *do not constitute an opinion*; they both correspond to three inconsistent elementary judgments.

Once this is done, and once the poll has given us the attitude of each voter, it is possible to construct the table of results by writing in three columns titled x, y, and z the signs corresponding to each ballot. We count in each column the number of votes for (+) and against (–), and finally we compare these two numbers. This amounts to the same thing as performing one algebraic addition and looking at the sign of the result. If the result is positive, it means that there is a majority in favor; if it is negative, there is a majority against.

The table of results for the example we studied first looks like this:

	x	y	z
(1)	0	0	0
(2)	– 23	– 23	+ 23
(3)	– 2	+ 2	+ 2
(4)	– 16	+ 16	– 16
(5)	+ 19	+ 19	– 19
(6)	0	0	0
Total	– 22	+ 14	– 10
Opinion of majority	–	+	–

We conclude by attributing to the majority the opinion (– + –), that is, opinion (4) or ($C > B > A$).

In the second example we have:

	x	y	z
(1)	– 23	– 23	+ 23
(2)	0	0	0
(3)	– 10	+ 10	+ 10
(4)	– 8	+ 8	– 8
(5)	+ 17	+ 17	– 17
(6)	+ 2	– 2	– 2
Total	– 24	+ 10	+ 6
Opinion of majority	+	+	+

and the result is not a consistent opinion (Condorcet effect).

In general, if the numbers for the six opinions are respectively

$N_1, N_2, N_3, N_3\ N_4, N_5$ and N_6

we have to compute three algebraic sums:

$$x = N_1 - N_2 - N_3 - N_4 + N_5 + N_6$$
$$y = - N_1 - N_2 + N_3 + N_4 + N_5 - N_6$$
$$z = N_1 + N_2 + N_3 - N_4 - N_5 - N_6$$

The Condorcet effect results when the three numbers *x, y, z* have the same sign.

Now we can give an algebraic form to the Condorcet paradox. It is sufficient to represent each individual opinion by an arrangement of three numbers, each of these numbers taking on the value 1 or -1. Then we have to define a law of composition (or aggregation) combining several opinions into one by adding the columns and replacing each of the results with 1 or -1, depending on its sign.

Symbolically:

$$\hat{x} = \text{Sgn} \sum (x_i)$$
$$\hat{y} = \text{Sgn} \sum (y_i)$$
$$\hat{z} = \text{Sgn} \sum (z_i)$$

(x_i, y_i, z_i,) being the opinion of individual i, $\sum$ meaning a sum over all the individuals in the assembly, and Sgn being an operation which consists of replacing every positive number with + 1 and every negative number with –1; $(\hat{x}, \hat{y}, \hat{z})$ is the resulting opinion.

The Condorcet effect is essentially the following: we define a law for aggregating a set of objects (the triplets of + 1 and –1); then we see that the fact that some values, (– 1, – l, – 1) or (+ l, + l, + 1), are prohibited for component objects is not a sufficient condition for avoiding these same values in the resulting object: the consistency of judgments in each individual opinion is *not sufficient* to ensure a similar consistency for judgments resulting from the application of the majority rule.

Quételet's Paradox and the Iron Rule

A phenomenon of a more extensive kind can be seen in the *Condorcet effect*. For instance, we immediately see analogies with the very famous Quételet paradox. The "average man" theory was strongly criticized for more than a century; but critics often allowed themselves to be drawn in on the ground of sociological science before solving the preliminary question, which is a technical one. It is this preliminary question (and this one only) which will be summarily treated here[46]. As early as 1843, Cournot presented very clearly the fundamental objection[47]: "Lorsqu'on applique la détermination des moyennes aux diverses parties d'un système compliqué, il faut bien prendre garde que ces valeurs moyennes peuvent ne pas se convenir : en sorte que l'état du système, dans lequel tous les éléments prendraient à la fois les valeurs moyennes déterminées *séparément* pour chacun d'eux serait un *état impossible*[48]." (When determination of the mean (average) is applied to the different parts of a complicated system, great care should be taken to remember that these mean values might be inconsistent with each other, for the system might be in an *impossible* state if each of the elements took on its mean value, determined *separately*.)

This is a well-known mathematical fact which was clearly recognized by Gauss and Laplace in their theories of errors of observation and their explication of the method of least squares. If one measures several quantities that are not independent, corrections cannot be independent either. Cournot suggests some simple examples inspired by triangulation techniques.

Take some *right* triangles and compute the means of the lengths of their sides. The three means can be used to construct a new triangle which we could call "average" but *which is not a right triangle*. Here is a numerical example which stresses, by its presentation, analogies with the Condorcet paradox; we have taken four right triangles (the lengths of the sides must satisfy the Pythagorean rule) and computed the three means:

	First side	Second side	Hypotenuse
	5	12	13
	15	8	17
	3	4	5
	7	24	25
Sum	30	48	60
Mean	7 ½	12	15

The result is not a right triangle, because the sum of the squares of the first two sides is

$$\left(7\frac{1}{2}\right)^2 + (12)^2 = 200\frac{1}{4}$$

and the square of the hypotenuse is

$$(15)^2 = 225.$$

Here again, by combining several objects, each having the same property, we obtain an object which does not have this property.

More generally, suppose we measure the three sides and three angles of several triangles and compute the mean for each of the six elements. The system of six computed numbers cannot form a triangle unless all the triangles chosen are similar. In this case (similar to the case of unanimity) an average triangle exists[49].

46 The best introduction for a reader who wants to know more about the question is the lecture given by Fréchet under the title "Réhabilitation de la notion statistique de l'homme moyen" (*Les Conférences du Palais de la Découverte*. Paris. 1950, 24 pp.). There is also the mathematical memorandum on typical elements by Fréchet,quoted footnote 56,p.20.

47 As on many occasions, J. Bertrand satisfied himself by following closely Cournot's text and by adding to the scientific objections his own sprightly style. The result is that Bertrand has been quoted more often than Cournot.

48 *Exposition de la théorie des chances* (A. A Cournot, Paris, 1843, p. 213).

49 For right triangles it is possible to compute quadratic means: the result will be all right. But it is impossible to find a satisfactory operation for ordinary (not right) triangles. Here we touch on the classical problem of aggregation in econometric

"Généralement, il n'y aura pas de triangle dans lequel les valeurs moyennes pour les angles et pour les côtés puissant se correspondre. La moyenne des aires ne coïncidera pas (non plus) avec l'aire du triangle construit sur les valeurs moyennes des côtés, et ainsi de suite. Donc, même si l'on mesurait, sur plusieurs animaux de la même espèce, les dimensions des divers organes, il pourrait arriver et il arriverait vraisemblablement que les valeurs moyennes seraient incompatibles entre elles et avec les conditions pour la viabilité de l'espèce. Nous insistons sur cette remarque bien simple, parce qu'elle semble avoir été perdue de vue dans un ouvrage[50], fort estimable d'ailleurs, où l'on se propose de définir et de déterminer *l'homme moyen*, par un système de moyennes tirées de la mesure de la taille, du poids, des forces, etc., sur des individus en grand nombre. L'homme moyen ainsi défini, bien loin d'être en quelque sorte le type de l'espèce, serait tout simplement un homme impossible, ou, du moins, rien n'autorise jusqu'ici à le considerer comme possible" [51]. (Generally, there will not exist any triangle in which the average values for angles and sides correspond. The average of the areas will not coincide with either the area of the triangle based on the average of the sides, and so forth .... Thus even if one measures the dimensions of different organs in several animals of the same species, it could very likely happen that the average values would be incompatible with each other and with the conditions for the survival of the species. We insist on this very plain remark because it seems to have been forgotten in work, very estimable from another point of view, where the author intends to define and determine the *average man* by a system of means obtained by measuring the size, weight, and other characteristics of a great number of individuals. The average man so defined, far from being the archetype of the species, would plainly be an impossible man, or at least nothing allows us to consider him possible).

Another famous quarrel has some basic traits in common with the problem raised by Cournot concerning Quételet's text. The invention of printing modified in a very profound way the attitude of the human mind toward the written word; it was probably the necessity to print the Bible which made people aware of the particular nature of problems with the manuscript tradition. The different manuscripts available did not always agree. By the middle of the sixteenth century some scholars recommended authenticity rules which call to mind the rules of popular suffrage, the degree of truth being measured by the number of witnesses. Editions appeared provided with as complete critical apparatus as possible, raising *ipso facto* some problems very close to those of statistical induction. Finally, during the nineteenth century, a transformation of the critics' state of mind occurred which was quite similar to the one already mentioned concerning collective judgments. This corresponded to a kind of strategic retreat approaching careful positivism, at the same time reevaluating the "subjective" as an object of science. With Karl Lachmann a true science of the manuscript tradition was born; it was no longer a matter of starting with value judgments (the authority of the witnesses or even their antiquity) but of determining first the genealogy and relationships. In the same way that scientific criticism of judicial decisions was to lead to some paroxysms so excessive that they would be easy targets for the "subtle minds" who would come later, textual criticism produced, in emulation of Condorcet and Poisson, erudite espousers of "the geometric way" who (perhaps only by the reactions they caused) served philology well. Even more than

models, which is very close to the problem we are studying. For instance, one can compare the results of A. Nataf (*Econometrica*, 1948, pp. 232-44) and those of M. Fleming (*Quaterly Journal of Economics,* 1952. pp. 366-84). The same mathematical structure is discussed.

50 This is probably Quélelet's book, published in Paris, 1835 (Brussels, 1836) with the title *Sur l'homme et le dévelopement de ses facultés, ou Essai de Physique Sociale*. But the "average man" theory already had appeared in an essay of 1831: *Recherches sur la loi de croissance de l'homme* (Nouveaux mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles. Vol. VII (1832).

51 Cournot, *Exposition de la théorie des chances* (Paris, 1843), p.214. Compare the moderation of Cournot with J. Bertrand's tone, *Calcul des Probabilités* (Paris, 1889), pp. 61-63.

Lachmann, editor of Lucrèce and then of the Greek text of the New Testament (1831), we have to mention Dom Quentin, editor of the Latin version of Genesis who wrote in 1926, "Je ne connais ni erreurs, ni fautes communes, ni bonnes, ni mauvaises leçons, mais seulement des formes diverses du texte…, par une méthode qui s'appuie sur des statistiques rigoureuses…je classe les manuscrits…de ce classement résulte un canon critique qui impose pour l'établissement de ce texte une règle de fer"[52]…(I recognize neither common errors or mistakes, nor good or bad passages, but only different forms of text.… By a method founded on rigorous statistics ... I classify manuscripts… From this classification results a critical canon which imposes an iron rule for the establishment of the text…)

One can imagine the storms raised: "Si l'on peut calculer les mouvements de machines, on ne peut pas calculer ceux d'une volonté humaine!…Les scribes n'ont pas été des machines à copier!…" (Though it may be possible to compute the movements of machines, it is impossible to compute those of a human will! Scribes were not copying machines!) Dom Quentin's problem or challenge consisted of building an algebra: once. the voices of the different witnesses have been weighed and counted, one finally decides by looking at statistics alone, without letting one self be influenced by a subjective criticism suspected of bias. That is precisely, in a more sophisticated version, Condorcet's problem. Complication comes from the fact that, for Condorcet, voters were contemporary men, the members of a single assembly, and hypothetically equally right, while in the other case the voters — that is, manuscripts — can be very unequally right depending on how the copy descended. Still, the question is to choose between several divergent opinions for each section of text. However that which makes it difficult to construct an absolute and acceptable rule which will always permit an appeal to intelligent internal criticism is that the repeated application of the rule, whatever it is, does not guarantee the consistency of the final product. A set of preferences constitutes a real opinion, provided that some logical regularities are respected. ln the same way, once a Latin or Greek text has been reconstructed, it must show a unity, however difficult to define, of vocabulary, syntax, style, tone, and thought.

In the three cases we have presented, the problem is always to combine several individual representations into a single one and to see if the result satisfies some internal requirements of consistency. Many other illustrations of the conflict between external criticism (which combines the votes) and internal criticism (which examines the product of this combination) could be given; we will see some of them later on. The preceding is sufficient to help us to state the preliminary logical problem in its most general terms. However, the provocative character of the theses of Condorcet, Quételet, Dom Quentin, and their few supporters should be noted; the problems stated are not purely mathematical or logical. Il should be very clear that preliminary logical analysis does not allow us to draw conclusions, but it should be evident that drawing conclusions without it is not allowed. This mistake was made by the majority of their opponents. In their eagerness to condemn this psychology or that philosophy, they jumped to conclusions and did not try to understand whether something of the method could be saved. "*Multi pertransibunt*" Pierre de Fermat liked to repeat, "*ut augeatur scientia* "[53]. This implies a lot of waste to gather a little gold.

52 *Essais de critique textuelle* (Paris, 1926), p. 37.

53 A contemporary of the sympathetic mathematician from Toulouse said more brutaly "Eviter soigneusement la précipitation et la prevention." (Headlong haste and prejudice should be carefully avoided.)

Means and Associative Operations

The idea of a law of composition is one of the most primitive in mathematics: it occurs naturally in concrete and familiar forms as soon as we start to compute, either with numbers or measurable quantities: addition and multiplication are the two primary examples. Each of the successive extensions of the notion of number has required a parallel extension of the operations. It was quickly noticed that the form of computations and operations could remain the same while the nature of the numerical objects subject to the computation evolved. In this way a kind of general grammar of operations, often called modern algebra (or abstract algebra), was set up[54]. This grammar went back to earlier formal logic, with its combinatorial procedures, which were seen quite early to be similar to arithmetic processes. For instance, the use of the conjunctions *and* and *or* is quite close to the signs + and ×. The universal notion which subtends the progress of arithmetic as well as that of logic is the law of composition or operation. Consider a set of objects which have no specified nature, and between which relations have been set up in order to define an operative or "algebraic" structure: the setting up of a composition law or *operation* consists of associating with a group of elements called *terms* one unique element called the *result*. In most fields of application, and in particular those we are interested in (composition of votes, judgments, testimonies, etc.), one should be able to apply the operations of composition to a variable number of terms; thus addition for numbers (ordinary or otherwise) or vectors allows us to compute

$$x_2 = a + b$$
$$x_3 = a + b + c$$
$$x_4 = a + b + c + d$$
etc.

In spite of analogies, we take a point of view quite different from that of the definition of a *function* of several variables. When we write

$$x = f(a,b,c)$$

it indicates that the function f requires three and only three arguments. If we use the functional language, we will have to say that the problem is to define a whole battery of functions such as

$$x_2 = f_2(a, b)$$
$$x_3 = f_3(a, b, c)$$
$$x_4 = f_4(a, b, c, d)$$
etc.

Each function gives a composition law for a definite number of terms. Of course, the set of these functions possesses a certain unity: one must be able to say that, in some sense, the composition law remains the *same* whatever the number of terms being composed.

One of the most frequently used processes in abstract algebra to assure this unity is the recurrence process which defines f_3 by means of f_2 ; f_4 by means of f_2 and f_3, etc. For instance, as soon as we know how to add two terms in some set, we can, without innovating, add any number of terms just by repetition: addition is said to be *associative*. It is proper to note that in addition the different terms play the same role: in systems where the suffrage is said to be universal, voters are all treated equally and the composition law of the votes possesses the same property of total symmetry as ordinary addition. But this complete Indifference to the

54 Leibniz and even Lulle are precursors. Boole is the initiator: *Laws of Thought*, 1847.

arrangement of the component terms is not always required; it could be useful to consider cases where the different terms are treated differently, each one having its own role and being unable to change places with another[55].

The most interesting case of composition by association for this study is the case of the *mean*; as is well known, it is one of the most common processes of statistics, among those which substitute one typical representative object for a set of objects. From the algebraic point of view[56], which we will take here in order to define the arithmetical mean of a set of objects, we have to know how to write

$$m_2 = \frac{a+b}{2}$$
$$m_3 = \frac{a+b+c}{3}$$

etc.,

that is, to add the objects a,b,c, ... and to divide them by an integer.

The operation is really associative, but it is possible to miss this by not computing cautiously. Let us compare the means of five terms: a,b,c,d,c. The associative rule allows us to operate progressively, for instance by computing first the mean of (a,b,c) and then the mean of (d, e) and finally the mean of the two results. Thus wc have

$$x = \frac{a+b+c}{3} \qquad y = \frac{d+e}{2}$$

But it is clear that the final mean is not $\frac{x+y}{2}$. The true mean is $\frac{3x+2y}{5}$.

Indeed, contrary to what one may be led to believe by an unfortunate expression in certain elementary manuals, there is no such thing as a nonweighted mean, only *equally weighted* means, for every averaging operation implies that a weight is given to each of the combined objects. The previous way of writing is convenient because of its conciseness, but it is fallacious.

If we write ($a;p$) to indicate the object a weighted by p (which is a number), the composition law is

$$(a;p) + (b;q) = \left(\frac{ap+bq}{p+q};p+q\right),$$

which indicates, in addition to the arithmetical rule for computing the mean,

$$\frac{ap+bq}{p+q} \ ,$$

55 However. the variety of situations obtained by repeating any binary operation is almost always too large; this leads us to reduce that variety by some restriction. This is the meaning of the associative axiom, which states:

$f_2(a, f_2(b, c)) = f_2(f_2(a, b), c) = f_2(a, b, c)$

Modern algebra which is not too easily freed from numerical models has surely been a little negligent about studying nonassociative operations.

56 The idea of the mean has been extensively studied from a topological point of view by M.Fréchet; see, first. "Les éléments aléatoires de nature quelconque dans un espace distancié" (*Annales de l'Institut Henri Poincaré*. Vol. X. fasc. 4, pp. 215-308. Paris, 1948); second, "Une propriété générale des valeurs typiques d'un nombre aléatoire" (*Publications de l'Institut Statistique de l'Université de Paris*, Vol. l, fasc. 1. Paris. 1952, 47 pp.); and also for an elementary and rapid view, the conference at the Palais de la Découverte mentioned footnote 46.

the weight that must be assigned to this mean, which is the sum of the weights of the components.

With this convention, the case previously mentioned takes the form

$$(x;3) = (a;1) + (b;1) + (c;1)$$

$$(\mathrm{y};2)=(d;1) + (e;1)$$

and the general mean is indeed as it must be:

$$(x;3) + (y;2) = \left(\frac{3x+2y}{5};5\right).$$

Thus the operation of averaging is really associative. It is even commutative, that is, perfectly symmetrical, as is addition. This is not the case for another very common procedure, namely the *median*. It is known that to define a median[57], it is not necessary to know how to add the objects treated by statistics; it is sufficient to know how to rank them in a "scale," or linear ordering, in order to be able to make meaningful the statement that one object is located *between* two other objects. The operation of determining a median can therefore be used in qualitative areas[58]. If we have a particular set of objects which can be ordered, we shall call the cut which divides the set into two groups with an equal number of objects (or more exactly, of equal weight) the median[59].

It is clear that (in contrast to the computation of a mean) the determination of a median cannot be made progressively. If the median of a group of seven numerical measures (assumed to be equally weighted) is $x = 15$ and if the median of another group of three is $y = 10$, it is not possible to say very much, as long as we do not have other information, about the median of the group formed by combining the ten measures, except that it is somewhere between 10 and 15. When a group increases through the addition of new members, we have to start computing all over again; the operation is not associative.

A third classical procedure leading to the choice of a typical object is the *mode*; in a collection of objects to which weights have been assigned (in statistics, these weights represent the number of observed repetitions), a mode (also called a dominant or most frequent value) is an object which has the greatest weight[60]. The mode corresponds exactly to the rule of the majority; in this case objects do not have to be added or even ordered.

These three methods are all laws of composition which, applied to objects of a specific type, lead to a result of the same type. One could figure out others; in fact, the mean has been generalized (quadratic, geometric, harmonic, etc.). We note that Quételet used means. For philologists the question is naturally more subtle, yet it is possible to relate the critical (external) rules to a process of determining the median; an important reservation should be

57 This word was introduced with that meaning by Cournot: *Exposition de la théorie des chances*, pp. 63 and 120.

58 Strictly speaking. means and medians do not have the same range of application. To consider a mean the set of objects studied must have a structure of the kind called vectorial (possessing an internal addition and multiplication by a field of operators, here the weights, which are ordinary numbers). To consider a median. the set must have an ordered structure. The set of real numbers possesses both these structures, so that it is possible to speak of mean and median and compare these two composition laws. but this is an exceptional case.

59 If the number of objects in a collection is odd, the cut consists of one of the objects of the collection itself; if the number is even, it is an interval which might contain objects of the set, but which may also be empty.

60 Obviously, there will be more than one mode if two weights arc equal and greater than all the other weights. This causes certain problems (analogous to a standoff in voting).

that testimonies are not ranked in linear order but as far as possible are arranged in a genealogical tree. All efforts are made to find the source or archetype, and it is no longer a matter of choosing an *intermediate* situation like the median but still of choosing some situation defined in a more complex structure.

The analytical study of the Condorcet paradox will lead us to see how *medians* can be defined in partially ordered structures of various types. From now on, let us note that the majority rule can be used with the mode just as well as with the median. If a variable can only be in two states "yes or no", " + or –", the mode is obviously defined as the most frequent reply. To speak of a median we must order, but then when there are only two objects there is really no major difference between ordering and simply distinguishing. We can thus imagine the two states as two points of an arbitrary space: if we want to apply the median rule, it is clear that we will place the median in the point that received the greatest number of suffrages.

Every operation is defined *within* a certain set: the result of the operation is of the same nature as each of the terms; one does not go outside the set of all objects of the nature considered. Thus the sum of several numbers is a number; the mean of a set of (weighted) numbers is, in the same way, a (weighted) number. But if we restrict in any way the range of variability of component terms, it is not always true that the operation gives a result located in the new restricted range. The sum of several integers is still an integer, but their mean may not be. We will say that the set of integers is *closed*[61] for addition, but not for the mean.

The problem stated by Cournot with regard to the theory of Quételet is the following: given the definition of the mean of a certain set of objects (triangles or any geometrical objects, or even, as Quételet wanted, numerical descriptions of humans), is this set closed under the operation or will the operation make external, impossible, and absurd objects appear? The Condorcet paradox also consists of observing that a certain operation applied to certain objects (individual opinions in the form of preference judgments) gives birth to objects which do not have the same form (a set of inconsistent judgments). The paradox in both cases arises because the composition operator has been mathematically defined on a set larger than the set to which it can be effectively applied. The mean of Quételet is applicable to every system of numbers, even if they do not describe any real object; the counting of the votes recommended by Condorcet could be done even if the ballots expressed inconsistent judgments. In one case as in the other, the typical, representative object was not directly defined to be a mean or a mode. In both cases the beginning was the analysis of each of the objects of the collection. According to Quételet, a descriptive card giving a series of measures is set up for each individual of the studied population; anthropometry is the first step, which will reduce our knowledge to a system of numbers, and for each of these "dimensions" or "coordinates" a mean will be set up. From Condorcet, an analogous principle directs us to reduce opinion to a series of preference judgments, answering yes or no to a battery of questions, and it is for each of these "dimensions" that the "will of the majority" is established.

The general frame in which our paradoxes and problems of "aggregation" will be contained takes form by itself. The objects for which we are trying to define composition are complex objects formed by pooling several components. Thus (x,y.z ...) is such an object which has for components x,y,z, etc. Given a population

$$(x_i,y_i,z_i,\ldots)$$

61 Stable has the same meaning.

(i = 1,2,...,n), we try to define a mean, median, mode, or any other operation on the complex objects by doing separately such operations on each of the populations of components. If the different components have the same nature (e.g., real numbers), it may seem natural to perform the same operation on each component:

$$\hat{x} = M(x_1, x_2, x_3 \ldots)$$
$$\hat{y} = M(y_1, y_2, y_3 \ldots),$$

but it is possible that in some cases we will find it advantageous to do one operation on the x's and another on the y's.

In the case in which all the components are numerical -the ideal case for Quételet's anthropometry- the study of such operations on complexes of numbers is a natural extension of simple algebra. A complex object (x,y,z) can be symbolized by a point. in an adequate space, and if the M operation is the arithmetical mean, the point $(\hat{x},\hat{y},\hat{z})$ is the center of gravity of the population of given points[62]. Thus the problem stated by the paradoxes is this: given a part of the space containing points representing real (or possible) objects, does the center of gravity of those points always lie in that subspace? In other words, is the possibility space closed for the operation of the mean[63]?

If we return to Cournot's example and we take as objects right triangles and for coordinates the lengths of the three sides, we always have

$$x^2 + y^2 = z^2.$$

The possibility space is the surface of a cone. It is not closed; the center of gravity is inside the cone. The conditions for closure of the operation of the median or any other technique would be studied in the same way, but we will not linger on it. The study of qualitative, not quantitative, situations will make us better able to grasp the nature of the problems.

The Logical Problem

The problems of abstract algebra arise as soon as we have to compose or aggregate *several* objects into one of the *same* kind; the two fundamental characteristics of the algebraic game were just outlined. On the one hand, the operation of composition is not a juxtaposition but a real reduction of a multitude of component objects to one, the result having the same nature as each of the terms. On the other hand, the number of component terms should be kept indeterminate; it must be possible to compound any number of terms. These two aspects are obviously present in the voting problem, where the result looks like an individual opinion once it has been announced and the procedure has to be adaptable to any variation in the number of votes cast. That is not all. As was clearly seen by Condorcet, the very nature of the terms and of the result is not strictly defined in advance; it even seems impossible, except in some exceptional cases, to predict the set of forms that the opinions expressed by the voters can take. In contrast, it is certainly possible to consider each individual *opinion* as a system of judgments, that is, of affirmations or negations, of acceptances or refusals, of a certain number of simple propositions. All that each member of the collectivity may say at the

62 Quélelet referred to his average man as a "center of gravity" but for him the word gravity had an active meaning (attraction).

63 A closed part in this sense is also called convex.

moment of the ballot on a given subject can be represented by the answers yes and no to a series of questions.

Thus the general scheme will be:

Questions	Individual responses				
	N° 1	N°2	N° 3	…	…
a………..	+	–	+	…	…
b………..	+	+	–	…	…
c………..	–	–	+	…	…
d………..	–	+	+	…	…
…………..	…	…	…	…	…

But, in the same way that the number of individuals must be left arbitrary and rules valid for any number have to be figured out, the number of elementary questions must not be fixed in advance. Thus the algebraic or combinatorial problem comes up in two ways. Condorcet solves this problem by proposing, first, to define a composition rule for every series of simple judgments, a rule which could be applied to every row of the preceding table; and second, to apply this rule to all the rows and to compute in this manner a new column which will be taken as an expression of the collective opinion. It is easy to see, that in this case, the number of elements in each column could be arbitrary and that the method would remain applicable. Without discussing here the practical applications, let us admit that Condorcet was right to believe that some institutions fit the model. An assembly, for instance, or a committee, often has to give its opinion successively on several questions[64]; the collective answer for each question is determined *separately* by a procedure of discussions, votes, or the like, and we often recognize after some time that the different decisions are logically interconnected and should present a consistent appearance. Experience, in fact, proves that this consistency is not always very strong. But can we charge an assembly with "contradicting itself"? The idea of contradiction applies to the logic of the individual: what logic (is it the same?) results from the mechanisms of voting?

Let us first consider what the application of the rule of the majority leads to. Once each individual opinion has been analyzed into simple elements by means of a series of questions (columns of the preceding table), a statistic is established separately for each line and we keep the dominant answer. However, the example already studied (p. 10ff), in which elementary questions were related to preference judgments, warns us that for each individual opinion there exist certain connections among the different answers. This interdependence, incidently, is not functional and does not resemble the interdependence used by Cournot in his criticism of Quételet: in a right triangle the three sides are interdependent, meaning that each side is a function of the other two; knowing the lengths of two sides allows us to compute the length of the third side. Here it is different. Knowing the answers to the two questions

" $A > B$?"

" $B > C$?"

does not *always* allow us to predict the answer to the third question,

" $C > A$?"

64 The procedures of disjunction, of voting "by articles", etc., are of the same family.

All we can say is that it is impossible to have the same answer (either yes or no) for all three questions. The interdependence of the component elements, or the logical consistency of an individual opinion, is expressed by excluding some arrangements of signs (see Table 1, p.14).

The general logic of propositions, a development of part of the older theory of syllogisms, teaches us that the problem is universal. Given several propositions or questions, every logical relation between them can be expressed by establishing the list of possible arrangements of signs and the list of impossible arrangements[65]. For instance, the implication "*A* implies *B* " or, more familiarly, "no *A* without *B*" is nothing but the impossibility of the one arrangement

$$(A = +, B = -).$$

In the same way, the conjunction "*A* implies *B* and *B* implies *C*," which forms the framework of the traditional syllogism, is equivalent to the impossibility of the four arrangements:

A	+	+	–	+
B	+	–	+	–
C	–	+	–	–

and the possibility of four others:

A	+	–	–	–
B	+	+	–	–
C	+	+	+	–

As a consequence, each time a logical connection among several questions appears, we may be sure that certain arrangements are missing. However, as Condorcet's example shows, the rule of the majority may very well lead to a forbidden arrangement. It is easy to see that this phenomenon, that I propose to call, also in this much more general case, the *Condorcet effect*, does not occur for two propositions, but it can appear as soon as we consider three propositions that are not completely independent[66]. In order to analyze this easily, a geometrical model can be used. The different answers to the three questions can be schematized by the eight vertices of a cube, the first question being represented by the left-right alternative; the second, upper-lower; the third, front-back. Condorcet's method of deriving a collective opinion consists of considering opposite faces and choosing the face having the majority. The *effect* consists of the following: it is possible to distribute a population among the vertices of a cube, leaving empty some corners so that the three majority faces meet at an empty vertex. In spite of the very large number of possible cases, it is always possible to reduce them to combinations of situations analogous to the following:

Individual opinions			Resulting opinion
+	+	–	+
+	–	+	+
–	+	+	+

This radical configuration is obtained easily if we start at the empty corner and find the three points joined to this empty point by an edge of the cube; if we put one individual at each of these three points, there will be two individuals (and thus a majority) on each face. It is

65 Sec any treatise on logic: for instance, J. Piaget, *Traité de logique* (Paris, 1949). p. 225ff. or J. Dopp, *Leçons de logique formelle* (Louvain. 1950). 2d part. "Logique moderne" Vol. l, pp. 37ff.

66 With the meaning of formal logic: three propositions are independent if the eight arrangements are *a priori* possible.

precisely this configuration of opinions which is always pointed out as an example of the voting paradox. This effect can occur for any number of propositions, provided there are more than two. It can also occur for propositions having more than two values. For instance, it could be assumed that besides the answers yes and no, indifference[67] is a possible response.

In any case, it is possible to state the following result. If the individual opinions are analyzed into elementary attitudes or answers to a series of questions, and if we determine the dominant attitude separately for each question, the system of dominant attitudes may *not be represented* in the population and may even be judged *contradictory* by universal agreement.

Since in many cases these contingencies (mainly the second one) will be considered serious impediments, the occurrence of the Condorcet effect should be prevented. A first line of research is to note that the Condorcet effect can be produced only by a certain distribution of individual opinions, but experience proves that the distribution (which could be called opinion, in the singular) is not of any particular type. Certain individual interdependencies exist which in fact make up one of the subjects for study in the sociology of opinions. If all the opinions are alike (unanimity), there is no danger. We are led to consider that the distributions which produce the Condorcet effect are those which are the farthest from unanimity, and that the effect itself should be considered a pathological symptom of unusually deep social division. It would be vain to try to remedy this evil through a complicated electoral system. It is an evil which should be prevented before it is manifested in voting. The way Condorcet considered the problem[68] is not very different from what we have just summarily described, which brings to mind Rousseau's theory of the general interest.

But we should realize that although, from a political and social point of view, such views appear reasonable, they leave untouched the fundamental logical problem. We have seen that if the law of the majority is applied to a set of opinions, each opinion consisting of a preference judgment, it can lead to inconsistencies. But let us see if in the population considered there is, besides the universal logical rules[69] , a certain community of view which, while not being unanimous, ensures that we no longer risk the Condorcet effect. We know the role played by logical connections among the components of an individual opinion; the question now is to consider the role that could be played by the connections among the individual opinions and in particular to see whether these new connections could not, al least in certain circumstances, reduce the disturbing effect of the first ones. Let us start by considering an example.

Suppose that an assembly debates, not different propositions of some kind, but the choice of a number — for instance. the amount of credit, expenditure, tax, sale price, or the like. It is very likely that individual opinions will have something in common. If there are three choices, a low, medium, and high figure, say

$$A = 1{,}000,$$

$$B = 1{,}500,$$

$$C = 2{,}000$$

67 However, we must carefully distinguish between equivalence and ignorance. A blank ballot can mean either "no opinion" or "the two proposed solutions have the same value" (cf. p. 47).

68 Of course this way of thinking was mocked. The same people who reproached Condorcet for using mathematics in political and human affairs now reproach him for not using enough mathematics in his conclusions.

69 Lack of a cycle ($A > B$, $B > C$ ... $Z > A$) of any length.

the voters will be divided naturally into three classes:

(a) Those who prefer the low figure
(b) " " " " medium figure
(c) " " " " high figure

But although it may be all right to subdivide class (b) according to whether A is judged better than C, or vice versa, it is doubtful whether the persons who prefer A would judge C better than B. Thus we can be led (at least under certain circumstances) to exclude a priori some orderings of preference: that is, admit that *subjective* individual opinions are related to the natural and *objective* ordering of the proposed figures.

The same type of considerations will occur each time the options proposed for deliberation have an *objective* order. If there was a fourth option,

$$D = 3000,$$

these orderings would be possible:

$A > B > C > D$	$D > C > B > A$
$B > A > C > D$	$C > B > A > D$
$B > C > A > D$	$C > B > D > A$
$B > C > D > A$	$C > D > B > A$

but others would be excluded, such as

$$B > A > D > C,$$

by the following argument: B is preferred to everything else, and C is closer to B than to D, so C should be preferred to D. It is seen that subjective preference orderings do not directly follow the objective order, but rather certain rules of proximity which derive from this objective order.

Let us treat the general case by designating choices by the letters A. B, C, ... , X,Y,Z in such a way that the objective order is the alphabetic one. Every opinion will be represented by a complex of simple judgments:

$$A < D, B < F, ...$$

but, we admit that these simple judgments are not completely independent. Not only must they follow the general logic of preference; they must make allowance for the logical ordering. Thus, when an opinion contains the judgment:

$$D < K,$$

it indicates that this opinion locates its optimum either between D and K or at K or beyond K. The result is that all options preceding (objectively) D must be judged inferior (subjectively) to all the options preceding K:

$$\begin{array}{llll} & C < K & B < K & A < K \\ D < J & C < J & B < J & A < J \\ D < I & C < I & B < I & A < I \\ \text{etc.} & & & \end{array}$$

In the same way, as soon as one allows a judgment which inverts the objective order,

$$H > M,$$

it indicates that the optimum precedes M and as a consequence we must have:

$$\begin{array}{llll}
H > N & H > O & H > P & \\
I > M & I > N & I > O & I > P \\
J > M & J > N & J > O & J > P
\end{array}$$

etc.

These observations focus attention on a sort of *hierarchy* of judgments — one judgment dominates several others. This subordination is easy to designate in the form of an ordered network[70]. A triangular table such as the following is constructed:

$$\begin{array}{ccccccccc}
(A<B) & \leftarrow & (A<C) & \leftarrow & (A<D) & \leftarrow & (A<E) & \leftarrow & (A<F)\ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & (B<C) & \leftarrow & (B<D) & \leftarrow & (B<E) & \leftarrow & (B<F)\ldots \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & (C<D) & \leftarrow & (C<E) & \leftarrow & (C<F)\ldots \\
 & & & & & & \uparrow & & \uparrow \\
 & & & & & & (D<E) & \leftarrow & (D<F)\ldots \\
 & & & & & & & & \uparrow \\
 & & & & & & & & (E<F)\ldots
\end{array}$$

Note that the affirmation of any one of these judgments implies the affirmation of all the "consequences": that is, the affirmation of those located either in the same row and to the left, or in the same column and above, and thus of all the judgments located to the left and above.

Analogously, the negation of any judgment in the preceding table implies the negation of all the "antecedents": that is, all those to the right and below.

If affirmation is indicated by a plus sign and negation by a minus sign, the result is that the complete opinion of any individual can be represented by a triangular table like the following:

	B	*C*	*D*	*E*	*F*	*G*	*H*	*I*
A	+	+	+	+	+	+	+	–
B		+	+	+	+	+	–	–
C			+	+	+	+	–	–
D				+	–	–	–	–
E					–	–	–	–
F						–	–	–
G							–	–
H								–

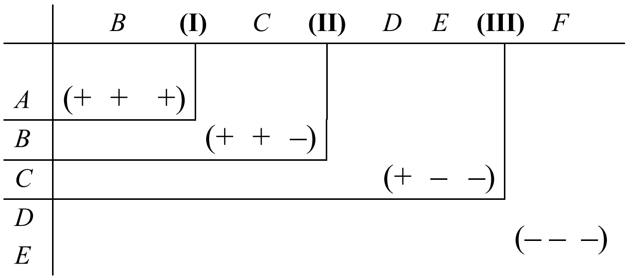
There are two distinguishable regions, one of plus signs and one of minus signs, which can be separated by a border as has been done in the above table. This border must respect the mentionned implications: all the arrows of the network that cross the border pass from the plus region to the minus region.*

70 This is a partially ordered structure called a lattice. The elements of the theory will be found in V. Glivenko. *Théorie générale des structures* (Paris. Hermann. (938): "Actualités scientifiques", n° 652. 51 pp., and in H. H. Curry. *Leçons de logique algébrique* (Paris and Louvain. 1952, 163 pp.).

* *[Editors' note: A plus sign in the above table thus indicates that the column option is preferred to the row option by the individual whose opinion the table represents, whereas a minus sign signifies the opposite preference. The supposed objective order of the options and the assumptions about the behavior of the individual respondents ensure that entries above or in the left of a plus sign will be plus, whereas entries below or to the right of a minus sign will be minus. Then the above table represents the preference I <A < H < B < C < G < F < D < E].*

Now let us see how several opinions of this type can give birth to a collective opinion when the majority rule is applied to each of the component judgments.

Superimposing the different schemata and counting the votes is sufficient: let us take for example the simplest case, that of three voters. The three borders might not cross each other as below:

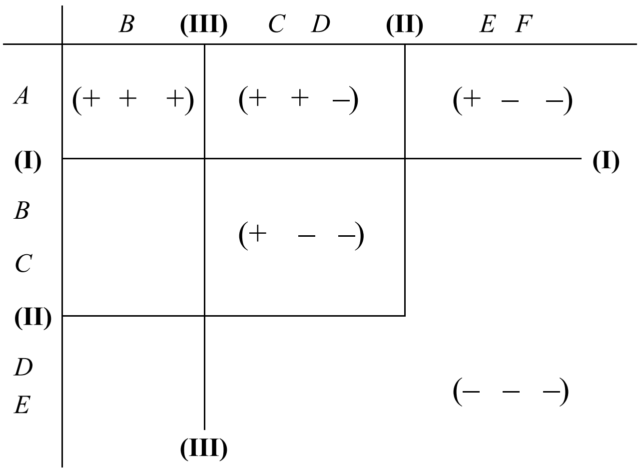


thus determining four regions. In the first there is unanimity (three plus signs), in the second a majority (+), in the third a majority (–), and in the fourth unanimity (–)*.

If the law of the majority is imposed, we will institute the plus sign in the first two regions and the minus sign in the last two.

Finally, the majority law leads us to adopt the *intermediary* opinion represented by the border (II), namely, $F < E < D < A < B < C$.

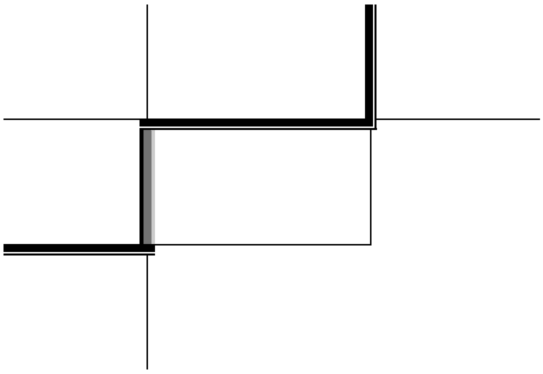
In the cases that we must also examine where the borders cross, like the following**:



* *[Editors' note : the three signs in each cell of this array indicate the preference of each of the three voters between two options X and Y (with X before Y in the alphabetical order). For instance (+ + –) in row B and column C indicates that voters 1 and 2 prefer C to B, whereas voter 3 prefer B to C].*

** *[Editor' note: Then the preferences of voters 1, 2 and 3 are respectively $A < F < E < B < C < D$, $F < E < A < D < C < B$ and $F < E < D < C < A < B$.]*

the law of the majority leads us to adopt for the resulting opinion a border that includes fragments of each of the initial borders, namely, $F < E < A < D < C < B$ — but it is still convenient to say that it is *intermediary*:



In every conceivable case, this new border respects the desired conditions for representing an opinion: all the arrows in the network that cross it do so in the same direction.

Thus we obtain two important results: first, majority law leads to the adoption of a set of judgments which possess the same consistency as the components of each individual opinion; and, second, this same law permits the definition of a "median" or "intermediate" opinion as the collective opinion. It would not be difficult to show that the situation is the same if we compound five, seven, or any odd number of individual opinions. For an even number, exact balances of the number of votes "for" (+) and "against" (-) may occur. But whatever decision is taken with regard to these litigious cases[71], the result is always the same: the result still deserves the name *opinion* because of the consistency of its component elements and the epithet *intermediate* because of its location. The first conclusion to draw from this examination[72] is the lack of any Condorcet effect. If the individual opinions respect, with the indicated meaning, the same *objective* order, the law of composition by majority leads to an opinion which has the same form. The set of opinions taken into account is *closed* for the composition law adopted; there is no risk of having to leave this set.

The second conclusion is just as important. The problem is to understand the reasons for the vanishing of the Condorcet effect. We have seen that the name "median" or intermediate opinion presented itself very naturally during the analysis. It is, indeed, because the set of opinions that we have studied possesses a certain order, at least partial, which is a direct consequence of the partial order present in the set of component judgments, and is symbolized by the triangular network (lattice) drawn earlier (p.28).

Let us briefly describe the structure which prevails in the set of acceptable opinions. First, there are two extreme opinions[73]; the alphabetic order *ABCDE* and the inverse, *EDCBA*. Then, given any opinion — for example, *CBDAE* — it is possible to construct an opinion

71 For example, a preponderant voice to the president, or any other analogous rule.

72 This result was recognized by Duncan Black, "The Decisions of a Committee", *Econometrica*, 16, 1948, 245-61. For a deeper analysis, the book of the same author can be consulted: D. Black and R. A. Newing, *Committee Decisions* (London, Hodge. 1952), p. 59, See also K. J. Arrow, *Social Choice and Individual Values,* pp. 6 and 75-80.

73 Let us limit ourselves to seeing the phenomenon in the example of choosing between five options: *A*, *B*, *C*, *D* and *E*. Nevertheless, the conclusions are general.

which is *intermediate* between this one and one of the two extremes: for instance, *ABCDE* affirms *all* the elementary judgments, and *CBDAE* affirms only

$$AE, BD, BE, CD, CE, DE$$

and negates all others.

We will say that the opinion *BCADE* is "intermediate", since it affirms *more* than *CBDAE* and *less* than *ABCDE*. These different kinds of relations can be symbolized by an oriented network on which we will verify that *BCDAE* is really on one of the paths which lead from *CBDAE* to *ABCDE*:

$$\begin{tikzcd}
 & & & & BACDE \arrow[r] & ABCDE \\
 & & & CBADE \arrow[r] & BCADE \arrow[u] & \\
 & DCBAE \arrow[r] & CDBAE \arrow[r] & CBDAE \arrow[r] \arrow[u] & BCDAE \arrow[u] & \\
 & DCBEA \arrow[r] \arrow[u] & CDBEA \arrow[r] \arrow[u] & CBDEA \arrow[r] \arrow[u] & BCDEA \arrow[u] & \\
 & DCEBA \arrow[r] \arrow[u] & CDEBA \arrow[u] & & & \\
EDCBA \arrow[r] & DECBA \arrow[u] & & & &
\end{tikzcd}$$

Using this schema, it is possible to interpret the construction of the collective opinions based on some given individual opinions. If the various individual opinions are located on the same chain, the ordinary definition of the *median* is satisfactory: otherwise it is a generalized median.

We know now that there are two cases where the Condorcet effect does not occur and where the application of the majority law will consequently never lead to absurdity.

The first case is when no combination of affirmations and negations of the elementary propositions is *a priori* impossible. If all combinations of plus and minus have meaning, it is clear that the resulting opinion will be acceptable. The example we just studied belongs to the second case, where the 'individual opinions are restricted to a preferential set which has been defined here by referring to an "objective" value; in order to explain that this set is closed, it is sufficient, as we have seen, to note that the set is *ordered* and that the majority law is equivalent to the choice of an *intermediate* opinion. The two cases are actually very similar, in spite of appearances. The first can be put in the form of the second, and vice versa.

Let us begin by presenting the second case in a form analogous to the first. Again take the example of the five options A,B,C,D,E. Here an opinion consists of an order of preference established among the options, but not just any order: we have supposed that the subjective preferences are subordinate to an objective order and that there are only sixteen possible orders (the orders of the above network) [74]. To know an opinion it is not necessary to ask all the questions " $A > B$?", " $A > C$?" ,…, "$D > E$?". There is a more economical way to conduct the inquiry. Begin by asking if the option put in the last place is A or E. The answer to this question is seen to divide the table of opinions in half. Proceed by asking if the option put in last place and the one before it are naturally contiguous (as for *CBADE*) or not (as for *CBDAE*). Then ask if options put in two successive places in the subjective opinion are

74 Generally for n options, there will be 2^{n-1} acceptable orders.

contiguous in the objective order. There are four questions whose answers are absolutely independent; therefore the opinions are represented by all possible combinations of yes and no. This is the same form as the first case.

Inversely, the first case (answers completely independent) can be presented in the form of the second (ordered system of opinions). The set of all obtained opinions can be ordered by arranging the plus and minus signs in all possible ways. Begin by choosing two extreme opinions that we will call respectively, the "first" and the "last"; it is sufficient that they be contradictory – for instance, if the first opinion is the one that answers no to all the questions, the last will be the one that answers yes to all. To go from the first to the last, we can construct different chains or sequences of opinions so that each opinion is inferred from the preceding one by a "concession," replacing a no with a yes:

$$
\begin{array}{ll}
\text{First} \ldots\ldots\ldots\ldots\ldots & \begin{array}{ccccc} - & - & - & - & - \end{array} \\
\text{Intermediate} \ldots\ldots & \left\{\begin{array}{ccccc} - & + & - & - & - \\ - & + & - & + & - \\ - & + & + & + & - \\ + & + & + & + & - \end{array}\right. \\
\text{Last} \ldots\ldots\ldots\ldots\ldots & \begin{array}{ccccc} + & + & + & + & + \end{array}
\end{array}
$$

For two elements of such a chain it is possible la talk of antecedents and consequences; the result is again an oriented network.

So we perceive what we will call the *reasons* for the Condorcet phenomena. The system of acceptable opinions must show an internal consistency linked to the two images of order and completeness. 1t is not necessary to go further into the algebraic technique[75], but it is important to underline the significance of the fundamental logic.

Condorcet clearly saw that the majority rule seems to be an improvement on the statistical determination of a typical element which by itself, has the greatest weight[76]. If the majority opinion were chosen, a large part of the information given by the vote (or statistical survey) might be neglected. In fact, very often the "palette" of opinions is not totally devoid of structure: one feels that one opinion is more or less "close" to another. Why do we not recognize the elements of the vote which, being common to the minority opinions, may have more real importance than the majority opinion considered alone? Thus the whole problem consists of beginning to be conscious of what we vaguely called the structure of the palette of opinions. It is interesting to see that the analysis of an opinion as responses of yes and no to a series of questions, followed by the adoption of the opinion built up from the majority responses, leads us to define this structure as *partially ordered* and to recognize that the collective opinion finally adopted is a generalization of the *median* of a *completely* ordered linear set.

If the elementary questions were independent, every combination of responses would be possible: the set of all these combinations is ordered, but if a part of this set is not allowed

75 It would be necessary to clarify the relations between the "lattice" structures and the "ring" structures.

76 The other meaning of mode, the maximum density of probability, is relevant to a completely different logic. In order to speak of density, the distribution has to have a certain internal consistency.

(because of some criterion of internal consistency) then the rest of the set has to possess the same type of structure — all the cases which can occur can be predicted by looking for those parts of a structure which have a structure analogous to that of the whole.

A certain internal harmony of the *possible* opinions is the general form of the conditions imposed in order for a series of decisions taken by the majority of votes to form a unique consistent opinion. It should be recognized that voting and majority rule are often used to resolve conflicts which are too profound to depend on this procedure. By examining the vote closely enough, we would be led to some useful reflections on majorities which are "more or less strong." It is not the strength of numbers which gives more or less weight to a voted decision, but whether the vote has revealed little diversity of individual opinion: then we are undoubtedly closer to the ideal conditions which allow the application of the majority law. Voting operates at first as a test of the degree of unity in the electoral body.

By the way, it can be seen that the usual language does not adequately express the questions we raise here; it is tempting to say — and it is said sometimes — that a series of decisions made by the majority of voters cannot claim to represent a true collective opinion if the individual opinions are too deeply divergent: for aggregation to be allowed, the variety of opinions must not be too large. But this is not the point: if the variety is the largest possible, if all the combinations of yes and no are acceptable, then the majority law performs its task and correctly determines a median opinion[77]. But if a gap occurs, if an opinion is forbidden, then all the logic collapses; and if we want to restore this coherence, it is necessary to restrict the variety even more until it takes on a stable form. Thus it is not a question of the magnitude of the ideal population of possible opinions. To know whether it is stable and whether it has a *meaning*, we should not wonder whether it is too numerous or not numerous enough – we should find and mark the forms and wonder if they have the property of stability (or if they are closed). These forms are set off from each other by profound gaps – it is not possible to go from one to the other through a series of stable positions, but we must "jump" from one to the other. We are in the realm of discontinuity[78], of *quantification*, to use the modern jargon. In this way the particular nature of the problem is revealed. It is nowadays called "algebraic", a term which does not appeal much to the imagination, but is dedicated to, and designates the study of operational and combinatorial structures.

What is a Majority?

The preceding analysis indicates only what is required for one to be able to apply the rule of the majority without danger of contradiction. When opinions are arranged in order of preference, we know that there is a danger if the preferences are free, but that this danger vanishes if the preferences refer to a universal objective order. But we can ask for more. In the case of preferences, for instance, could we not, by *abandoning the majority rule*, find a law of composition of individual opinions which always gives an acceptable result? In other words, if there is a danger of contradiction, why blame the heterogeneity of individual opinions rather than the rule of the majority?

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77 It is *meaningful*.

78 The name that would be the mast suitable here would be "harmony" — in all its connotations and echoes. This text — "une distribution de couleurs…système…problème complet…groupe fermé de relations qui a sa logique, ses opérations propres…c'est un tout qui peut subsister de soi…" (a distribution of colors…system … complete problem…closed group of relations which has its own logic. its own operations if is a whole which can exist by itself ...) — and many others by Valéry evoke this science, so sadly called algebra. (P. Valéry, *Lettres à quelques-uns*, Paris, N.R.F., 1952. p. 142.) The wonderful papers of Hermann Weyl, *Symmetry* (Princeton University Press. 1952, 168 pp.) should be read also.

We see where the problem has taken us and why it has been raised: we have seen that it was possible to consider the contradictions revealed by the Condorcet effect symptoms of the social division which made it impossible to form a stable collective opinion. But one could say that it is the instrument used to detect collective opinion which is too crude. Replace the rule of the majority with a more minute examination of the state of opinion and perhaps a noncontradictory resultant opinion would appear. *Non numerentur sed ponderentur* has been repeated everywhere about the quest for testimonies — it being well understood that the pondering required is not the brutal pondering of the modern statistician, but "weighing" ("pesée") which is also "thinking" ("pensée"), a qualitative and personal appreciation.

Now we should see whether other laws of composition of individual opinions can be constructed in such a way as to escape the threat of the Condorcet effect.

Although some research in formal logic has approached this fundamental problem, thus unconsciously providing useful material, the first results seem to have been reached by the economists' studies on the general interest. The great contribution of Arrow lay in discerning a logical problem in the literature of welfare and bringing it out as clearly as possible[79].

Let us consider a set of individuals whom we will continue to call voters, because we try to learn their wishes by asking each of them a series of questions. The answers of all the voters to all the questions can be arranged in a rectangular table, where a row is assigned to each question and a column to each individual.

For each row or question, we try to construct a global response. The set of global responses will constitute the resultant opinion. We must now study the different possible ways of aggregating. The point which especially interests us is this: assuming that all the individual opinions obey some logical constraints – that is, some combinations of answers never appear in any column – is it possible for these forbidden (or "absurd") combinations to appear as the resultant opinion, as happens, according to Condorcet, for the majority rule? Are there not, on the contrary, rules better adapted to avoid absurdity?

A very important theorem has been proved by Arrow which establishes the impossibility of a rule of composition of opinions when they are preference judgments and the rule is chosen in a well-defined way. We will find this result again further on; however, the statement and proof of the theorem are presented by their author in a negative form. It is quite laborious to talk in an indeterminate way of a rule that will eventually be proved to be nonexistent; the reduction to absurdity which is sometimes necessary is not the most enlightening method of mathematical logic.

In what follows, a "constructive" style has been sought systematically — perhaps at the cost of ponderousness in the progress of ideas, but with a guarantee of comprehension.

Let us search for a method of *building* an acceptable rule of composition. Such a rule must allow us to arrive at a "global" opinion in all cases where the individual opinions are known. This rule must also guarantee that absurdities never arise in the global opinion.

For each question, if the individual answers are known, it must be possible to "compute" the global opinion. If for question A the response of individual i has been a, the global opinion $\hat{a}$, will be determined by the different a_i

$$\hat{a} = R_A(a_1, a_2, \ldots, a_n)$$

79 Once the logical problem is solved, if that is possible, it that is possible, it is clear that this would only *prepare* the ground for the economic analysis.

Consider first the case (to which all others can be reduced (cf. below page 47) where the responses can only be yes or no. Then the rule R_A consists of saying which $\hat{a}$ results from each combination of the signs for a_i. For two individuals, a rule R_A can easily be written in tabular form.

	$a_1 = +$	$a_1 = -$
$a_2 = +$		
$a_2 = -$		

Here we will write in each of the four cells the value $\hat{a}$ chosen to represent the set (a_1,a_2). If we do not want to leave out any possibilities, we should consider all the possible choices, all imaginable rules R_A. There are 2×2×2×2 = 16 ways to fill four cells with plus or minus signs, and thus there are 16 possible rules for two individuals. Their enumeration is easy. To shorten the writing, it is convenient to condense the preceding table. We will write in sequence the four values of $\hat{a}$ taken in a conventional order, say

1	3
2	4

Thus the symbol

$$(+ - + + -)$$

means:

$$R\,(+\,+) = +$$
$$R\,(+\,-) = -$$
$$R\,(-\,+) = +$$
$$R\,(-\,-) = -$$

The complete list of all imaginable rules is:

1.	(+ + + +)	7.	(+ – – +)	13.	(– – + +)
2.	(+ + + –)	8.	(+ – – –)	14.	(– – + –)
3.	(+ + – +)	9.	(– + + +)	15.	(– – – +)
4.	(+ + – –)	10.	(– + + –)	16.	(– – – –)
5.	(+ – + +)	11.	(– + – +)		
6.	(+ – + –)	12.	(– + – –)		

We could then study the significance of these diverse ways of compounding two answers given by two persons. Rule 1 and rule 16 consist of not taking into account individual answers and imposing an answer a priori. Rule 4 consists of adapting the point of view of the first voter, rule 6 the point of view of the second; while 13 and 11 do the exact opposite, respectively, of 4 and 6. Rule 7 consists of saying yes when the two voters agree and no otherwise; 10 does the opposite; and so on. But once the family of 16 rules for two voters has been detailed, we pass to the case of three voters. Now every rule must specify a function of three individual responses:

$$\hat{a} = R(a_1,a_2,a_3)$$

There are therefore eight values to examine. We consider these in an arbitrary order; for example:

a_1	+	+	+	+	–	–	–	–
a_2	+	+	–	–	+	+	–	–
a_3	+	–	+	–	+	–	+	–

Each rule will be well defined by a sequence of eight signs. For instance, according to this convention the rule of the majority would be expressed by the sequence

(+ + + – + – – –).

Similarly, the rule consisting of taking into account only the first voter if his ballot is positive and otherwise using the ballot of the second voter (an imaginary example) would be denoted by the sequence

(+ + + + + + – –).

If we want to *enumerate* all the rules which are possible *a priori*, however, we have to form all possible combinations of eight plus or minus signs: there are $2^8 = 256$ of these. Establishing the complete list would be very troublesome, and the interpretation of each rule would take too long[80]. There would be some 65,000 rules for four voters. Even if we had the patience to write the list[81], we would wonder how to use it. And do not speak of the four billion cases for five voters[82]. Let us console ourselves by realizing that five is still a very small number of voters!

Complete enumeration is certainly impractical. Anyway, we would like to know if there exists any privileged rule among this gigantic multitude which avoids contradictions. Better, we will look for all such rules.

An Example of the Systematic Resolution of the Problem of Noncontradiction

Suppose we had chosen a rule composing the responses of a large number of voters. Then we would have a list such as

x_1	x_2	x_3	…	x_i… x_n	$\hat{x}$
+	+	–		+ … –	+
–	+	+		– … –	–

Represent each row by x, a set of individual answers x_i followed by the global response $\hat{x}$ prescribed by the rule. We write

80 The list of ternary operations of two-valued logic must be constructed. A summary enumeration can be found in M. Boll, *Manuel de logique scientifique* (Paris, Dunod, 1948), pp. 92-93, 108-9, 139-61; and a more thorough description in J. Piager, *Essai sur les transformations des opérations logiques* ; *les 256 opérations ternaires de la logique bivalente des propositions* (Paris, P.U.F., 1952). Compare this point of view with that of the theory of switching functions in, for example, *Synthesis of Electronic Computing and Control Circuits* (Harvard Univ. Press, 1951), pp. 15ff., 259ff.

81 Which would fill four or five fascicules comparable to the present edition of *Economie Appliquée*.

82 Each time we increase the electoral body by one unit, the number of rules is squared: 16 × 16 = 256, 256 × 256= 65,536; etc.

$$\hat{x} = R(x) = R(x_1, x_2, \ldots x_n)$$

Let us now ask two questions *A* and *B*. We will obtain two ballots a and b and the rule[83] will give two answers $\hat{a}$ and $\hat{b}$. If the two questions are independent, that is if each voter is free to choose among the four "opinions":

$$\begin{cases} a_i = + \\ b_i = + \end{cases} \quad \begin{cases} a_i = + \\ b_i = - \end{cases} \quad \begin{cases} a_i = - \\ b_i = + \end{cases} \quad \begin{cases} a_i = - \\ b_i = - \end{cases}$$

we do not have to fear any contradiction. But what if the two questions are related? Condorcet, who was interested in juries, chose this example:

A). Is it proven that the defendant is guilty?

B). Is it proven that the defendant is not guilty?

For this example the answers

$$\begin{cases} a_i = + \\ b_i = + \end{cases}$$

must be rejected.

The problem is then the following: what precautions have to be taken in the construction of the rule *R* so that when, for instance, the response

$$\begin{cases} a_i = + \\ b_i = + \end{cases}$$

cannot occur on a single ballot, the resulting opinion

$$\begin{cases} \hat{a} = + \\ \hat{b} = + \end{cases}$$

cannot occur.

There are four cases to consider, according to which of the opinions is proscribed:

3) $a = +$ $b = +$ (incompatibility)	1) $a = +$ $b = -$ (implication of *B* by *A*)	2) $a = -$ $b = +$ (implication of *A* by *B*)	4) $a = -$ $b = -$ (disjunction)

We will start with the cases of implication which are homologous to each other and easier to translate into ordinary language.

83 We suppose that we use the same rule for both counts of the vote. See below p.46 for the more general case.

If the opinion

$$a = +$$

$$b = -$$

is forbidden, it means that when going from question A to question B, none of the voters who had answered *yes* can switch to *no*, but switching the other way is allowed. The yes party does not lose any of its members and has a chance to gain some. In other words, the "partisans of A" constitute a party which is contained[84] in the party of the "partisans of B ".

We will write[85]:

$$\text{Partisans of } A \leq \text{Partisans of } B.$$

It will be convenient to use a summary symbolism to designate the two "parties" on a ballot.

From now on we will write a^+ to designate the set of individuals who have voted + in the ballot, and a^- for the others.

We can conclude in these terms:

If proposition A implies Proposition B — that is, if it is impossible to answer no to B after having answered yes to A — then every opinion

$$\begin{cases} a_i = + \\ b_i = - \end{cases}$$

is impossible and consequently[86] we have

$$\mathrm{a}^+ \leq \mathrm{b}^+$$

"the party in favor of A is included in the party in favor of B ".

Let us return to the rule of composition of individual answers. To each system x, there corresponds a unique answer $\hat{x}$:

$$\hat{x} = R(x).$$

This is equivalent to dividing the set of all possible x (that is, the set of all possible ballots or sequences of plus and minus) into two classes, those which give $\hat{x}$ = + and those which give $\hat{x}$ = –, the positive and negative ballots. But we have just seen that when the A proposition implies the B proposition, the positive party a^+ is “contained” in b^+:

$$a^+ \leq b^+$$

On the other hand, there would be a contradiction if, when A implies B, we nevertheless obtain the result

84 According to the mathematical convention, this word includes the extreme case in which the yes party would not gain any member. Thus the symbol ≤ is used, not <.

85 Note that this is not a numerical relation between the number of members in each party: of course the party of B cannot have fewer members, but we mean by the preceding symbol every person of A is a member of the party B. The relation is essentially qualitative; it will be seen even more clearly in what follows. This is the third distinct use that we have made of the same sign (cf. Footnote 38, p.11).

86 We could just as well say "$a^- \geq b^-$".

$$\hat{a} = +$$
$$\hat{b} = -$$

The contradiction that we wish to avoid consists of the simultaneous occurrence of

$$(1)\ a^+ \leq b^+$$
$$(2)\ \hat{a} = +,\ \hat{b} = -$$

For a rule *R* to avoid this contradiction, it is necessary and sufficient that the "plus" party of a positive ballot never be contained in the "plus" party of a negative ballot.

One would treat the three other cases in the same manner. If it is *B* which implies *A*, we have necessarily

$$a^+ \geq b^+$$
$$(\text{or } a^- \leq b^-)$$

and we want to avoid the rule saying that $\hat{a}$ is negative when $\hat{b}$ is positive. "

If two yes responses[87] are forbidden (case three), we have

$$a^+ \leq b^-$$
$$(\text{or } a^- \geq b^+)$$

and we want to avoid having $\hat{a}$ and $\hat{b}$ both positive. Finally, if the relation between the propositions *A* and *B* excludes two no responses[88] we have

$$a^+ \geq b^-$$
$$(\text{or } a^- \leq b^+)$$

and we want to avoid having $\hat{a}$ and $\hat{b}$ both negative.

Finally, if we want to avoid *simultaneously* the four risks of contradiction, we have to make it impossible for the following four results to appear at the same time:

1)	$\hat{a} = +,\ \hat{b} = -$	and $a^+ \leq b^+$
2	$\hat{a} = -,\ \hat{b} = +$	and $a^- \leq b^-$
3)	$\hat{a} = +,\ \hat{b} = +$	and $a^+ \leq b^-$
4)	$\hat{a} = -,\ \hat{b} = -$	and $a^- \leq b^+$

If these four kinds of simultaneous occurrences are impossible, we are sure that the rule will avoid contradiction whenever two ballots are related in one of the four ways considered.[89]

We now have to determine the rules satisfying these exigencies. First of all such rules certainly exist. We have only to think of the trivial solution where we would decide to adopt the opinion of one of the voters and would always stick to this opinion without taking into consideration the opinions of the other voters. It is clear, indeed, that the rule will then respect ail the logical connections of the opinion of the chosen voter. But the question is to find out whether there are other rules which satisfy the requirement of contradiction and whether it is

87 Incompatibility.

88 This is the logical "disjunction".

89 We could introduce other types of relations which would forbid note one joint opinion, but two. But it can be easily established that these types do not raise new problems.

possible to know all of them[90]. Let us call a rule *acceptable* if it never leads to a contradiction, avoiding the four kinds of events mentioned above.

If a rule states that a ballot, say a, must lead to a positive decision

$$R(a) = \hat{a} = +,$$

we may say that this rule decides *in favor* of the positive party, a^+: that is, in favor of the individuals who have voted yes (+). But if the same individuals have voted no (–), or if we consider the ballot b in which b^- is identical with a^+ (the ballots a and b are contradictory), we then have simultaneously

$$b^- \geq a^+$$
$$a^+ \leq b^-;$$

but as $\hat{a} = +$, it is necessary, if we want to avoid contradiction (see case three above) that we have

$$\hat{b} = -.$$

Thus, if the rule is acceptable, and the decision is in favor of some coalition if it votes +, then this rule will also decide in favor of this coalition whenever it votes −. Such a coalition will be called *efficient*.

If e is efficient,

$$a^+ = e$$

has for a consequence

$$\hat{a} = +$$

and $b^- = e$ has the consequence $\hat{b} = -$.

Consider now a coalition which contains e:

$$f \geq e.$$

If we state that $c^+ = f$, we will have

$$c^+ \geq a^+ \qquad \hat{a} = +$$
$$c^+ \geq b^- \qquad \hat{b} = -,$$

and in order to avoid the contradictory simultaneous occurrences (1 and 4), the rule must say

$$\hat{c} = R(f) = +$$

Thus the coalition f is efficient.

As we might have expected, one of the conditions of noncontradiction is the following:

an efficient coalition does not cease being efficient if new members are added.

90 We could state this problem in terms of a network in the following way: consider the set of possible ballots; the rule divides this set into two parts, positive and negative. On the other hand the eventuality of a relation between the two questions sets up certain relations between the ballots (such as the mutual inclusion of the positive or negative parts). The problem is then to demand that the partition made by the rule conform to the structure revealed in the relation. The graphic image appears at once. Let us represent the ballots by small circles drawn in a plane and the relations by arrows linking the circles: the question is then to divide the set into two categories — white circles and black circles — in such a way that none of the arrows of the first category goes from white to black, none of the arrows of the second category goes from black to white, etc.

It can be easily proved that if e is efficient, the complementary coalition (which groups all the voters who are not contained in *e*) is *not* efficient. Then, if *g* is not efficient, it will not become efficient if it loses members. Finally, the coalition which contains all the voters is surely efficient.

We are led to the following conclusions: if a rule is *acceptable*, the efficient coalitions which this rule defines are such that:
– Every coalition which contains an efficient coalition is efficient.
– Every coalition contained in a nonefficient coalition is nonefficient.
– The complementary coalition of an efficient coalition is nonefficient, and inversely.

On the other hand, it is easily proved that these conditions are sufficient. Any set of individuals can be divided into two parties in a number of ways. If for each partition one states arbitrarily which of the two parties will be the efficient one, a rule for determining a resultant opinion is set up, and this rule will be *acceptable* (free from contradiction) if, first, unanimity is efficient: second, every efficient party does not cease to be efficient by adding new members: and third, every inefficient party does not cease being inefficient by losing members.

It is clear, indeed, that the four simultaneous occurrences to avoid (p. 39) all state that the efficient party must not be contained in an inefficient party.

It is easy to enumerate, for each case, the rules which satisfy the preceding conditions — that is, to find the *acceptable* rules.

In the case of two voters, since one of the two must be efficient by himself, we come to the rule: always take the opinion of one of the voters (always the same one, evidently). It is clear that we thus avoid any contradiction, and the rule is trivial.

In the case of three voters, we must distinguish between two cases; First, if one of the three voters is efficient by himself, call him (1), then the coalition (2,3) is not efficient, nor (2) nor (3) — but clearly (1,2), (1,3), and (1,2,3) will be efficient. The solution is given by the trivial rule: take the opinion of Primus.

Second, if none of the three is efficient, then every coalition of two is efficient and we arrive at the rule: take the opinion of the majority.

For four voters, if we discard the trivial solution, none of the voters can be right by himself; thus three voters are always right against the fourth. As for the coalitions of two voters, they cannot all be efficient or all inefficient, being complementary two by two. It is then necessary to decide in one way or another: for example, between coalitions (1,2) and (3,4). If the first is declared to be efficient, it is then necessary to choose between (1,3) and (2,4), and then between (1,4) and (2,3). If we enumerate all the possible choices, we observe that they are of two types. Either the three coalitions of size two which are efficient all contain a common voter — that is the well-known rule giving the deciding vote to one particular member in case of a tie — or the three inefficient coalitions have a common member. This means an analogous but opposite rule: in case of a tie, the value of one of the votes is diminished or, what would have the same result, a louder voice is given to three of the four members of the jury[91].

91 The fourth has, in some sense, a "consulting" voice.

We can continue the analysis: for five voters, the diversity is greater. The trivial solution being discarded, as well as the solutions which do not take into account the vote of one or two particular voters[92], and the well-known solution of the ordinary majority (where every three-member party is efficient), there remain three types of new rules in which some parties of two voters can be efficient (either only one of this kind or all except one, or only two). If we examine these new solutions, we perceive a close relation with the usual majority rules: it is a type of weighted majority. For instance, it is possible to choose as efficient coalitions

(1,2)	(1,3,4)	(2,3,4)
	(1,3,5)	(2,3,5)
	(1,4.5)	(2,4,5)

and it could be said that, in this system, each of the two privileged individuals (1 or 2) is "worth" two of the others, since the coalition (1,3,4) has the same "strength" as the coalition (1,2).

In this case we could get the same result by giving two votes to 1 and 2, and one vote to each of the others.

But we should beware of believing that this is always the case. With six voters some rules appear which *cannot be reduced* to a *numerical weighting*. It is still possible to speak of a weighted majority (but not of a plurality). It is a matter of basically qualitative structures, and one can say that the opposed forces are no longer measurable. Let us give a curious example of this.

Suppose that in a committee of seven voters it has been decided that five votes always outweigh the two others — but the two losing members can reverse the situation if they form a coalition with a third voter, chosen so as to form a "harmonious" trio. It is sufficient to write the list of these winning trios. *A* and *B* being any two members, they need a third, say *C*: *ABC* is an efficient coalition. *D* being one of the other four, it is necessary to be able to form a trio with *A*, *D* and another, say *ADE*; the same for *BDF* and *CDG*. Il then is clear that *AFG* is a winning coalition; likewise for *BEG* and *CEF*. Thus there are[93] seven winning trios,

ABC, *ADE*, *BDF*, *CDG*, *AFG*, *BEG*, *CEF*,

and it can be seen that the set is constructed in such a way that it is impossible to say that any member is privileged — or even that any pair is privileged. Only the trios have a real existence. It is impossible to attribute the strength of such a trio to the sum of the strengths of the individual components. Il is impossible to give numerical weights to each individual.

We have to introduce a definition of the concept of majority, beyond that usually conceived, which will include all the usual cases and also some other significant cases[94]. We will say that a decision is a majority decision (in the broad sense) when, in a collectivity, the efficient

92 But not of three, for only two voters would remain, and there is no other rule acceptable for two, as we have seen, except following the opinion of the same voter.

93 It would also be possible to present the system by beginning with three elements which do not form a trio, for instance *ABD:* it will be said that *C*, harmonizing *A* and *B*, is "intermediary" between *A* and *B*, and likewise *F* lies between *B* and *D*, and *E* between *A* and *D*: this looks like harmonious colors. It is possible to associate in a trio the three intermediates *CEF* or an intermediate and its "complement" along with the seventh element, *G* (which plays the role of the color white).

94 These schemata are found in theory of what O. Morgenstern and J. von Neumann have called Simple Games. See their work *Theory of Games and Economic Behavior* (Princeton, 1947), pp. 420-70. The authors discard *a priori*, the solutions that we have called trivial in which a single individual can constitute a majority.

coalitions have been fixed in advance, that is, for each "partition" we know which one of the two parties outweighs the other; the several "majorities" chosen in this way will satisfy the conditions of consistency which could be summarized by the axiom: no majority is *contained* in a minority. With this definition it is possible to state the following theorem:

If we wish to compose the opinions of several voters into one, guarding against any contradiction between two successive votes, we have to use a rule of majority in the broad sense, and this is sufficient (as long as it can be guaranteed that the various opinions of a particular voter will not be contradictory).

The Inevitable Contradictions

The law of the majority, so expanded, gives all the solutions to the problem slated: *two* decisions could be made by the majority, and *whatever the logical relation* existing between the two questions asked on the two ballots, we can be sure that if each of the voters respects this relation, the result will also respect it.

But, as we know, consistency between *two* ballots is not always enough. If the successive ballots carry judgments of preference, any two ballots are logically independent. It is possible to answer as we please to the two questions " $A < B$? " and " $B < C$?"; but if we have answered *yes* twice or *no* twice, then we find ourselves restricted on the third question, "$A < C$?"

Consideration of the binary relation is not sufficient, for there are logical relations which are *essentially* ternary. On the other hand, if we want to respect *all* the ternary relations, we have first to respect *all* binary relations which are a part of ternary relations. Thus it is among the laws of the majority (in the broad sense) that we must search.

We already know that the law of the ordinary majority or plurality is not suitable. It does not respect the ternary relations, and this is precisely the Condorcet effect. But one can legitimately wonder whether among the other majority laws (which we have seen are very diverse) there would not be any which would respect the ternary relations (or even relations of higher order which must also be anticipated).

Given three questions A,B,C, let us suppose that we have obtained, after composition, the resultant responses:

$$\hat{a} = +$$
$$\hat{b} = +$$
$$\hat{c} = -$$

This means that a "majority" answered yes to A, a "majority" (the same or another) answered yes to B, and a "majority" answered no to C. It can be concluded that any individuals whose opinions conform to the resultant opinion belong simultaneously to all these three majorities. The same result is verified for any number of questions: in order that the collective opinion (which results from the application of a majority rule) be the opinion of a given individual, it is necessary and sufficient that this individual belong to all the majorities. Thus *the Condorcet effect manifests itself as soon as the majorities* on the several questions have *no common elements*: *for then the majority opinion is the opinion of no one*.

We are back to a question of structure. Among all the laws of majority that can be imagined, do any exist for which any three majorities always have at least one common element for which any number of majorities have at least one common element?

Let us examine the singular 1aw, valid for an assembly of seven voters, which we have already mentioned and which is defined by majority trios.

ABC forms a majority (against *DEFG*)

but also

ADE,*ADF*,*CDG*

and finally

AFG,*BEG*,*CEF*

are majorities.

If we add that there are no other groupings where three are efficient, the law is completely known[95]. But any two majorities must have a common member[96], as:

ABC

ADE

and by adding a third "majority coalition" we obtain either *A* in all three:

ABC

ADE

AFG

(*A* in common)

or no member in all three majorities, for example:

ABC

ADE

BDF

This last eventuality proves the possibility of Condorcet effects. If, for instance, three questions are asked which cannot all be answered in the affirmative, then we only have to form a majority of yes on each question in order to end up with an absurd result.

A	*B*	*C*	*D*	*E*	*F*	*G*	Majority	Results
+	+	+	–	–	–	–	(*ABC*)	+
+	–	–	+	+	–	–	(*ADE*)	+
–	+	–	+	–	+	–	(*BDF*)	+

95 Every coalition of five members contains a majority and is itself a majority, and so every coalition of two is a minority; for the coalitions of four, it is sufficient to distinguish those which arc complementary to majorities of three, as *DEFG*, which is a minority, and those which contain a majority, like *ABCD*, and so are a majority.

96 And, in the present example, never more than one common member.

In order to construct a law of the majority which excludes every Condorcet effect. we have to make another arrangement. Return to the case of any number of voters. Notice first that two majority parties must necessarily have members in common; indeed, if in the assembly

ABC, ... ,*XYZ*

ABC is a majority, then *DE*,.., *Z* is a minority and consequently every fraction of this party is a minority, *DEF* for instance. It would be absurd to state that *ABC* and *DEF* are both majorities.

Then, given two distinct majorities:

ABCDEF

ABCKLMN

what can be said about their common part (*ABC*)? If it is a minority, its opposite (*DEF* ... *XYZ*) is a majority and we have three majority parties without common members:

*ABCDE*F

ABCKLMN

DEF, ... , *XYZ*

and so the Condorcet effect may occur.

To avoid the Condorcet effect it is necessary that the common part of two majority parties be itself a majority coalition. Therefore the set of *all* majority parties must have a common part which is itself a majority, and, being contained in every majority coalition, represents in a certain way the minimum or limiting majority. But this is not sufficient, for, once a majority has been found, say:

ABC,

to decide whether it is a minimum we have to examine whether its fractions, for example *AB*, are minorities. But if *AB* is a minority, *CDE*, ..., *Z* is a majority and the two majorities *ABC* and *CDE* ... have a common part *C* which must be a majority.

Thus either *AB* or *C* is a majority. By repeating the same reasoning it is seen that there is no other way to operate than the following: choose a single member who by himself constitutes a majority.

Thus the only majority laws which completely avoid the Condorcet effect are the trivial ones, according to which a single voter constitutes the "majority"[97], that is, the opinion of one member determines the opinion of the collectivity (the same member, of course, for all questions asked). It is clear that consistency is assured in that case. We can state the following result:

If we want to avoid having the collective opinion take a form not chosen by any of the voters, the only universal rules valid for all imaginable circumstances are those which give the privilege of deciding to a single individual chosen once and for all.

Such a result is rather disappointing: the only solution we have found is illusory although formally correct. We looked for a rule for aggregating opinions: we can consider that we have found nothing. Probably the ambitions were too great and the demands too strong. It is then

97 These are the solutions that K. J. Arrow calls "dictatorial".

natural to reconsider the problem by examining the role of each requirement *a priori*. One can direct attention to three possible aspects.

(1°) We wanted to apply the same voting rule to all the questions that were submitted to the voters.

(2°) We imposed that the answers to the questions would always be "yes" or "no".

(3°) We wanted to guarantee coherency for all possible logical relationships between an arbitrary numbers of ballots.

It is weakening the condition (3°) that constitutes the real problem: in which case is it possible to construct an inter-subjectivity, that is, a collective choice that really reflects individual wishes by using successive votes treated independently from each other? Which fixed and which variable relationships authorize other solutions than the trivial one which gives no consideration to more than a single opinion? We already know that the question is not asked in vain because we have previously (p.27ff) constructed an acceptable example in which the composition rule is the ordinary simple majority rule and contradictions could be completely avoided as long as all the subjective preferences were in conformity with a certain objective order. More generally, we also know that for it to be possible to aggregate the subjective preferences, these as a whole must have a certain (algebraic) structure that can be said to be “fragile” in the sense that it may be lost both due to excess and due to lack (see above p.33). This being said, the problem can nevertheless be stated and solved in the very terms that we have used so far. But as the conditions (1°) and (2°) remind us, these terms are probably too restrictive: we should consider extending the study in these two directions.

Let us begin by examining the first point. It is common practice to have several types of ballots in assemblies, commissions, and associations and to use one or the other depending on the nature of the debate. For example, modifying the statutes would require a numerically greater majority than the one demanded for an ordinary decision etc. One should therefore allow for the case where the aggregation of the votes is done differently depending on the category of the question at hand.

For the question (A) we have the ballot:

$$(a_1, a_2, a_3, \ldots, a_n)$$

Where each a_i is + or -, so that the result $\hat{a}$ will be a function not only of the a_i but also of some of the qualifications of the question (A):

$$\hat{a} = R(a_1, a_2, \ldots, a_n, a', a'', \ldots)$$

We can always assume that these qualifications result from a previous poll with answers "yes" and "no", that is, we assume that the a', a'',…, also take the values "+" or "–"[98]. Thus, in a sense, the method consists in introducing "ideal" voters who guard the constitution and whose task is to qualify the ballot by their answers a', a'',… But this addition does not change the nature of the reasoning. It only changes the result. Because the only way to avoid the Condorcet paradox is to always choose the opinion of one voter but it could be a real voter (= dictatorship) just as well as an ideal voter. The latter hypotheses correspond to what Arrow related to "Platonic" conceptions and which he referred to as *imposed*: the collective decisions are independent of individual preferences and only result from an answer provided a priori by the constitution. We may simply bear in mind from now on that it is useful to add to the group

98 See further on our analysis of how to reduce the problem to obtain bivalence.

of real voters, guards of the constitution whose answers should be present when the ballot result is established. But if we want a universal law *completely* without contradictions in *all* circumstances here once again, the only possible choices are between the "objective" solution imposed *a priori* by a set of rules — and the tyranny of a single individual. There is not any room at all for "inter-subjectivity".

Let us now examine the second point: it would be possible, as we have done so far, to always consider the alternatives "yes-no". For every question A, it is sufficient to list the various possible answers and when they are greater than two, replace "A?" by a series "A^1?", "A^2?", etc., obtained by arranging the answers by alternating categories. However, this reduction of all multivalent models to bivalent ones will very often introduce *objective* "connections" between the individual answers.

Consider for example, the common case where a question "A?" has 3 answers. Yes, no and abstention; to reduce this to a bivalent formulation, it is sufficient to ask two question instead of one

A^1 : Are you in favor of A?

A^2 : Are you against A?

And to consider that a double "no" is equivalent to abstention; but then the double "yes" must be excluded as an absurdity. Generally, every analogous reduction could result in new fixed relationships. Then the variable logical relationships between different ballots that we have to worry about (to avoid the Condorcet effect) are no longer all the possible relationships — and the general theorem is not necessarily applicable.

In other words, the risk for contradiction may very well be reduced. But another more serious reason requires that we directly consider the possibility of more than two answers (multivalence instead of bivalence): it is the fact that the whole difficulty of the Condorcet effect comes precisely from the fact that we have *decomposed* every individual opinion into more simple elements.

Let us return to Condorcet's example: the question is how to take decisions about 3 questions A, B and C while taking into account all the individual opinions. There are 6 possible opinions, listed above (p.10): if we decide to settle for the alternative among these 6 opinions that obtained the greatest number of votes[99] there is no risk of contradiction. But, as Condorcet correctly pointed out, one would not give any consideration to how preferences were distributed among the 5 remaining alternatives — and it is in order to use all the information provided by the ballot that the idea of decomposition appeared. Thus decomposition modifies the law according to which the votes of the voters are aggregated: indeed one sees that a legislator has less liberty if he organizes two ballots with answers "yes" or "no" than with a single ballot with 4 possible answers[100]. Therefore, we should also consider the question in the multivalent case. Moreover, there are no major additional difficulties — and the general results are analogous. Suppose for example that the answers to a question on which a poll is organized are numbered and that the counting of the votes translates into a law

$$R(x_1, x_2, \ldots, x_n) = \hat{x}$$

99 With some additional rules in case there is a tie.

100 In the first case we have, if there are N voters, 2^{2N} possible laws for each vote, thus in total 4^{2N} and in the second case 4^{4N} possible laws.

analogous to the one on page 35 above, but in which the different x, can take a certain number of values indexed in the usual way by 1,2,3,4 etc instead of just two values (+ and –). If we want to avoid contradictions between the ballots, and do so in every possible case that could arise, we must ensure that a logical contradiction between values of two answers such that:

$$\begin{cases} a = 1 \\ b = 2 \end{cases} \quad \text{or} \quad \begin{cases} a = 2 \\ b = 4 \end{cases} \qquad \text{etc.}$$

are respected by $\hat{a}$ and $\hat{b}$ whenever it is respected by all the individual answers (a_i and b_i). Then arguments that are completely analogous to the one we gave for the bivalent case (above p. 37) show that only the majority rule (in the wide sense) completely guarantees the absence of contradictions between two ballots[101].

Now, if we want to be sure to avoid the contradictions that may appear with *three* ballots, the result is still the same: one must follow the opinion of a single voter who completely dominates everyone else. This establishes the fact that the fundamental result is not linked to the bivalent character of the voters' answers. We can provide an interesting illustration. Suppose that we are faced with a certain number n of options $A,B,C,\ldots,Z$, and that we successively ask the voters what preference ranking they give to each of these options — every ballot would be a series of numbers taken from the sequence 1, 2,…n. Can we imagine an aggregation rule that establishes a collective preference ranking based on the individual rankings?

$$R\,(X)=F[R_1(X),\, R_2(X),\ldots]$$

It is readily shown[102] that the only ones are:

$$R = R_1, \text{ or } R = R_2, \text{ etc.}$$

That is, the trivial solution which gives all the power to one of the voters.

The irreducibility of the general interest

And so we arrive at the point where the theory of collective choice and the theories of the general interest, which had developed independently for a long time, meet. Does not the previous example immediately bring to mind the formalization of choice by utility functions — utility or "ophelimity" being an indicator of preference ranking (and nothing else according to those who are faithful to Pareto's interpretation). Then one may wonder what is meant by a social utility, defined as suggested by A. Bergson[103] by a composition of individual utilities:

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101 We can give an abstract formulation of the voluntarily intuitive reasoning in the pages 36 to 43 and prove that this form is perfectly general. It is sufficient to note that if the ban concerns the constitution $a = i$, $b = j$ we want to obtain that $R(a_1, a_2,\ldots) = i$ and $R(b_1, b_2,\ldots) = j$ necessarily implies either that $a_1 = i$ and $b_1 = j$, or that $a_2 = i$ and $b_2 = j$, etc., and this for all possible values of i and j. Then if one effectuates a substitution of values in a ballot ($a_1, a_2,\ldots$), one shows that this same substitution also acts on the result; symbolically: $R\,(S\,.a) = S.\,R\,(a)$.
Now what is invariant under the substitution S is the repartition of the voters in parties. This leads us to study the partitions that give rise to the same result $R = j$ and to show that their general form is that of the law of the majority.

102 The method remains the same; the individual answers to each ballot are aggregated into a single answer by the function F. We must take care not to assign the same rank to two different options: the value of the function F must thus change if all the variables change. If we call those modifications of the individual decisions that alter the final verdict *efficient,* we show that this efficiency must be attached to parties or groups of voters and we conclude as above.

103 We refer in particular to Abram Bergson's article, entitled significantly Socialist Economies, in *A Survey of contemporary Economics*, edited by H.S. Ellis, Blakiston Cy, Philadelphia, 1948, p. 412-448. A bibliography is contained in the text.

$$W(X) = F[U_1(X), U_2(X), U_3(X), \ldots]$$

a formula in which each X is an option presented to society, U_i the utility indicator of individual i, and W the collective utility (Welfare) that one wants to construct using the function F.

The word *interest* generally refers to[104] that which is important, which makes a difference; it is an advantage, a comparative weight, but hardly intended for knowledge alone. In its widest and most banal sense, interest is the driving force of action. This being the case, the word seems suitable to describe various interpretations of human actions by suggesting an expansion of the usual framework of utility theory. Because the "utilities" of Bentham, of Dupuit, of Gossen or of Walras — to mention only the most ancient ones — presented themselves as *possible* forms of the elements that underpin decisions and actions. Utility spontaneously took a numerical form; either because one was concerned with a theory of decisions motivated by monetary calculations, as was the case for Daniel Bernoulli, or, as for Bentham, because the analogies with monetary computations and the maturation of decisions which we still refer to as calculations are justified because in both cases, it is about comparing what is likely to be gained and lost. Pareto was well aware that there was a possible fallback, that it would be sufficient — thus returning to an Aristotelian position — to perceive a form common to all these deliberations of an agent, and that this common form was the preference. It should then always be possible to represent individual interest by a coherent system of preferences. The required coherency is that of transitivity which allows a completely ordered set to be formed. Thus the notion of *order* provides a mathematical image of interest. But the notion of order does not imply that of the number: the various numerical sets constructed in mathematics are ordered but the order is not an exhaustive description of their structure. Thus it is possible to wish for a way to present the theory completely independently from any numerical apparatus. Pareto never succeeded in this, on one hand, due to the insufficiency of his mathematical knowledge, but also, and more importantly because he established his theory very gradually[105] and never managed to disentangle his choice theory from the theories of utility, nor to clearly formulate the relationship between these two points of view. Sometimes he claimed that his own construction had the merit of explaining everything — but without approaching the central problem: if, which seems likely, everything that had been said about utility was not completely erroneous, attempts to synthesize contributions from older theories of utility and valid insights from the more recent ordinal conception should have been made. But there were only polemics and while these were probably not completely devoid of interest they did not contribute much to the necessary clarifications. Due to certain technical advantages, the economists who called themselves mathematicians preferred Pareto's construction which they sometimes presented as a further development of utility theories while it is just *one* theory of individual interest among others[106]. According to Pareto and his successors, interest could be represented by an ordered ranking[107] of the possible elements of choice[108]. But this order is given: one does not think of explaining how it comes about — thus

104 Disregarding, of course its use in the sense of a compensation or a capital return.

105 Cf. his *Manuel d'Economie politique*, édit. Française, Paris, 1909, p.543, n°1 ; and his article "Economie mathématique" in the *Encyclopédie des sciences mathématiques,* édit Française, Paris, 1911, t. I, vol. IV, p.596, n° 9. We know that the first correct mathematical representation of Pareto's conception was due to E. Slutsky (1915).

106 Hicks, *Value and Capital*, Oxford, 1946 (2nd ed.), goes as far as saying that Pareto clarified the difficulties of Marshall (p. 12) and he confirms: "given wants can be quite adequately defined as a given scale of preference" (P.18). Robbins (*An essay on the nature and significance of economic science*, London 2nd ed. , 1936, p. 75) does not hesitate to confound value and rank.

107 For the theoretically minded : it is not necessarily a conscious representation for the subject who may only be acting "as if". The theory in question is not at all psychological.

108 "Un habile homme doit régler le rang de ses intérêts et les conduire chacun dans son ordre" (A shrewd man must be able to

one will meet with considerable difficulties when one has to speak of the interest of a collectivity. Because although the general interest can also be represented by an ordered ranking of means and ends, it can no longer be considered as given. One must examine how it is formed. This prompted Pareto to construct a general interest given the name "ophelimity maximum for a collectivity": the question is how to determine what it is beneficial for a collectivity to do and not to do. Pareto answers that: that which is advantageous for a collectivity is that which is advantageous for each one of its members or which is at least advantageous for some of them without being disadvantageous for anybody. Thus the general interest is defined by *unanimity*, it is the totality of the interest that are *shared* by the individuals who form the collectivity. But then this general interest is no longer an ordered set: between two options *A* and *B*, one cannot determine which one is better if some individuals prefer *A* and others *B*. The general interest no longer contains anything but common preference judgments, it is a *partial* order. As such, it is not sufficient to guide our actions.

This made it necessary to go further. That is what modern theories of Welfare do. They search for acceptable models for determining what is advantageous for the collectivity that are not as Pareto's one limited to some particular comparisons, in other words, they look for a general interest that forms a complete system. They *also* want to construct this general interest only from individual interests. This is what is sometimes called "the sovereignty" of the consumer (at the very most it is an autonomy). It is then inevitable to encounter the fundamental logical problem: to compose objects of a certain type to obtain a single object of the same type, and thus to encounter the combinatorial difficulties that Condorcet had faced. If one is to aggregate the individual utilities, one must know in a precise manner the nature of the used objects: if this nature is numerical and can be assimilated to a sum of money, there is a simple and natural idea due to Pigou:

$$W = U_1 + U_2 + U_3 + \ldots$$

which can be generalized to:

$$W = F(U_1, U_2, U_3, \ldots)$$

But if one maintains, as Pareto did, that individual interest is nothing but an order — then one must compose the orders which is not as simple.

A. Bergson believed that the previous formula could be written giving the indicators their signification in the Pareto sense, while interpreting the symbol *F* in the usual way — as a numerical function. This is certainly incorrect which can be seen in a number of ways.

According to a well-known principle[109], it is not forbidden to represent the system of preferences by a numerical indicator:

"*A* is preferred to *B*"

being equivalent to: "$U(B) < U(A)$".

rank his interests and take on each one in its order) said La Rochefoucauld. And as we can see, Condorcet consistently used the ordinal conception. This was not new but it had been forgotten.

109 This principle was stated perfectly clearly in a letter that Henri Poincaré wrote to Léon Walras in 1901 (published by the latter in an appendix to his article: Economique et Politique, *Bull. Soc. Sc. Nat.* , XLV, 166, 1909): "Vous pouvez définir la satisfaction par une fonction arbitraire pourvu que cette fonction croisse toujours en même temps que la satisfaction qu'elle représente…vous devez donc vous efforcer d'*éliminer* ces fonctions arbitraires…" (You may define satisfaction by an arbitrary function as long as this function always increases at the same time as the satisfaction that it represents…you should thus strive to *eliminate* these arbitrary functions…)

This is a way to establish a correspondence between the options A, B,…, and the numbers U. But since the structure of the set of numbers is much richer than an order structure one must take some precautions. It is indeed possible to replace the U by V by means of an arbitrary increasing transformation function:

$$V = f(U)$$

The order, which is the only thing that matters is not modified; thus all the conditions obtained through the calculation must be absolutely independent from the arbitrary transformation f. If we posit an aggregation rule in the form suggested by Bergson:

$$W = F(U_1, U_2, U_3, \dots)$$

then the possible transformations of each U_i which do not modify the individual preference rankings should not modify the collective ranking expressed by W either[110].

If we define a law F for aggregating the U, we must then construct a law for the V as well which gives equivalent results. Since U and V can be modified in an infinite number of ways, we see that the question is less simple than it appears to be. The difficulty stems from the ambiguity of the symbols[111] such as F that designate the functions.

Writing $$W = F(U_1, U_2, U_3, \dots)$$

the intention is to say that the general interest (represented by W) is a function of the individual interests (represented by the U_i). But here every U_i is not a number but a collection of numbers assigned to various options: A, B, C. What is given is thus

Tableau des intérêts particuliers	$U_1(A) \dots U_1(B) \dots U_1(C) \dots$ $U_2(A) \dots U_2(B) \dots U_2(C) \dots$ $U_3(A) \dots U_3(B) \dots U_3(C) \dots$ … … … …

And similarly, W also represents a function $W(X)$, a number assigned to each option X.

General interest … . $W(A)$ $W(B)$ $W(C)$

But to say that the general interest is a "function" of particular interests is not necessarily the same thing as affirming that the collective utility of a given option, $W(A)$ for example, can be calculated using only the individual utilities related to that same option: $U_1(A)$, $U_2(A)$, $U_3(A)$, etc. Thus we see that Bergson and Samuelson as well as Marshall and Pigou draw on a hypothesis that considerably limits the relationship between the general interest and the individual interests; they assume that the *same* numerical operation determines the index W for every option *independently* of the other options:

$$\begin{cases} W(A) = F[(U_1(A), U_2(A), \dots)] \\ W(B) = F[(U_1(B), U_2(B), \dots)] \\ \dots \quad \dots \quad \dots \end{cases}$$

110 This invariance condition was stated by Samuelson, *Foundations of Economic Analysis*, Harvard University Press, 1948, p. 228; but the author had not seen the very restrictive conclusions that ensued.

111 Or of the expression "function of a function" which refers either to a functional or to a composed function.

While this *separate* procedure may be conceivable for theoreticians who acknowledge the notion of numerical utility, it is difficult to accept for those who completely reject “cardinal” representations and who view the indicators U and W only as intermediaries towards the ordinal concepts: one senses that it is not easy to treat each option A independently from the others if all one knows is a ranking which is a relationship between A and B. As we will see, it must be acknowledged that Bergson’s suggestion leads to a dead end; when one wants the best of both worlds, to keep the technical advantages of the utility and to attain the logical purity of preference rankings, there is a risk of contradiction — and in order to avoid it all that is left is trivial inefficiency. In the language of voting, using a function W such as the one suggested by Bergson and Samuelson is to ask each voter i to give a numerical evaluation U_i of all the possible options:

	Options				Preference orders
	A	B	C	…	
U_1	0	1	3	…	$\ldots < A < B < C < \ldots$
U_2	1	2	0	…	$\ldots < C < A < B < \ldots$
U_3	4	1	3	…	$\ldots < B < C < A < \ldots$
…	…	…	…	…	… … …

and then to obtain by a calculation F a table of evaluations W. Imagine that we decide to add the indicators (as in Pigou), we find: W : 5,4,6

that is to say $B < A < C$

But if U_1 had been 0,3,4 (which does not alter the order $A < B < C$) we would find: $W = 5, 6, 7$,

that is $A < B < C$.

Thus the resulting order changes although the individual components have not been modified. Therefore it is necessary to change the law (F) — here addition — as soon as the numerical scale of U changes for some individual: the aggregation law (F) must be adapted to the numerical representations (U_i) of the objects that are to be aggregated (they are essentially preference rankings). This means that $W(A)$ is no longer a function only of the $U_i(A)$ and that the $U_i(B)$, $U_i(C)$,etc. must be used as well in order to calculate it. But then one ends up losing all the advantages of a *separate* procedure.

Moreover, these conclusions are well-known: consider the case of an examination, for example. Let (A, B, C) be the candidates and (1, 2, 3) the jury members who give the notes U. The final ranking will be the result of a certain combination W of the notes U which must according to the rules be calculated separately (most often a weighted average). But then it is obvious that each examiner cannot be allowed to grade in any way he pleases: there must be a pre-established standardization, for example all the numbers U must be chosen within a certain specified range, moreover, one cannot use any number contained in the continuum between 0 and 20 but only one among a finite number of possibilities etc.

And finally the rather vague idea of what the grades represent is not strictly limited to an ordinal: certainly, the examiners may not all believe that they are *measuring* some aspect of ability but most of them attribute a meaning to phrases such as "the candidate A lies between

B and C but is closer to B than C ". If the evaluation of each examiner had indeed been purely ordinal, it would not be possible to separately compute the grade of each candidate[112].

Obviously, the numerical example given above is not sufficient to rigorously establish the impossibilities that we just alluded to. This requires a mathematical analysis. We can begin by analyzing the case of addition.

$$W(A) = W = U_1(A) + U_2(A) + U_3(A) + \ldots$$

We can suppose that each voter has at his disposal a ranking of the options of the type

$B < A < C\ldots$which expresses his preferences.

We will then suppose that he uses a numerical operation to translate this order into

$$U_i(B) < U_i(A) < U_i(C) < \ldots$$

Then the resulting system W will in its turn provide a ranking of the options. But every given ranking has an infinity of corresponding choices for the system U. One can then show that if one varies the individual U_i (while respecting, nevertheless, the basic information that is given, that is the preferences) there are a vast number of possible collective rankings W that can be obtained. The only rankings that are not possible are those that contradict a unanimous preference affirmed by all the voters. This means that the only *valid* information provided by W is the one given by Pareto's rule of unanimity. Thus all we are allowed to say if we add the preference indicators is that if *all* the members of the collectivity prefer A to B:

$$A > B$$

So that the same holds for the derived order. But if the collectivity is split on an issue, some of them considering $B > C$ and the others $C > B$, then there is always a way of evaluating so as to make one group or the other emerge as the winner.

Moreover this result can be generalized beyond the case where the U are simply added, to any arbitrary numerical operation no matter how complicated it is. Indeed, it is very generally impossible to find a function that really[113] depends on several variables:

$$W = F(u_1, u_2, \ldots)$$

such that the order of two particular values w' and w'':

$$w' > w'' \text{or } w' < w''$$

only depends on the order of the corresponding couples of arguments.

$$u'_1 > u''_1 \text{ or } u'_1 < u''_1,\ u'_2 > u''_2 \text{ or } u'_2 < u''_2, \text{ etc}$$

It is readily shown that the existence of such a function would provide a general solution to the Condorcet paradox and we know that there is none. Moreover, it is possible to give a geometrical illustration of why it is impossible in the space of points whose coordinates are the u_i. We will use the established language of three dimensional space and place ourselves in an arbitrary point which we can always assume to be the origin

$$\bar{u}_1 = \bar{u}_2 = \ldots 0.$$

112 As we have seen above, it would not even be possible to determine an aggregate order by *separately* studying the binary relations : "$A <$ B?".

113 Indeed, because there are always the trivial solutions $w = u_1$ or $w = u_2$, etc.

The various combinations $u_1 > 0$ or $u_1 < 0$, $u_2 > 0$ or $u_2 < 0$, $u_3 > 0$ or $u_3 < 0$ give rise to 8 trihedrons in each of which the sign of w must remain constant. Therefore, the surface $w = 0$ cannot be one of the coordinate planes. Thus the only solutions are the trivial ones.

And so we must conclude that

(1) with the formula of Bergson and Samuelson

$$W = F(U_1, U_2, U_3, \ldots)$$

if we intend to use this computation separately for each option:

$$W = F[(U_1(A), U_2(A), U_3(A), \ldots)]$$

then we cannot allow the indicators U_i to be nothing but a numerical representation of an ordinal structure because as we have just observed, the resulting ordinal described by W could vary although the rankings in the input do not.

(2) We must thus be prepared to adapt the computation F to numerical scales of U that are chosen to express the ranking structures. If every scale depends only on a certain number of parameters[114], h,k, etc. we could correct the formula to

$$W = F[U_1, U_2, \ldots, \ldots; h_1, k_1, \ldots, h_2, k_2, \ldots]$$

But is we remain faithful to a strictly *ordinal* interpretation as did Hicks, Samuelson, Bergson, and Arrow, it is not possible[115] to parameterize. The only parameters are *all* the numerical values of the U_i and one must therefore write:

$$W = F[(U_1(A), U_2(A), \ldots U_1(B), U_2(B), \ldots U_1(C), U_2(C), \ldots]$$

which means that in the initial formula

$$W = F(U_1, U_2, U_3, \ldots)$$

F must not be interpreted as a relationship between numbers but as a relationship between *functions*.

(3) Then the dilemma that arises is that:

– either the general interest is constructed from particular interests which are considered to be indecomposable objects: society's judgments of an alternative does not only depend on what the members of the society think of this alternative but depends on all their opinions.

– or alternatively, each particular interest is nothing but a simple ordinal structure, which of course does not mean that we must postulate a measurable utility but at least that we should look for a pathway between order structures and measurable quantities (ordered groups).

(4) The first direction is impossible to take. It would be necessary to begin by constructing the universe of all possibilities and never take a decision on an isolated question! The second one consists, as we have already seen in the Condorcet problem, in recognizing that there is always something in common in different individual opinions, rather than attempting to make impossible constructions between completely — independent subjectivities. To speak of a

114 For instance a measure unit, an origin, etc.
115 At least without restricting the acceptable field of ordinations: for finite domains, this proposition is obvious — for infinite sets we must be more precise but is seems quite unnecessary to attack here the well known difficulties of continuous ordination!

resemblance is insufficient. The "something" in question is not necessarily a consensus, even a partial one. It is a structure of the set of possible opinions. Then it could often be possible to see the individual opinion as a kind of *approximation* of an objective judgment. And a collective opinion would then be constructed in accordance with the statistical theory of *errors*. This was the direction that Condorcet himself took in order to resolve his paradoxes, it is along these lines that one can provide a pertinent critique of Quetelet's research. In the political domain, there is a famous saying: "Quand on propose une loi dans l'assemblée du peuple, ce qu'on leur demande n'est pas précisément s'ils approuvent la proposition ou s'ils la rejettent, mais si elle est conforme ou non à la volonté générale…quand l'avis contraire au mien l'emporte, cela ne prouve autre chose *sinon que je m'étais trompé*[116]…" (When a law is proposed in the people's assembly, what one asks is not exactly whether or not they approve of the proposition but whether or not it is in conformity with the general will…when it is the opinion contrary to my own one that wins, all this proves is *that I was mistaken*.)

But there is a third way.

Often the difficulty has been presented in the following seemingly uncontestable way: it is claimed that there is no way to "compare the satisfaction that different individuals experience". But it must be clarified what is meant by a "comparison". Pareto himself and Edgeworth[117] and many others replied: life in a society effectuates these forbidden comparisons on a daily basis! That is to say that every society constructs, by the very decisions that are taken a "general interest", what escapes us is the "rationality" behind it. The negative theorems that we have quoted throughout this article allow us to reject a large number (if not all) of those models that give rise to the general interest through a simple *ex ante* combination of particular interests. In other words, it is highly difficult to imagine an arbitrage that respects everyone's autonomy if there is neither a will to reach consensus nor struggle. Isn't there indeed some resemblance between the logical analyses we have developed and the picture presented by Rousseau: "plus le concert règne dans les assemblées, c'est à dire plus les avis approchent de l'unanimité, plus aussi la volonté générale est dominante; mais les longs débats, les dissensions, le tumulte annoncent l'ascendant des intérêts particuliers…; les citoyens n'ayant qu'un intérêt, le peuple n'a qu'une volonté…à l'autre extrémité, l'unanimité revient; c'est quand les citoyens, tombés dans la servitude, n'ont plus ni liberté, ni volonté"[118] (the greater the consensus in the assemblies, that is the closer they are to unanimity, the more dominant is the general will but long debates, dissent, and tumult announces the ascendency of individual interests… ; when the citizens have only one interest, the people has only one will…at the other extreme, unanimity appears again, in this case because the citizens have fallen into servitude and have neither liberty nor a will.)

Only these two extremes are coherent: consensus or servitude. They are also the only two solutions provided by the computations. If the unanimity requirement may seem somewhat naïve, similar criticism could be aimed at all the constructions post-Pareto where the general interest is only a result of algebra. When unanimity is not granted, struggle determines the outcome. To speak of conflicting interests is a metaphor. It is only actions that can really be combined. This is what happens and what can be studied in detail in society games: each player has value judgments about all the possible outcomes — but the outcome bears no resemblance to a unique value judgment which could be said to be collective. Mathematical game theory has the merit of analyzing this irreducible character of collective actions[119].

116 *Contrat social*, book IV, chap. II.

117 Pareto, *Cours*, n^{os} 613-652. Edgeworth *Papers*, v. II, p. 475.

118 *Contrat social*, IV, 2

119 In his book (*The problem of summation in economic science*), G. Nyblen rightly drew a parallel between the theorems of

While it might, in the case of a single individual[120], be possible to conceive of a rationality that precedes all actions and takes the form of deliberations or calculations that determine the decision, it is (in the general case) vain to attempt to obtain a similar separation as soon as several wills are involved. The discussion itself, the debate and the negotiation is an integral part of the struggle. To engage in this direction[121] is to recognize the precariousness of agreements and the specific character of the notion of the *adversary*. From Pareto's point of view, there was a simple link between taste and decision. Everyone makes his best possible choice and is limited only by invincible *obstacles*. In a representation of collective actions, it is important on the contrary to see that the interval between a wish and a realization is characterized by a game of alliances and oppositions in which it is impossible to determine once and for all the absolute borders between the possible and the impossible.
Among the models that have been exhaustively studied by mathematical game theory, there are very few that lead to a hierarchy of outcomes similar to the "taste" of a single individual. In most cases, we must give up this simplicity and acknowledge that the interaction between conflicting wills create structures of a new kind.

K. Arrow and those established by O. Morgenstern and J. Von Neumann.

120 Condorcet suggested rather, that in order to understand the deliberations of an individual we take as a model the deliberations of an assembly. This path gives some interesting results for instance in the choice between random outcomes. See my essay in the *Colloque international sur le risque*, C. N. R. S., Paris, 1951.

121 Which can, depending on the context, be done with or without mathematics.