Geometric Thickness of Complete Graphs

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Abstract

We define the *geometric thickness* of a graph to be the smallest number of layers such that we can draw the graph in the plane with straight-line edges and assign each edge to a layer so that no two edges on the same layer cross. The geometric thickness lies between two previously studied quantities, the (graph-theoretical) thickness and the book thickness. We investigate the geometric thickness of the family of complete graphs, $\{K_n\}$. We show that the geometric thickness of K_n lies between $\lceil (n/5.646) + 0.342 \rceil$ and $\lceil n/4 \rceil$, and we give exact values of the geometric thickness of K_n for $n \leq 12$ and $n \in \{15, 16\}$. We also consider the geometric thickness of the family of complete bipartite graphs. In particular, we show that, unlike the case of complete graphs, there are complete bipartite graphs with arbitrarily large numbers of vertices for which the geometric thickness coincides with the standard graph-theoretical thickness.

1   Introduction

Suppose we wish to display a nonplanar graph on a color terminal in a way that minimizes the apparent complexity to a user viewing the graph. One possible approach would be to use straight-line edges, color each edge, and require that two intersecting edges have distinct colors. A natural question then arises: for a given graph, what is the minimum number of colors required?

Or suppose we wish to print a circuit onto a circuit board, using uninsulated wires, so that if two wires cross, they must be on different layers, and that we wish to minimize the number of layers required. If we allow each wire to bend arbitrarily, this problem has been studied previously; indeed, it reduces to the graph-theoretical thickness of a graph, defined below. However, suppose that we wish to further reduce the complexity of the layout by restricting the number of bends in each wire. In particular, if we do not allow any bends, then the question becomes: for a given circuit, what is the minimum number of layers required to print the circuit using straight-line wires?

These two problems motivate the subject of this paper, namely the *geometric thickness* of a graph. We define $\overline{\theta}(G)$, the geometric thickness of a graph G, to be the smallest value of k such that we can assign planar point locations to the vertices of G, represent each edge of G as a line segment, and assign each edge to one of k layers so that no two edges on the same layer cross. This corresponds to the notion of "real linear thickness" introduced by Kainen [15]. Graphs with geometric thickness 2 (called "doubly-linear graphs) have been studied by Hutchinson *et al.* [13], where the connection with certain types of visibility graphs was explored.

A notion related to geometrical thickness is that of (graph-theoretical) thickness of a graph, $\theta(G)$, which has been studied extensively [1, 3, 8, 9, 10, 14, 16] and has been defined as the minimum number of planar graphs into which a graph can be decomposed. The key difference between geometric thickness and graph-theoretical thickness is that geometric thickness requires that the vertex placements be consistent at all layers and that straight-line edges be used, whereas graph-theoretical thickness imposes no consistency requirement between layers.

Alternatively, the graph-theoretical thickness can be defined as the minimum number of planar layers required to embed a graph such that the vertex placements agree on all layers but the edges can be arbitrary curves [15]. The equivalence of the two definitions follows from the observation that, given any planar embedding of a graph, the vertex locations can be reassigned arbitrarily in the plane without altering the topology of the planar embedding provided we are allowed to bend the edges at will [15]. This observation is easily verified by induction, moving one vertex at a time.

The (graph-theoretical) thickness is now known for all complete graphs [1,

2, 4, 17, 19], and is given by the following formula:

$$\theta(K_n) = \begin{cases} 1, & 1 \le n \le 4 \\ 2, & 5 \le n \le 8 \\ 3, & 9 \le n \le 10 \\ \lceil \frac{n+2}{6} \rceil, & n > 10 \end{cases} \tag{1.1}$$

Another notion related to geometric thickness is the *book thickness* of a graph G, $bt(G)$, defined as follows [5]. A *book with k pages* or a *k-book*, is a line L (called the *spine*) in 3-space together with k distinct half-planes (called *pages*) having L as their common boundary. A *k-book embedding* of G is an embedding of G in a k-book such that each vertex is on the spine, each edge either lies entirely in the spine or is a curve lying in a single page, and no two edges intersect except at their endpoints. The book thickness of G is then the smallest k such that G has a k-book embedding.

It is not hard to see that the book thickness of a graph is equivalent to a restricted version of the geometric thickness where the vertices are required to form the vertices of a convex n-gon. This is essentially Lemma 2.1, page 321 of [5]. It follows that $\theta(G) \le \overline{\theta}(G) \le bt(G)$. It is shown in [5] that $bt(K_n) = \lceil n/2 \rceil$.

In this paper, we focus on the geometric thickness of complete graphs. In Section 2 we provide an upper bound, $\overline{\theta}(K_n) \le \lceil n/4 \rceil$. In Section 3 we provide a lower bound. In particular, we show that $\overline{\theta}(K_n) \ge \left\lceil \frac{3-\sqrt{7}}{2}(n+1) \right\rceil \ge \lceil \frac{n+1}{5.646} \rceil$. This follows from a more precise expression which gives a slightly better lower bound for certain values of n.

These lower and upper bounds do not match in general. The smallest values for which they do not match are $n \in \{13, 14, 15\}$. For these values of n, the upper bound on $\overline{\theta}(K_n)$ from Section 2 is 4, and the lower bound from Section 3 is 3. In Section 4, we resolve one of these three cases by showing that $\overline{\theta}(K_{15}) = 4$. For $n = 16$ the two bounds match again, but they are distinct for all larger n.

Section 5 briefly addresses the geometric thickness of complete bipartite graphs; we show that

$$\left\lceil \frac{ab}{2a + 2b - 4} \right\rceil \le \theta(K_{a,b}) \le \overline{\theta}(K_{a,b}) \le \left\lceil \frac{\min(a, b)}{2} \right\rceil.$$

When a is much greater than b, the leftmost and rightmost quantities in the above inequality are equal. Hence there are complete bipartite graphs with arbitrarily many vertices for which the standard thickness and geometric thickness coincide. We also show that the bounds on geometric thickness of complete bipartite graphs given above are not tight, by showing that $\overline{\theta}(K_{6,6}) = 2$ and $\overline{\theta}(K_{6,8}) = 3$.

Section 6 contains a table of the lower and upper bounds on $\overline{\theta}(K_n)$ established in this paper for $n \le 100$ and lists a few open problems.
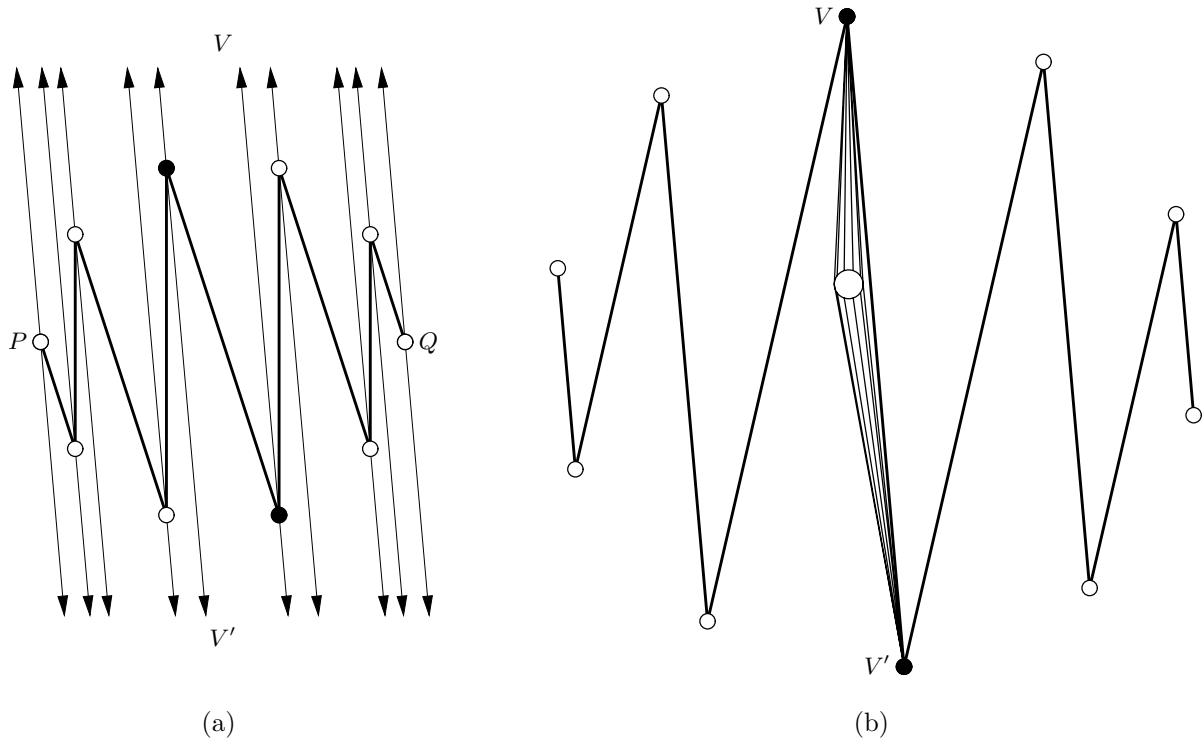
Figure 1: Construction for embedding K_{2k} with geometric thickness of $k/2$, illustrated for $k = 10$. (a) The inner ring. (b) The outer ring. The circle in the center of (b) represents the inner ring shown in (a).

2   Upper Bounds

Theorem 2.1 $\overline{\theta}(K_n) \leq \lceil n/4 \rceil$.

Proof   Assume that n is a multiple of 4, and let $n = 2k$ (so, in particular, k is even). We show that n vertices can be arranged in two *rings* of k vertices each, an *outer ring* and an *inner ring*, so that K_n can be embedded using only $k/2$ layers and with no edges on the same layer crossing.

The vertices of the inner ring are arranged to form a regular k-gon. For each pair of diametrically opposite vertices P and Q, consider the zigzag path as illustrated by the thicker lines in Figure 1(a). This path has exactly one diag-

onal connecting diametrically opposite points (namely, the diagonal connecting the two dark points in the figure.) Note that the union of these zigzag paths, taken over all $k/2$ pairs of diametrically opposite vertices, contains all $\binom{k}{2}$ edges connecting vertices on the inner ring. Note also that for each choice of diametrically opposite vertices, parallel rays can be drawn through each vertex, in two opposite directions, so that none of the rays crosses any edge of the zigzag path. These rays are also illustrated in Figure 1(a).

By continuity, if the infinite endpoints of a collection of parallel rays (e.g., the family of rays pointing "upwards" in Figure 1(a)) are replaced by a suitably chosen common endpoint (so that the rays become segments), the common endpoint can be chosen so that none of the segments cross any of the edges of the zigzag path. We do this for each collection of parallel rays, thus forming an outer ring of k vertices. This can be done in such a way that the vertices on the outer ring also form a regular k-gon. By further stretching the outer ring if necessary, and by moving the inner ring slightly, the figure can be perturbed so that none of the diagonals of the polygon comprising the outer ring intersect the polygon comprising the inner ring. The outer ring constructed in this fashion is illustrated in Figure 1(b).

Once the $2k$ vertices have been placed as described above, the edges of the complete graph can be decomposed into $k/2$ layers. Each layer consists of:

1. A zigzag path through the outer ring, as shown in Figure 1(b).

2. All edges connecting V and V' to vertices of the inner ring, where V and V' are the (unique) pair of diametrically opposite points joined by an edge in the zigzag path through the outer ring. (These edges are shown as edges connecting the circle with V and V' in Figure 1(b), and as arrows in Figure 1(a)).

3. The zigzag path through the inner ring that does not intersect any of the edges connecting V and V' with inner-ring vertices. (These are the heavier lines in Figure 1(a).)

It is straightforward to verify that this is indeed a decomposition of the edges of K_n into $k/2 = n/4$ layers. ∎

3   Lower Bounds

Theorem 3.1 *For all $n \geq 1$,*

$$\overline{\theta}(K_n) \geq \max_{1 \leq x \leq n/2} \frac{\binom{n}{2} - 2\binom{x}{2} - 3}{3n - 2x - 7}. \tag{3.1}$$

In particular, for $n \geq 12$,

$$\overline{\theta}(K_n) \geq \left\lceil \frac{3 - \sqrt{7}}{2} n + 0.342 \right\rceil \geq \left\lceil \frac{n}{5.646} + 0.342 \right\rceil. \tag{3.2}$$

Proof   We first prove a slightly less precise bound, namely

$$\overline{\theta}(K_n) \geq \frac{3 - \sqrt{7}}{2} n - O(1). \tag{3.3}$$

For graph G and vertex set X, let $G[X]$ denote the subgraph of G induced by X. Let S be any planar point set, and let $T_1, \ldots T_k$ be a set of straight-line planar triangulations of S such that every segment connecting two points in S is an edge of at least one of the T_i. Find two parallel lines that cut S into three subsets A, B, and C (with B the middle set), with $|A| = |C| = x$, where x is a value to be chosen later. For any T_i, the subgraph $T_i[A]$ is connected, because any line joining two vertices of A can be retracted onto a path through $T_i[A]$ by moving it away from the line separating A from B. Similarly, $T_i[C]$ is connected, and hence each of the subgraphs $T_i[A]$ and $T_i[C]$ has at least $x - 1$ edges.

By Euler's formula, each T_i has at most $3n - 6$ edges, so the number of edges of T_i not belonging to $T_i[A] \cup T_i[C]$ is at most $3n - 6 - 2(x - 1) = 3n - 2x - 4$. Hence

$$\binom{n}{2} \leq 2\binom{x}{2} + k(3n - 2x - 4). \tag{3.4}$$

Solving for k, we have

$$k \geq \frac{\binom{n}{2} - 2\binom{x}{2}}{3n - 2x - 4},$$

and hence

$$k \geq \frac{n^2 - 2x^2}{6n - 4x} - O(1). \tag{3.5}$$

If $x = cn$ for some constant c, then the fraction in (3.5) is of the form $n(1 - 2c^2)/(6 - 4c)$. This is maximized when $c = (3 - \sqrt{7})/2$. Substituting the value $x = (3 - \sqrt{7})n/2$ into (3.5) yields (3.3).

To obtain the sharper conclusion of the theorem, observe that by choosing the direction of the two parallel lines appropriately, we can force at least one point of the convex hull of S to lie in B. Hence, of the edges of T_i that do not belong to $T_i[A] \cup T_i[C]$, at least three are on the convex hull. If we do not count these three edges, then each T_i has at most $3n - 2x - 7$ edges not belonging to $T_i[A] \cup T_i[C]$, and we can strengthen (3.4) to

$$\binom{n}{2} - 3 \leq 2\binom{x}{2} + k(3n - 2x - 7),$$

or

$$k \geq \frac{\binom{n}{2} - 2\binom{x}{2} - 3}{3n - 2x - 7}. \tag{3.6}$$

Since (3.6) holds for any x, (3.1) follows.

To prove (3.2), let $f(x)$ be the expression on the right-hand side of (3.6). Consider the inequality $f(x) \geq x_0$, where x_0 is a constant to be specified later. After cross-multiplication, this inequality becomes

$$-x^2 + x + \frac{n^2}{2} - \frac{n}{2} - 3 - (3n - 7 - 2x)x_0 \geq 0. \tag{3.7}$$

The expression in the left-hand side of (3.7) represents an inverted parabola in x. If we let $x = x_0$, we obtain

$$x_0^2 + (8 - 3n)x_0 + \frac{n^2}{2} - \frac{n}{2} - 3 \geq 0, \tag{3.8}$$

and if we let $x = x_0 + 1$ we obtain the same inequality. Now, consider x_0 of the form $An + B - \epsilon$. Choose A and B so that if $\epsilon = 0$, the terms involving n^2 and n vanish in (3.8). This gives the values $A = (3 - \sqrt{7})/2$ and $B = \sqrt{7}(23/14) - 4$. Substituting $x_0 = An + B - \epsilon$ with these values of A and B into (3.8), we obtain

$$\sqrt{7} \cdot \epsilon \cdot n + (\epsilon^2 - \frac{23\epsilon}{\sqrt{7}} - 3/28) \geq 0. \tag{3.9}$$

For $\epsilon = 0.0045$, (3.9) will be true when $n \geq 12$. Therefore, for all $x \in [x_0, x_0 + 1]$, $f(x) \geq x_0$, when $\epsilon = 0.0045$ and $n \geq 12$. In particular, $f(\lceil x_0 \rceil) \geq x_0$. Since k is an integer, (3.2) follows from (3.6). ∎

4 The Geometric Thickness of K_{15}

The lower bounds on geometric thickness provided by equation (3.1) of Theorem 3.1 are asymptotically larger than the lower bounds on graph-theoretical thickness provided by equation (1.1), and they are in fact at least as large for all values of $n \geq 12$. However, they are not tight. In particular, we show that $\overline{\theta}(K_{15}) = 4$, even though (3.1) only gives a lower bound of 3.

Theorem 4.1 $\overline{\theta}(K_{15}) = 4$.

To prove this theorem, we first note that the upper bound, $\overline{\theta}(K_{15}) \leq 4$, follows immediately from Theorem 2.1.

To prove the lower bound, assume that we are given a planar point set S, with $|S| = 15$. We show that there cannot exist a set of three triangulations of S that cover all $\binom{15}{2} = 105$ line segments joining pairs of points in S. We use the following two facts: (1) A planar triangulation with n vertices and b convex hull vertices contains $3n - 3 - b$ edges; and (2) Any planar triangulation of a given point set necessarily contains all convex hull edges. There are several cases, depending on how many points of S lie on the convex hull.

Case 1: 3 points on convex hull. Let the convex hull points be A, B and C. Let A_1 (respectively, B_1, C_1) be the point furthest from edge BC (respectively AC, AB) within triangle ABC. Let A_2 (respectively, B_2, C_2) be the point next furthest from edge BC (respectively AC, AB) within triangle ABC.

Lemma 4.2 *The edge AA_1 will appear in every triangulation of S.*

Proof   Orient triangle ABC so that edge BC is on the x-axis and point A is above the x-axis. For an edge PQ to intersect AA_1, at least one of P or Q must lie above the line parallel to BC that passes through A_1. But there is only one such point, namely A. ∎

Lemma 4.3 *At least one of the edges A_1A_2 or AA_2 will appear in every triangulation of S.*

Proof   Orient triangle ABC so that edge BC is on the x-axis and point A is above the x-axis. For an edge PQ to intersect A_1A_2 or AA_2, at least one of P or Q must lie above the line parallel to BC that passes through A_2. There are only two such points, A and A_1. Hence an edge intersecting A_1A_2 must necessarily be AX and an edge intersecting AA_2 must necessarily be A_1Y, for some points X and Y outside triangle AA_1A_2. Since edges AX and A_1Y both split triangle AA_1A_2, they intersect, so both edges cannot be present in a triangulation. It follows that either A_1A_2 or AA_2 must be present. ∎

Now let Z be the set of 12 edges consisting of the three convex hull edges and the nine edges pp_1, pp_2, p_1p_2 (where $p \in \{A, B, C\}$). Each triangulation of S contains 39 edges, and since any triangulation contains all three convex hull edges, it follows from Lemmas 4.2 and 4.3 that at least 9 edges of any triangulation must belong to Z. Hence a triangulation contains at most 30 edges not in Z. Thus three triangulations can contain at most $30 \cdot 3 + 12 = 102$ edges, and hence cannot contain all 105 edges joining pairs of points in S.

Case 2: 4 points on convex hull. Let A,B,C,D be the four convex hull vertices. Assume triangle DAB has at least one point of S in its interior (if not, switch A and C). Let A_1 be the point inside triangle DAB furthest from the line DB. By Lemma 4.2, the edge AA_1 must appear in every triangulation of S, as must the 4 convex hull edges. Since any triangulation of S has 38 edges, three triangulations can account for at most $3 \cdot 33 + 5 = 104$ edges.

Case 3: 5 or more points on convex hull. Let h be the number of points on the convex hull. A triangulation of S will have $42 - h$ edges, and all h hull edges must be in each triangulation. So the total number of edges in three triangulations is at most $3(42 - 2h) + h = 126 - 5h$, which is at most 101 for $h \geq 5$.

This completes the proof of Theorem 4.1.

5   Geometric Thickness of Complete Bipartite Graphs

In this section we consider the geometric thickness of complete bipartite graphs, $K_{a,b}$. We first give an upper bound, (Theorem 5.1); it is convenient to state this bound in conjunction with the obvious lower bound on standard thickness

that follows from Euler's formula. It follows from this theorem that, for any b, $\overline{\theta}(K_{a,b}) = \theta(K_{a,b})$ provided a is sufficiently large (Corollary 5.2). Hence, unlike the situation with complete graphs, there are complete bipartite graphs with arbitrarily many vertices for which the standard thickness and geometric thickness coincide. We show that the lower bound in Theorem 5.1 is not a tight bound for geometric thickness by showing that $\overline{\theta}(K_{6,8}) = 3$. A pair of planar drawings demonstrating that $\theta(K_{6,8}) = 2$ can be found in [16]. Finally we show that the upper bound in Theorem 5.1 is also not tight, since $\overline{\theta}(K_{6,6}) = 2$ while Theorem 5.1 only implies that $\overline{\theta}(K_{6,6}) \leq 3$.

Theorem 5.1 *For the complete bipartite graph $K_{a,b}$,*

$$\left\lceil \frac{ab}{2a + 2b - 4} \right\rceil \leq \theta(K_{a,b}) \leq \overline{\theta}(K_{a,b}) \leq \left\lceil \frac{\min(a, b)}{2} \right\rceil. \tag{5.1}$$

Proof   The first inequality follows from Euler's formula, since a planar bipartite graph with $a + b$ vertices can have at most $2a + 2b - 4$ edges. To establish the final inequality, assume that $a \leq b$ and a is even. Draw b blue vertices in a horizontal line, with $a/2$ red vertices above the line and $a/2$ red vertices below. Each layer consists of all edges connecting the blue vertices with one red vertex from above the line and one red vertex from below. ∎

Corollary 5.2 *For any integer b, $\overline{\theta}(K_{a,b}) = \theta(K_{a,b})$ provided*

$$a > \begin{cases} \frac{(b-2)^2}{2}, & \text{if } b \text{ is even} \\ (b-1)(b-2), & \text{if } b \text{ is odd} \end{cases} \tag{5.2}$$

Proof   If $a > b$, the leftmost and rightmost quantities in (5.1) will be equal provided $ab/(2a + 2b - 4) > (b - 2)/2$ if b is even, or provided $ab/(2a + 2b - 4) > (b - 1)/2$ if b is odd. By clearing fractions and simplifying, we see that this happens when (5.2) holds. ∎

Theorem 5.3 $\overline{\theta}(K_{6,8}) = 3$.

Proof   It follows from the second inequality in Theorem 5.1 that $\overline{\theta}(K_{6,8}) \leq 3$, so we need only show that $\overline{\theta}(K_{6,8}) > 2$. Suppose that we did have an embedding of $K_{6,8}$ with geometric thickness 2, with underlying points set S. Since $K_{6,8}$ has 14 vertices and 48 edges, and since Euler's formula implies that a planar bipartite graph with 14 vertices has at most 24 edges, it follows that each layer has exactly 24 edges and that each face of each layer is a quadrilateral.

Two-color the points of S according to the bipartition of $K_{6,8}$. We claim that there must be at least one red vertex and one blue vertex on the convex hull of S. Suppose, to the contrary, that all convex hull vertices are the same color (say red). Then because each layer is bipartite and because the convex hull contains at least three vertices, the outer face in either layer would consist of at least 6
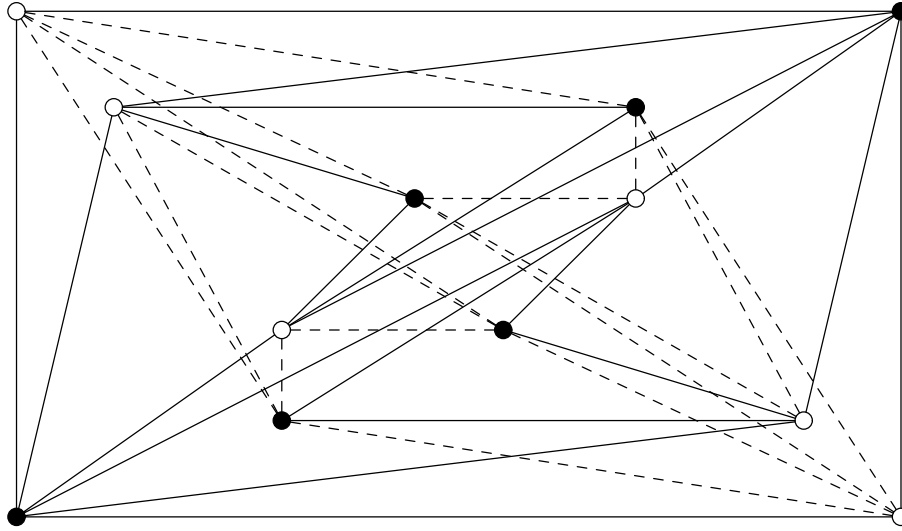
Figure 2: A drawing showing that $\overline{\theta}(K_{6,6}) = 2$. The solid lines represent one layer, the dashed lines the other.

vertices (namely the convex hull vertices and three intermediate blue vertices), which is impossible because each face is bounded by a quadrilateral. The claim implies that one of the layers (say the first) must contain a convex hull edge. But then this edge could be added to the second layer without destroying either planarity or bipartiteness. Since the second layer already has 14 vertices and 24 edges, this is impossible. ∎

Figure 2 establishes the final claim of the introduction to this section, namely that $\overline{\theta}(K_{6,6}) = 2$.

6 Final Remarks

In this paper we have defined the geometric thickness, $\overline{\theta}$, of a graph, a measure of approximate planarity that we believe is a natural notion. We have established upper bounds and lower bounds on the geometric thickness of complete graphs. Table 1 contains the upper and lower bounds on $\overline{\theta}(K_n)$ for $n \le 100$.

Many open questions remain about geometric thickness. Here we mention several.

1. Find exact values for $\overline{\theta}(K_n)$ (i.e., remove the gap between upper and lower bounds in Table 1). In particular, what are the values for K_{13} and K_{14}?

Table 1: Upper and lower bounds on $\overline{\theta}(K_n)$ established in this paper.

n	LB	UB
1- 4	1	1
5- 8	2	2
9-12	3	3
13-14	3	4
15-16	4	4
17-20	4	5
21-24	5	6
25-26	5	7
27-28	6	7
29-31	6	8
32	7	8
33-36	7	9
37	7	10

n	LB	UB
38-40	8	10
41-43	8	11
44	9	11
45-48	9	12
49-52	10	13
53-54	10	14
55-56	11	14
57-60	11	15
61-64	12	16
65	12	17
66-68	13	17
69-71	13	18
72	14	18

n	LB	UB
73-76	14	19
77	14	20
78-80	15	20
81-82	15	21
83-84	16	21
85-88	16	22
89-92	17	23
93-94	17	24
95-96	18	24
97-99	18	25
100	19	25

Note: Upper bounds are from Theorem 2.1. The lower bounds for $n \geq 12$ are from Theorem 3.1, with the exception of the lower bound for $n = 15$ which is from Theorem 4.1. Lower bounds for $n < 12$ are from (1.1).

2. What is the smallest graph G for which $\overline{\theta}(G) > \theta(G)$? We note that the existence of a graph G such that $\overline{\theta}(G) > \theta(G)$ (e.g., K_{15}) establishes Conjecture 2.4 of [15].

3. Is it true that $\overline{\theta}(G) = O(\theta(G))$ for all graphs G? It follows from Theorem 2.1 that this is true for complete graphs. For the crossing number [11, 18], which like the thickness is a measure of how far a graph is from being planar, the analogous question is known to have a negative answer.

Bienstock and Dean [7] have described families of graphs which have cross-ing number 4 but arbitrarily high rectilinear crossing number (where the rectilinear crossing number is the crossing number restricted to drawings in which all edges are line segments).

4. What is the complexity of computing $\overline{\theta}(G)$ for a given graph G? Comput-ing $\theta(G)$ is known to be NP-complete [16], and it certainly seems plausible to conjecture that the same holds for computing $\overline{\theta}(G)$. Since the proof in [16] relies heavily on the fact that $\theta(K_{6,8}) = 2$, Theorem 5.3 of this paper shows that this proof cannot be immediately adapted to geometric thickness. Bienstock [6] has shown that it is NP-complete to compute the rectilinear crossing number of a graph, and that it is NP-hard to deter-mine whether the rectilinear crossing number of a given graph equals the crossing number.

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