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Fully Dynamic 3-Dimensional Orthogonal Graph Drawing

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Abstract

In a 3-dimensional orthogonal drawing of a graph, vertices are mapped to grid points on a 3-dimensional rectangular integer lattice and edges are routed along integer grid lines. In this paper, we present a technique that produces a 3D orthogonal drawing of any graph with n vertices of degree 6 or less, using at most 6 bends per edge route and in a volume bounded by $O(n^2)$. The advantage of our strategy over previous drawing methods is that our method is fully dynamic, allowing both insertion and deletion of vertices and edges, while maintaining the volume and bend bounds. The drawing can be obtained in $O(n)$ time and insertions/deletions are performed in $O(1)$ time. Multiple edges and self loops are permitted. Three related constructions are also presented: a more elaborate construction that uses only 5 bends per edge, a simpler, more balanced drawing that requires at most 7 bends per edge, and a technique for displaying directed graphs.

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1 Introduction and Previous Work

In this paper we describe a 3-dimensional orthogonal drawing strategy for graphs of maximum degree at most 6 (*i.e.* graphs containing no vertex of degree 7 or higher). The **3-dimensional orthogonal grid** consists of grid points whose coordinates are defined as (x', y', z') , where $x', y', z' \in \mathbf{Z}$, together with the axis-parallel grid lines determined by these grid points. A **3-dimensional orthogonal drawing** of a graph G is an embedding of G into the 3-dimensional orthogonal grid with the vertices located at grid points and each edge (u, v) represented as a sequence of contiguous orthogonal segments of grid lines and which join the 2 points corresponding to the vertices u and v . No pair of edge routes are permitted to intersect (except at common vertex endpoints). Thus as a consequence, each grid point corresponding to a vertex has **6 ports** to which edge routes may be attached, namely along the 3 grid lines incident on it. It is common practice to use the same terminology for both the graph and its drawing.

The **bounding box** of a 3-dimensional orthogonal drawing is the minimum axis-parallel box which contains all vertices and edges of the drawing. The dimensions of such a bounding box are measured in terms of the number of grid points rather than the length. The **volume** of the bounding box is the product of the 3 dimensions which is then exactly equal to the number of grid points on or in the box.

For background information in graph drawing, the reader is referred to the text by Di Battista, Eades, Tamassia and Tollis [3].

There have been several 3-dimensional drawing strategies proposed in the graph drawing literature for graphs of maximum degree at most 6 — see for example [1, 4, 6, 8, 9]. Note that 3-dimensional drawings of graphs of higher degree have also been investigated in for example, [2, 8, 10]; however in this paper we consider only 3-dimensional orthogonal drawings of graphs of maximum degree 6.

For the most part, previous results have focussed on the trade-offs among: overall volume of the resulting drawing, the maximum number of bends on an edge, the maximum length of an edge, *etc.* Eades, Stirik and Whitesides [5], building on earlier work by Kolmogorov and Barzdin [7], presented an algorithm that produces a 3-d orthogonal drawing in $O(n\sqrt{n})$ time with at most 16 bends per edge, edge lengths bounded by $O(\sqrt{n})$ and in an asymptotically optimum volume of $O(n^{1.5})$. They also showed the following three problems are NP-Complete. Given a graph, determine a 3-d orthogonal drawing that:

- minimizes the total edge length;
- minimizes the area/volume;
- contains 0 bends.

These results follow from the related 2-dimensional cases (see also [3]). Experimental comparisons of several 3-d orthogonal drawing algorithms can be found in the paper by Di Battista, Patrignani, and Vargiu [4].

Although there are some classes of graphs of maximum degree 6 that can be drawn with few bends (*e.g.* trees can be drawn with 0 bends), the introduction of multiple edges or self loops into the graph immediately implies that some edges will have at least three bends in the drawing. For a self loop, the necessity of three bends is evident. A simple but tedious case study can be used to demonstrate that a pair of vertices with 6 edges between them, cannot be drawn with at most two bends per edge. Whether all *simple* graphs can be drawn with 2 or fewer bends is still an important open problem; in [11] Wood established lower bounds on the total number of bends in any drawing of certain classes of graphs.

Our result is motivated primarily by a dynamic version of the drawing problem in which edges and vertices of the graph can be inserted or deleted over time. Papakostas and Tollis [8] introduced a semi-dynamic solution in which only vertices are permitted to be inserted, and at all stages the resulting drawing must correspond to a connected graph. No other algorithms in this area have discussed non-trivial dynamic graph drawing cases. For example, one of the classic drawing algorithms in this area is due to Eades, Symvonis and Whitesides [6]. Their algorithm achieves $O(n^{1.5})$ volume, uses 7 bends per edge, requires a matching and coloring pre-processing step, and is only suitable in a static context. The same general technique has recently been modified by the authors to examine volume/bend tradeoffs and in particular they achieve a drawing with at most 6 bends per edge and $O(n^2)$ volume.

Our main drawing strategy is *fully dynamic*, allowing insertion and deletion of vertices and edges. The volume of the drawing is bounded by $O(n^2)$, each edge has at most 6 bends, and insertions and deletions are accomplished in $O(1)$ time. The dynamic nature of our drawing algorithm relies on the fact that at any time, a free port on any vertex may safely be connected to a free port of any other vertex without disturbing the drawing, and this connection requires at most 6 bends. In one variant of our strategy, we show that the same volume bound can be achieved with at most 5 bends per edge.

Multiple edges and self-loops are also accommodated in our drawings. Multiple edges may prove to be an important consideration in applications that require some degree of “fault-tolerance”. Some previous 3-dimensional orthogonal constructions such as [8] have been restricted to simple graphs.

One feature of our drawing technique that may be useful in some applications such as 3-dimensional VLSI design, is that the width of the drawing is constant; the entire drawing lies on 7 planes between the planes $Y = -3$ and $Y = +3$.

Finally, note that all four of our drawing techniques are practical and easily implemented. We supply a software package (OrthoPak) for constructing 3-dimensional orthogonal drawings. Detailed descriptions of the drawing techniques, two brief animations and several VRML worlds are also available via the web page associated with this paper.

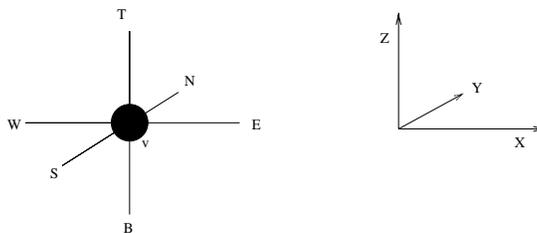


Figure 1: The 6 ports on a vertex

2 Drawing Graphs in $O(n^2)$ Volume with 6 Bends

2.1 Introduction and Definitions

Initially, we describe our drawing strategy in a static context, in which the entire graph (of maximum degree 6) is available. As will be seen, a dynamic version of the problem follows easily. We refer to this drawing strategy as the **staircase** model.

Let the vertices of the given graph be $v_1, v_2, v_3, \dots, v_n$ (labeled in an arbitrary order). The vertices will be located at grid points in a staircase fashion, separated by an appropriate amount to permit the crossing-free embedding of the edges. More formally, vertex v_i is located at $(7i, 0, 5i)$. For convenience, we denote these coordinates as $(X(v_i), Y(v_i), Z(v_i))$. Each vertex may have at most 6 incident edges attached at **ports** which will be denoted as N, S, E, W, T, B — refer to Figure 1. When viewed from above (*i.e.* from $+\infty$ along the **Z** axis), the compass designations effectively describe the ports.

Containing each vertex v is a small box of dimensions $6 \times 7 \times 5$, called its **neighborhood** and denoted by $\mathbf{N}(v)$. The two diagonal corners of $\mathbf{N}(v)$ have coordinates: $(X(v) - 3, Y(v) + 3, Z(v) + 3)$ and $(X(v) + 2, Y(v) - 3, Z(v) - 1)$. Contained entirely inside $\mathbf{N}(v)$ are 6 **pedestals** — one for each port of v . Each pedestal consists of a line segment parallel to the **Z** axis. The **Bottom pedestal** is directly attached to v beneath it, and is of length 1. Similarly, the **Top pedestal** attaches to the top port of v and is of length 3. The **North pedestal** is of length 2 and extends from $(X(v), +1, Z(v))$ to $(X(v), +1, Z(v) + 2)$. The **South pedestal** is of length 1 and extends from $(X(v), -1, Z(v))$ to $(X(v), -1, Z(v) + 1)$. The **West** and **East pedestals** are degenerate points at $(X(v) - 1, 0, Z(v))$ and $(X(v) + 1, 0, Z(v))$ respectively; they are introduced for consistency in the description and will be modified in a subsequent section describing a related drawing strategy.

The Neighborhood $\mathbf{N}(v)$ consists of 5 layers in the **Z**-dimension, the:

- **Top layer** located on the plane $Z = Z(v) + 3$,
- **North layer** located on the plane $Z = Z(v) + 2$,
- **South layer** located on the plane $Z = Z(v) + 1$,

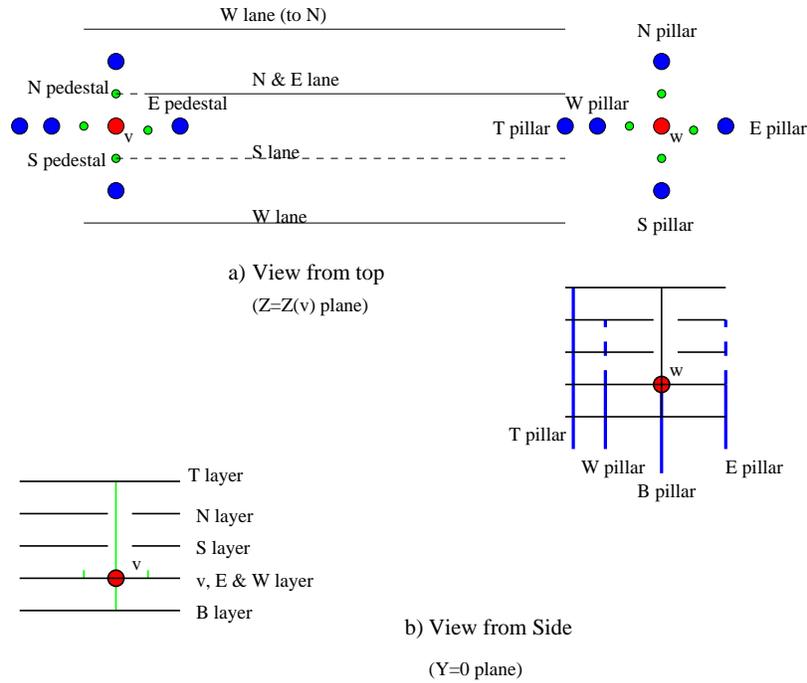


Figure 2: Two Neighborhoods

- **East, West, v layer** located on the plane $Z = Z(v)$,
- **Bottom layer** located on the plane $Z = Z(v) - 1$.

Edges are always considered as directed from the lower indexed vertex to the higher indexed vertex. Each pedestal is used only by *outgoing* edges from a vertex. *Incoming* edges are routed along **pillars**. The 6 pillars of a vertex v extend below the neighborhood of v to the plane $Z = -1$. More precisely, the

- **North pillar** extends from $(X(v), 2, -1)$ to $(X(v), 2, Z(v) + 2)$,
- **South pillar** extends from $(X(v), -2, -1)$ to $(X(v), -2, Z(v) + 2)$,
- **East pillar** extends from $(X(v) + 2, 0, -1)$ to $(X(v) + 2, 0, Z(v) + 2)$,
- **West pillar** extends from $(X(v) - 2, 0, -1)$ to $(X(v) - 2, 0, Z(v) + 2)$,
- **Bottom pillar** extends from $(X(v), 0, -1)$ to $(X(v), 0, Z(v))$,
- **Top pillar** extends from $(X(v) - 3, 0, -1)$ to $(X(v) - 3, 0, Z(v) + 3)$.

Note that the four side pillars (N, S, E, W) extend to $Z(v) + 2$ to accommodate self loops as will be described in a later section. Figure 2 displays two (perpendicular) views of the construction.

We refer to the orthogonal box directly below a vertex v (and containing the pillars of v), as the **airspace of v** . An edge from v to w passes safely from the neighborhood of v to the airspace of w through intervening airspaces by traversing *lanes*, which are guaranteed to avoid pillars. In particular, note that any line segment lying in the planes $Y = 1, Y = 3, Y = -1$, or $Y = -3$ intersects no pillar of *any* vertex. Each layer contains one or two lanes on one of these four planes, extending from $N(v)$, as far in the $+X$ direction as necessary. Let M be $X(v_n) + 2$, where v_n is the last vertex in the staircase. The lanes are then defined as:

- **North lane** from $(X(v), +1, Z(v) + 2)$ to $(M, +1, Z(v) + 2)$,
- **South lane** from $(X(v), -1, Z(v) + 1)$ to $(M, -1, Z(v) + 1)$,
- **East lane** from $(X(v) + 1, +1, Z(v))$ to $(M, +1, Z(v))$,
- **West lanes**
 - to N: from $(X(v) - 1, 3, Z(v))$ to $(M, 3, Z(v))$,
 - to all other destinations: from $(X(v) - 1, -3, Z(v))$ to $(M, -3, Z(v))$,
- **Bottom lanes**
 - to N, E, T: from $(X(v), +1, Z(v) - 1)$ to $(M, +1, Z(v) - 1)$,
 - to S, W, B: from $(X(v), -1, Z(v) - 1)$ to $(M, -1, Z(v) - 1)$,
- **Top lanes**
 - to N, E, T: from $(X(v), +1, Z(v) + 3)$ to $(M, +1, Z(v) + 3)$,
 - to S, W, B: from $(X(v), -1, Z(v) + 3)$ to $(M, -1, Z(v) + 3)$.

Figure 3 displays the pedestals, pillars and layers for three vertices on the staircase and an edge routing from the north port of the first vertex to the east port of the third.

Description of edge-routes. Typically, an edge from a vertex v to a vertex w leaves v from an arbitrary free port, travels to the associated pedestal, climbs the pedestal to the appropriate layer, is routed *on that layer* from the pedestal to a lane, along the lane to the airspace under w , and (still on that layer) joins to the pillar associated with the destination port of w , climbs this pillar and finally enters w . The source and destination ports are chosen arbitrarily by the algorithm — any unused port can be specified. Although the routings differ slightly for each port to port pair, this general technique applies, the critical observation being that the routing from the neighborhood of v to the airspace of w is done at some *layer of v* (*i.e.* the lower vertex). By using one of the lanes, this section of the edge passes safely through the airspace of any intervening vertices without intersecting any of its pillars (which may have edges routed from below). Since the E and W ports of v share the same layer as v , special care is required to ensure routings from these ports do not intersect each other,

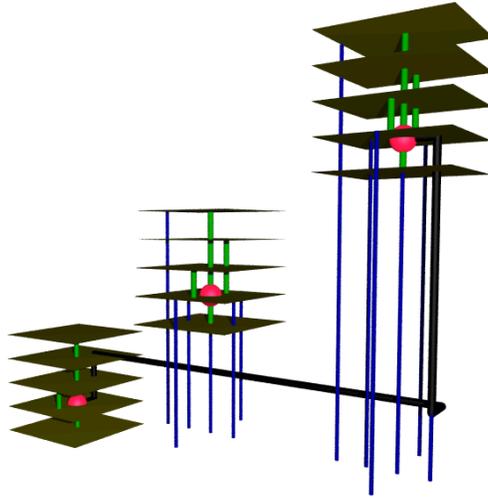


Figure 3: Pedestals, pillars, and layers

$(X(v), 0, Z(v))$	- vertex v
$(X(v), +1, Z(v))$	N pedestal of v
$(X(v), +1, Z(v) + 2)$	N lane
$(X(w) + 2, +1, Z(v) + 2)$	connect to E pillar of w
$(X(w) + 2, 0, Z(v) + 2)$	climb E pillar
$(X(w) + 2, 0, Z(w))$	
$(X(w), 0, Z(w))$	- vertex w

Table 1: Route from North to East

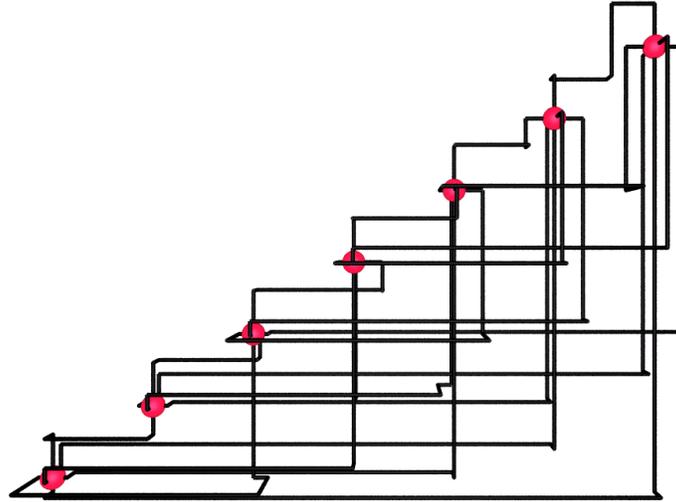


Figure 4: Drawing of K_7 using the Staircase method

or possible incoming edges to the N or S ports. A connection to the Top port of vertex w comes up the Top pillar, along w 's Top layer, and then directly down to w , thus safely passing above w 's West pillar.

For example, consider an edge routing from the N port of v to the E port of w . Table 1 shows the routing which consists of 6 segments and thus contains 5 bends.

Figure 4 displays a drawing of the graph K_7 using the staircase drawing method. For completeness, the 36 port-to-port routings are specified in Appendix A. Note that of these routings, 9 require six bends, 22 require five bends, and 5 require four bends.

Lemma 1 *No pair of edges in the construction intersect.*

Proof: Appendix A defines the paths for all port-to-port connections between a vertex v (at the lower level) to a vertex w ($\neq v$). To prove that edge (v, w) intersects no other edge in the drawing, note that the routing of each edge consists of three portions:

1. inside v 's neighborhood
2. along a lane through intervening airspaces
3. in w 's airspace.

Since the four lanes intersect no pillars, the second portion of edge (v, w) intersects no other edge.

Consider the portion of the edge (v, w) inside v 's neighborhood. Each of the 6 possible routings out of v intersect no pillar of v and so cannot intersect any

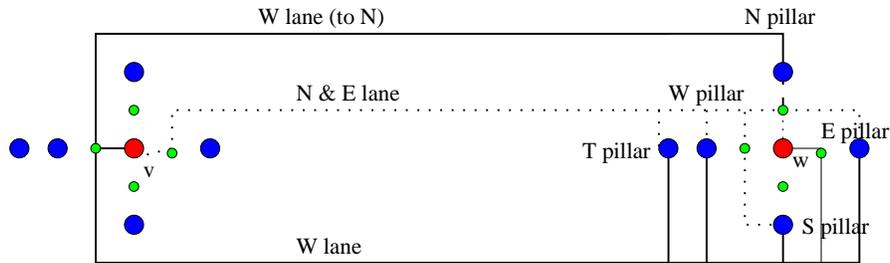


Figure 5: E and W routings do not intersect

incoming edge to v . And similarly, the portion in w 's airspace can intersect no outgoing edge from w .

It remains to consider the intersection of (v, w) with some outgoing edge from v (inside v 's neighborhood) or with some other incoming edge to w (inside w 's airspace). Without loss of generality, we consider the case of (v, w) intersecting another edge between v and w but with different port assignments. Since the N,S,T, and B ports each have their own layers inside the neighborhood of v , it is simple to verify that no intersections are possible if either of the two edges involve these ports at v . However, the E and W ports share a plane with v itself, and it is therefore critical to check that none of the 30 possible pairs involving these two ports from v intersect. In Figure 5, the 6 routings from the West port are shown as solid lines and the 6 routings from the East port are displayed as dotted lines; note that only the routings to the pillars of w are shown. That no pair of solid and dotted paths intersect either in the neighborhood of v nor in the airspace of w is easily determined. \square

2.2 Self-loops

The general drawing strategy for self loops is similar: from a port to a pedestal, up to the appropriate level, across to the specified pillar, down the pillar and into the vertex. A complete listing of the routes is presented in Appendix B, and on the web page.

2.3 Dynamic Case

The drawing strategy described in the previous section easily adapts to use in a dynamic context, in which both edges and vertices may be deleted or inserted. Insertion of a new vertex v_t is performed by placing it at $(7t, 0, 5t)$. It is clear that a new edge (u, v) can be added to the drawing in $O(1)$ time — any free port on u can be connected to a free port on v , and furthermore, the overall volume of the drawing remains the same. Deletion of a vertex is accomplished by replacing it by the top-most vertex and re-routing the (at most 6) incident edges. Let v_1, v_2, \dots, v_t be the vertices in the drawing at time t . To delete vertex v_i , the

edges on v_i are first deleted. Denote by w_1, w_2, \dots, w_6 the vertices adjacent to vertex v_t . These edges are all deleted in the drawing and vertex v_i is connected to w_1, w_2, \dots, w_6 , using arbitrary ports on vertex v_i . Vertex v_t is then removed from the drawing. The resulting drawing has a bounding box of dimensions $6(t-1) \times 7 \times 5(t-1)$. Note that we measure the number of grid points rather than the length in each dimension.

Theorem 1 *For every graph G of maximum degree at most 6 the staircase algorithm produces a 3D orthogonal drawing contained in a bounding box with volume $210t^2$, where t is the number of vertices in the drawing. Insertion or deletion of vertices or edges can be accomplished in $O(1)$ update time.*

3 Variants

In this section, we consider three related constructions - a *spiral* drawing technique that produces a more balanced drawing, a drawing strategy for *directed* graphs, and a more elaborate version of the *staircase* model that requires at most 5 bends per edge.

3.1 Ortho-Spiral Layout

One possible criticism of the drawings produced by the staircase model is that the dimensions of the resulting drawing are *unbalanced* - the construction is very narrow, with a Y dimension consisting of exactly 7 planes. Although there may be practical applications of this feature of our strategy (for example in future VLSI design), from an aesthetic standpoint a more balanced drawing may be preferable. In this section, we outline a strategy that produces a drawing of dimensions $O(\sqrt{n}) \times O(\sqrt{n}) \times O(n)$, but with a slight penalty on the number of bends: 7 bends are required for some port-to-port connections. The dynamic nature of the previous strategy is preserved; an implementation and a short video describing the routing is available on the web page associated with the paper.

Instead of placing the vertices in a linear staircase fashion, the vertices are embedded in an orthogonal spiral manner. When viewed from above, the vertices appear in a $O(\sqrt{n}) \times O(\sqrt{n})$ grid as displayed in Figure 6.

As in the previous method, each vertex v , is assigned to a unique Z-plane, thus forming a spiral of linear height. Associated with each vertex are 7 planes (one for each port, and one for the vertex itself). The degenerate E and W pedestals described in Section 2.1 are extended from the v layer to the appropriate plane.

Figure 7 shows a cross section (from the Top) of the modified neighborhood through a vertex. The four side pedestals (for N, S, E, W) have their bases on this plane and extend to their respective layers. The pedestals for the Bottom and Top layers originate at the vertex itself.

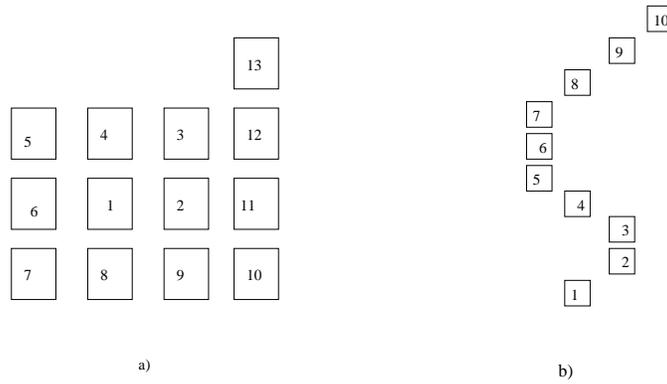


Figure 6: Spiral drawing: a) from above; b) from the side

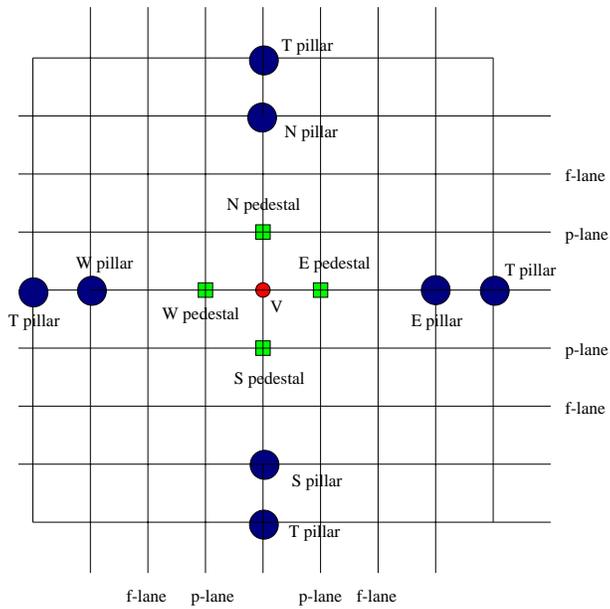


Figure 7: Spiral drawing: cross-section

$(X(v), Y(v), Z(v))$	- vertex v
$(X(v), Y(v) + 1, Z(v))$	N pedestal of v
$(X(v), Y(v) + 1, Z(v) + 4)$	p-lane (correct in X)
$(X(w) + 2, Y(v) + 1, Z(v) + 4)$	f-lane (correct in Y)
$(X(w) + 2, Y(w), Z(v) + 4)$	connect to E pillar of w
$(X(w) + 3, Y(w), Z(v) + 4)$	climb E pillar
$(X(w) + 3, Y(w), Z(w))$	
$(X(w), Y(w), Z(w))$	- vertex w

Table 2: Route from North to East — Ortho Spiral

Side pillars and the Bottom pillar (for incoming routings) are spaced 1 unit further from v , but otherwise remain the same. A total of four pillars for connection to the Top port are provided.

On each particular layer, there are 4 pairs of lanes used for routing:

- **f-lanes** - two pairs (horizontal and vertical) that intersect no pillar or pedestal on any level (*i.e.* free of any intersections), and
- **p-lanes** - two pairs (horizontal and vertical) that intersect only a single pedestal on (or passing through) this level.

More precisely, each lane is defined as the intersection of 2 planes: one of the layer planes and one of the 8 lane planes. The 6 layers lie on the planes: $Z = Z(v) - 1$ (Bottom), $Z = Z(v) + 1$ (West), $Z = Z(v) + 2$ (East), $Z = Z(v) + 3$ (South), $Z = Z(v) + 4$ (North) and $Z = Z(v) + 5$ (Top). The 4 p-lanes lie in the planes $Y = Y(v) + 1$, $Y = Y(v) - 1$, $X = X(v) + 1$ and $X = X(v) - 1$, and the 4 f-lanes lie in the planes $Y = Y(v) + 2$, $Y = Y(v) - 2$, $X = X(v) + 2$ and $X = X(v) - 2$.

The general edge routing technique remains the same as in the staircase model, however note that unlike the staircase model, neighborhoods may differ in both their X and Y coordinates. An edge from a vertex v at a lower level, to a vertex w , exits a specific port on v , joins its pedestal, climbs the pedestal to the appropriate layer, traverses the p-lane to correct in the X-coordinate (N, S layers) or the Y-coordinate (E, W layers), then traverses a f-lane to correct in the Y-coordinate or X-coordinate (respectively) in order to enter w 's airspace. Connection to the specified pillar of w then allows the edge to climb the appropriate pillar, and finally to enter w at the specified port. Edges from the Top or Bottom port climb/descend their pedestal and then safely join an f-lane, a second (perpendicular) f-lane and then connect to the specified pillar. As in the staircase model, connection into the Bottom port of w is directly from below

		destination		
		side	Bottom	Top
source	side	6	5	7
	Bottom	6	5	7
	Top	6	5	7

Table 3: Bends per edge

and connection to the Top port of w requires two bends and is performed via w 's Top layer. In each case, the connection from the pillar of w to a particular port of w itself intersects only the pedestal associated with that port which of course cannot be in use.

In Table 2 for example, the routing from the North port on vertex v to the East port of vertex w is described. Appendix C lists all 36 possible routings between an arbitrary pair of vertices. Table 3 summarizes the number of bends per edge for each routing in this spiral model.

The coordinate specification for the elements of this drawing is more elaborate than in the staircase model. For notational convenience assume the vertices are v_0, v_1, \dots, v_{n-1} and that vertex v_i is to be located at (x_i, y_i, z_i) . Then $z_i := 7i$ since 7 layers are required for each vertex. Let $c = \lceil \sqrt{i} \rceil$ and $f = \lfloor \sqrt{i} \rfloor$. Then the X and Y coordinates for vertex v_i can be determined depending on whether f is even or odd:

- f is even:
 - If $i \leq f^2 + f$ then $x_i := -\frac{9}{2}f$ and $y_i := 9(f^2 - i + \frac{f}{2})$.
 - If $i \geq f^2 + f$ then $x_i := 9(i - c^2 + \frac{c+1}{2})$ and $y_i := -\frac{9}{2}f$.
- f is odd:
 - If $i \leq f^2 + f$ then $x_i := 9\frac{f+1}{2}$ and $y_i := 9(i - f^2 - \frac{f-1}{2})$.
 - If $i \geq f^2 + f$ then $x_i := 9(c^2 - i - \frac{c}{2})$ and $y_i := 9\frac{f+1}{2}$.

The proof that no pair of edges intersect in the drawing is simpler than that of the staircase model since associated with each port is a distinct layer. There are no intersections possible within v 's neighborhood, within w 's airspace, and p-planes and f-planes pass safely through any intervening pillars.

Note that for each vertex, 7 layers are required and that the X and Y extents of each neighborhood are 9×9 (see Figure 7).

Theorem 2 *Any graph of maximum degree 6 can be drawn orthogonally in 3-dimensions with at most 7 bends per edge in a bounding box with dimensions $9\lceil \sqrt{n} \rceil \times 9\lceil \sqrt{n} \rceil \times 7n$.*

Again insertions and deletions of vertices and edges are accomplished in $O(1)$ time, and self-loops and multiple edges are easily accommodated. A brief animation and the implementation details are provided on the web page.

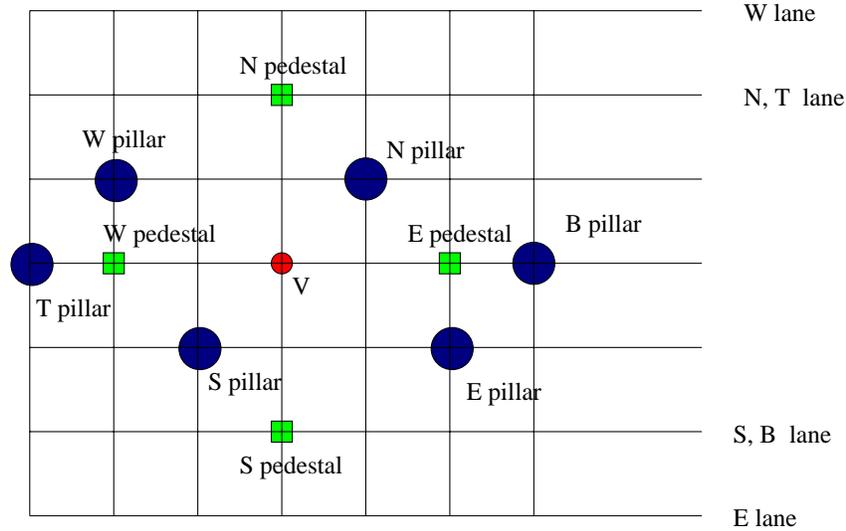


Figure 8: Directed Case

3.2 Drawings of Directed Graphs

Typically, 3-dimensional orthogonal drawings of graphs are visually difficult for humans to work with, even with the addition of colored edges, and both the staircase and orthogonal spiral techniques suffer from this drawback. In this subsection, we outline a modified staircase drawing for *directed* graphs which allows immediate recognition of the direction of each edge.

We refer to the orthogonal box directly below the neighborhood of vertex v , as the **lower-airspace** of v . The orthogonal box directly above v 's neighborhood is referred to as its **upper-airspace**.

In this variant, by repositioning the pillars and pedestals slightly (as illustrated in Figure 8), the number of layers required can be reduced to three and the overall volume is slightly smaller (70% of a staircase drawing), however *all* port to port routings require exactly six bends per edge. If v is lower than w on the staircase, then an edge directed from vertex v to vertex w is routed below the staircase, whereas an edge from vertex w to vertex v is routed above the staircase. Such a drawing allows for quick visual identification of the direction of an edge.

The staircase is formed with vertex v_i positioned at $(7i, 0, 3i)$. The neighborhood $N(v)$ has a size of $7 \times 7 \times 3$ and consists of three layers in the \mathbf{Z} -dimension. The:

- Top, South layer located on the plane $Z = Z(v) + 1$,
- West, East layer located on the plane $Z = Z(v)$,
- North, Bottom layer located on the plane $Z = Z(v) - 1$,

Outgoing edges are routed to the pedestal associated with the outgoing port, up or down the pedestal to the appropriate layer and then out to the lane. Let (v, w) be a directed edge to be drawn. If the vertex v is below the vertex w (on the staircase), the edge travels in the positive \mathbf{X} direction along the lane. If the vertex v is above the vertex w , the edge is routed in the negative \mathbf{X} direction. The North, South, Bottom and Top pedestals are each of length one; their extents are respectively: from $(X(v), +2, Z(v))$ to $(X(v), +2, Z(v) - 1)$, from $(X(v), -2, Z(v))$ to $(X(v), -2, Z(v)+1)$, from $(X(v), 0, Z(v))$ to $(X(v), 0, Z(v) - 1)$, from $(X(v), 0, Z(v))$ to $(X(v), 0, Z(v) + 1)$. The East and West pedestals are degenerate points at $(X(v) + 2, 0, Z(v))$ and $(X(v) - 2, 0, Z(v))$

An incoming edge ascends or descends a pillar. The six pillars of vertex v extend from the plane of $Z = -1$ through the neighborhood of v to the top of the drawing at plane $Z = Z(v_n) + 1$, where n is the index of the highest vertex in the drawing. The pillar extents are as follows:

- North: from $(X(v) + 1, 1, -1)$ to $(X(v) + 1, 1, Z(v_n) + 1)$
- South from $(X(v) - 1, -1, -1)$ to $(X(v) - 1, -1, Z(v_n) + 1)$
- East from $(X(v) + 2, -1, -1)$ to $(X(v) + 2, -1, Z(v_n) + 1)$
- West from $(X(v) - 2, 1, -1)$ to $(X(v) - 2, 1, Z(v_n) + 1)$
- Bottom from $(X(v) + 3, 0, -1)$ to $(X(v) + 3, 0, Z(v_n) + 1)$
- Top from $(X(v) - 3, 0, -1)$ to $(X(v) - 3, 0, Z(v_n) + 1)$

The routing of an edge from vertex v to w follows the standard pattern: from the specified port of v to the associated pedestal, up or down the pedestal to the associated layer, out to the assigned lane, along the lane to w 's airspace, into the specified pillar, up (if v is before w) or down (if v is after w) that pillar a correction to align with the port, and finally into the specified port of w . A detailed listing of all port-to-port connections is contained in Appendix D.

Theorem 3 *The algorithm produces a 3D orthogonal drawing of any directed graph with n vertices of maximum degree at most 6, in a bounding box of $7n \times 7 \times 3n$ and with 6 bends per edge.*

Proof: Refer to vertex v in Figure 8 and imagine an arbitrary *incoming* edge to v . The edge runs along one of the four lanes (note that the North and Top lanes and the South and Bottom lanes are actually on different layers). The edge turns and joins the appropriate pillar, climbs the pillar and enters v . Note that a route using the Top pillar safely passes over the West pedestal and a route using the Bottom pillar safely passes under the East pedestal. No pair of incoming edges can intersect and no incoming edge intersects an outgoing edge.

Outgoing edges from v intersect no pillars (except West and East which safely intersect their own pillars) on their way to their assigned lanes. Note that three pairs share levels but each pair has separate lanes, and thus do not intersect. Since no lane intersects a pillar, there can be no intersections caused by intervening vertices.

□

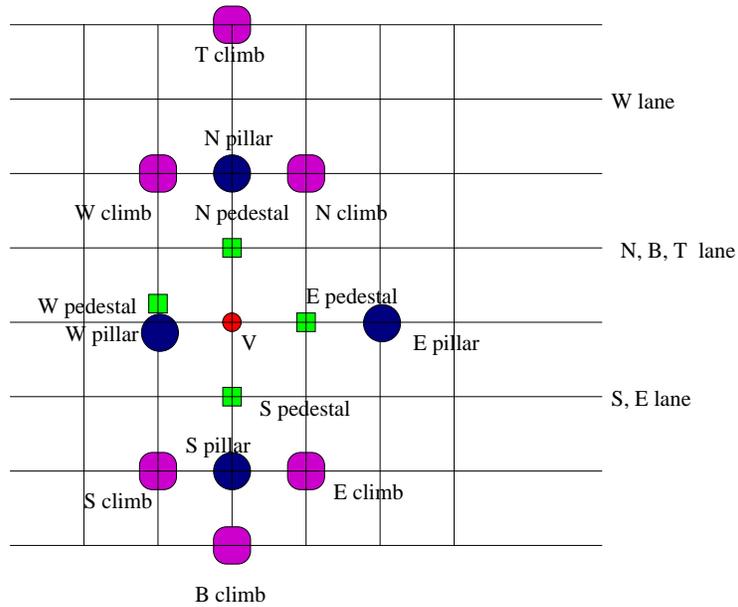


Figure 9: The 5 bend case

3.3 Achieving 5 Bends per Edge

The same basic format of the staircase models can be modified to produce a drawing with at most 5 bends per edge. In addition to using the region above the staircase, a new set of $2n$ planes called *under-planes* are introduced under the staircase. As a consequence of the under-plane construction, some of the dynamic nature of the drawing strategy is sacrificed. Although insertion and deletion of edges still takes $O(1)$ time, vertex insertions and deletions require $O(n)$ time. Note that in a *relative-change scenario* [8], such changes can be measured as $O(1)$; in this paper we assume the stronger *no-change* scenario.

In addition to the new construct of underplanes, we introduce **climbs** which extend only in the upper-airspace of v . Refer to Figure 9. These climbs are similar to the upper-airspace pillars of the digraph drawing; they are used in some routings instead of pedestals to climb to the level of w for routing at that level.

All routings are from a vertex v lower on the staircase to a vertex w that is higher. For the 36 possible cases there are three general types of routes in this model:

- using a pillar down to v 's underplane, across to w 's airspace and up its pillar (the 4 cases of: B to S, B to T, E to N, S to N),
- using a climb to a layer of w , and then across to w (the 10 cases of routes into T or S of w),

- using a pedestal to a layer of v , across to w 's airspace, up a pillar and into w (the remaining 22 cases).

The staircase is formed with vertex v_i placed at $(4i, 0, 4i)$. The size of each neighborhood $N(v)$ is $4 \times 8 \times 4$. As in the previous constructions, each vertex has a set of layers on which some local routing takes place. The four layers associated with a vertex v are: the **Top layer** located on the plane $Z = Z(v) + 2$, the **North, South layer** located on the plane $Z = Z(v) + 1$, the **East, West layer** located on the plane $Z = Z(v)$, and the **Bottom layer** located on the plane $Z = Z(v) - 1$.

The pedestals of vertex v are used by all routings into the W, E, B, and N ports of w except the 2 special cases of E to N and S to N. The **North pedestal** is of length 1 and extends from $(X(v), 1, Z(v))$ to $(X(v), 1, Z(v) + 1)$. The **South pedestal** is of length 1 and extends from $(X(v), -1, Z(v))$ to $(X(v), -1, Z(v) + 1)$. The **East** and **West pedestals** are degenerate points at $(X(v) + 1, 0, Z(v))$ and $(X(v) - 1, 0, Z(v))$ respectively. The **Bottom pedestal** is of length 1 and extends from $(X(v), 0, Z(v))$ to $(X(v), 0, Z(v) - 1)$. The **Top pedestal** is of length 2 and extends from $(X(v), 0, Z(v))$ to $(X(v), 0, Z(v) + 2)$.

The underplanes of the graph start at plane $Z = -2$ and extend downwards. There are two planes per vertex in the graph and are consecutive. The underplanes are arranged so that vertex v_n 's (highest on the staircase) under-planes are at $Z = -2$ and $Z = -3$. The underplanes for vertex v_i lie at $Z = -2(n-i) - 2$ and $Z = -2(n-i) - 3$. Only a portion of each underplane is actually used and is termed the **under-layer**; it extends from $X(v_n)$ to $X(v)$. The under-layers also form a staircase shape with the longest under-layers being the bottom planes of the graph. The underplane (and thus under-layer) positions are dependent on the number of vertices in the graph and thus must change when a vertex is inserted or deleted. Since there are $2n$ underplanes, this vertical shifting of the planes and the resultant lengthening or shortening of the routings that use these underplanes results in the $O(n)$ time penalty for vertex insertion or deletion.

The **North climb** extends from $(X(v) + 1, 2, Z(v))$ to $(X(v) + 1, 2, Z(v_n) + 2)$. The **South climb** extends from $(X(v) - 1, -2, Z(v))$ to $(X(v) - 1, -2, Z(v_n) + 2)$. The **East climb** extends from $(X(v) + 1, -2, Z(v))$ to $(X(v) + 1, -2, Z(v_n) + 2)$. The **West climb** extends from $(X(v) - 1, 2, Z(v))$ to $(X(v) - 1, 2, Z(v_n) + 2)$. The **Bottom climb** extends from $(X(v), -3, -2n)$ to $(X(v), -3, Z(v_n) + 2)$; this is a special case that runs from the lowest underplane of vertex v to the top of the drawing. The **Top climb** extends from $(X(v), 3, Z(v) + 2)$ to $(X(v), 3, Z(v_n) + 2)$.

The climbs of vertex v are used for all routes into the T and S ports of w except the 2 special cases of B to T and B to S. For a route to the South port of w , the edge leaves v and with one bend joins to the climb associated with v 's port, ascends the climb to the level of w , joins the S lane, traverses this lane to the base of w 's pedestal and finally enters w at the South port. The routings from v to the Top port of w are similar except the lane used is on the plane $Y = 0$, which can be safely used since at w 's level this lane intersects no intervening pillars.

The pillars of v have two purposes, namely for some routes into v and also for the 4 special case routes that use the underplanes. This causes no conflict since a port can only be used in one direction at a time (either to, or from another port). Some pillars need to extend to the bottommost underplane (located at $Z = -2n - 1$), and others extend only to the lowest plane of the staircase $Z = -1$. The **North pillar** extends from $(X(v), 2, -1)$ to $(X(v), 2, Z(v))$. The **East pillar** extends from $(X(v) + 2, 0, -2n - 1)$ to $(X(v) + 2, 0, Z(v))$ and is used to reach the bottom underplane of v . The **West pillar** extends from $(X(v) - 1, 0, -1)$ to $(X(v) - 1, 0, Z(v))$. The **Bottom pillar** extends from $(X(v), 0, -2n - 1)$ to $(X(v), 0, Z(v))$ and is used to reach the bottom underplane of v . The **South pillar** extends from $(X(v), -2, -2n)$ to $(X(v), -2, Z(v))$, and is only used to reach the topmost underplane associated with v . Note that the West pillar and West pedestal safely overlap. However, it is not possible to align the East pillar and pedestal in a similar fashion.

The edge routings are relatively elaborate and are listed in Appendix E. We also refer the interested reader to a series of web pages associated with this paper that describe the routings in more detail. The proof that no pair of edges intersects involves a lengthy case study since there are 3 general types of routes. Here we simply note that each edge has exactly 5 bends, but that self loops are not possible. A bounding box of dimensions $4n \times 8 \times 8n$ contains the drawing.

4 Implementation and Experimental Results

One important feature of the drawing strategy described in this paper is its simplicity. An implementation of the drawing is available in a package of 3-dimensional drawing tools, called OrthoPak, from:

www.cs.uleth.ca/~wismath/packages. This package was written in C++ and uses the LEDA library extensively. It runs under Solaris 2.6 or Linux, and produces either a VRML world that can be examined with an appropriate browser, or a PovRay file that can subsequently produce a static ray traced image of the drawing. OrthoPak is free for research and teaching purposes and is part of a larger suite of research tools developed at the University of Lethbridge.

A test suite of graphs for 3-D orthogonal drawing was described in [4]. Our time trials indicate that our staircase drawing strategy is very efficient: 7.013 seconds for the entire test suite of 1820 graphs containing from 5 to 95 vertices, running on a SPARC 5. On average, our staircase drawing had between 4.71 and 4.83 bends per edge over the entire distribution of graphs. Since our algorithm does not pre-process the graph, most of the time is spent on simple I/O operations.

5 Conclusion and Open Problems

A 3-dimensional orthogonal drawing of graphs with n vertices of degree at most 6 was presented in which each edge is routed with at most 6 bends in a volume

bounded by $O(n^2)$. The technique is fully dynamic, allowing insertion and deletion of edges and vertices in $O(1)$ time. Multiple edges and self-loops are permitted. Variants of the drawing strategy were also presented which may be more suitable in other contexts; for example the number of bends per edge can be reduced to 5 with the same volume bounds. It is an open problem to establish whether a volume of $O(n^2)$ is achievable with at most 4 bends per edge.

An implementation of the drawing strategy is available from the authors' web site. Demonstration VRML worlds, 2 short videos and several supporting web pages describing the techniques are supplied on the web page associated with this paper and maintained by the journal.

If the drawing strategy is not used in a dynamic setting then some pre-processing of the graph is possible and the number of bends can be reduced in the average case. Alternately, special properties of the graph may also be exploitable to reduce the number of bends. For example, trees of maximum degree 6 can be embedded with our staircase construction with no more than 5 bends. An interesting open problem is to determine if there are other classes of graphs that can be drawn using this strategy with a minimal amount of pre-processing.

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A Appendix: Routings - Staircase model

All 36 port-to-port routings from a vertex v to vertex w will now be enumerated. In each case, the initial and terminating points of the route are $(X(v), Y(v) = 0, Z(v))$ and $(X(w), Y(w) = 0, Z(w))$ respectively.

- $N \rightarrow S$ (6 bends):
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 2) \rightarrow (X(w) - 1, 1, Z(v) + 2) \rightarrow$
 $(X(w) - 1, -2, Z(v) + 2) \rightarrow (X(w), -2, Z(v) + 2) \rightarrow (X(w), -2, Z(w))$
- $N \rightarrow N$ (5 bends):
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 2) \rightarrow (X(w), 1, Z(v) + 2) \rightarrow$
 $(X(w), +2, Z(v) + 2) \rightarrow (X(w), +2, Z(w))$
- $N \rightarrow E$ (5 bends):
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 2) \rightarrow (X(w) + 2, 1, Z(v) + 2) \rightarrow$
 $(X(w) + 2, 0, Z(v) + 2) \rightarrow (X(w) + 2, 0, Z(w))$
- $N \rightarrow W$ (5 bends):
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 2) \rightarrow (X(w) - 2, 1, Z(v) + 2) \rightarrow$
 $(X(w) - 2, 0, Z(v) + 2) \rightarrow (X(w) - 2, 0, Z(w))$
- $N \rightarrow T$ (6 bends):
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 2) \rightarrow (X(w) - 3, 1, Z(v) + 2) \rightarrow$
 $(X(w) - 3, 0, Z(v) + 2) \rightarrow (X(w) - 3, 0, Z(w) + 3) \rightarrow (X(w), 0, Z(w) + 3)$
- $N \rightarrow B$ (4 bends):
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 2) \rightarrow (X(w), 1, Z(v) + 2) \rightarrow$
 $(X(w), 0, Z(v) + 2)$

- $S \rightarrow N$ (6 bends):
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w) - 1, -1, Z(v) + 1) \rightarrow$
 $(X(w) - 1, 2, Z(v) + 1) \rightarrow (X(w), 2, Z(v) + 1) \rightarrow (X(w), 2, Z(w))$
- $S \rightarrow S$ (5 bends):
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w), -1, Z(v) + 1) \rightarrow$
 $(X(w), -2, Z(v) + 1) \rightarrow (X(w), -2, Z(w))$
- $S \rightarrow E$ (5 bends):
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w) + 2, -1, Z(v) + 1) \rightarrow$
 $(X(w) + 2, 0, Z(v) + 1) \rightarrow (X(w) + 2, 0, Z(w))$
- $S \rightarrow W$ (5 bends):
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w) - 2, -1, Z(v) + 1) \rightarrow$
 $(X(w) - 2, 0, Z(v) + 1) \rightarrow (X(w) - 2, 0, Z(w))$
- $S \rightarrow T$ (6 bends):
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w) - 3, -1, Z(v) + 1) \rightarrow$
 $(X(w) - 3, 0, Z(v) + 1) \rightarrow (X(w) - 3, 0, Z(w) + 3) \rightarrow (X(w), 0, Z(w) + 3)$
- $S \rightarrow B$ (4 bends):
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w), -1, Z(v) + 1) \rightarrow$
 $(X(w), 0, Z(v) + 1)$
- $E \rightarrow N$ (5 bends):
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, 1, Z(v)) \rightarrow (X(w), 1, Z(v)) \rightarrow$
 $(X(w), 2, Z(v)) \rightarrow (X(w), 2, Z(w))$
- $E \rightarrow S$ (6 bends):
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, 1, Z(v)) \rightarrow (X(w) - 1, 1, Z(v)) \rightarrow$
 $(X(w) - 1, -2, Z(v)) \rightarrow (X(w), -2, Z(v)) \rightarrow (X(w), -2, Z(w))$
- $E \rightarrow E$ (5 bends):
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, 1, Z(v)) \rightarrow (X(w) + 2, 1, Z(v)) \rightarrow$
 $(X(w) + 2, 0, Z(v)) \rightarrow (X(w) + 2, 0, Z(w))$
- $E \rightarrow W$ (5 bends):
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, 1, Z(v)) \rightarrow (X(w) - 2, 1, Z(v)) \rightarrow$
 $(X(w) - 2, 0, Z(v)) \rightarrow (X(w) - 2, 0, Z(w))$
- $E \rightarrow T$ (6 bends):
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, 1, Z(v)) \rightarrow (X(w) - 3, 1, Z(v)) \rightarrow$
 $(X(w) - 3, 0, Z(v)) \rightarrow (X(w) - 3, 0, Z(w) + 3) \rightarrow (X(w), 0, Z(w) + 3)$
- $E \rightarrow B$ (4 bends):
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, 1, Z(v)) \rightarrow (X(w), 1, Z(v)) \rightarrow$
 $(X(w), 0, Z(v))$
- $W \rightarrow N$ (5 bends):
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, 3, Z(v)) \rightarrow (X(w), 3, Z(v)) \rightarrow$
 $(X(w), 2, Z(v)) \rightarrow (X(w), 2, Z(w))$

- $W \rightarrow S$ (5 bends):
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, -3, Z(v)) \rightarrow (X(w), -3, Z(v)) \rightarrow$
 $(X(w), -2, Z(v)) \rightarrow (X(w), -2, Z(w))$
- $W \rightarrow W$ (5 bends):
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, -3, Z(v)) \rightarrow (X(w) - 2, -3, Z(v)) \rightarrow$
 $(X(w) - 2, 0, Z(v)) \rightarrow (X(w) - 2, 0, Z(w))$
- $W \rightarrow E$ (5 bends):
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, -3, Z(v)) \rightarrow (X(w) + 2, -3, Z(v)) \rightarrow$
 $(X(w) + 2, 0, Z(v)) \rightarrow (X(w) + 2, 0, Z(w))$
- $W \rightarrow T$ (6 bends):
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, -3, Z(v)) \rightarrow (X(w) - 3, -3, Z(v)) \rightarrow$
 $(X(w) - 3, 0, Z(v)) \rightarrow (X(w) - 3, 0, Z(w) + 3) \rightarrow (X(w), 0, Z(w) + 3)$
- $W \rightarrow B$ (5 bends):
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, -3, Z(v)) \rightarrow (X(w) + 1, -3, Z(v)) \rightarrow$
 $(X(w) + 1, 0, Z(v)) \rightarrow (X(w), 0, Z(v))$
- $T \rightarrow N$ (5 bends):
 $(X(v), 0, Z(v) + 3) \rightarrow (X(v), +1, Z(v) + 3) \rightarrow (X(w), 1, Z(v) + 3) \rightarrow$
 $(X(w), 2, Z(v) + 3) \rightarrow (X(w), 2, Z(w))$
- $T \rightarrow S$ (5 bends):
 $(X(v), 0, Z(v) + 3) \rightarrow (X(v), -1, Z(v) + 3) \rightarrow (X(w), -1, Z(v) + 3) \rightarrow$
 $(X(w), -2, Z(v) + 3) \rightarrow (X(w), -2, Z(w))$
- $T \rightarrow E$ (5 bends):
 $(X(v), 0, Z(v) + 3) \rightarrow (X(v), +1, Z(v) + 3) \rightarrow (X(w) + 2, 1, Z(v) + 3) \rightarrow$
 $(X(w) + 2, 0, Z(v) + 3) \rightarrow (X(w) + 2, 0, Z(w))$
- $T \rightarrow W$ (5 bends):
 $(X(v), 0, Z(v) + 3) \rightarrow (X(v), -1, Z(v) + 3) \rightarrow (X(w) - 2, -1, Z(v) + 3) \rightarrow$
 $(X(w) - 2, 0, Z(v) + 3) \rightarrow (X(w) - 2, 0, Z(w))$
- $T \rightarrow B$ (4 bends):
 $(X(v), 0, Z(v) + 3) \rightarrow (X(v), -1, Z(v) + 3) \rightarrow (X(w), -1, Z(v) + 3) \rightarrow$
 $(X(w), 0, Z(v) + 3)$
- $T \rightarrow T$ (6 bends):
 $(X(v), 0, Z(v) + 3) \rightarrow (X(v), +1, Z(v) + 3) \rightarrow (X(w) - 3, 1, Z(v) + 3) \rightarrow$
 $(X(w) - 3, 0, Z(v) + 3) \rightarrow (X(w) - 3, 0, Z(w) + 3) \rightarrow (X(w), 0, Z(w) + 3)$
- $B \rightarrow N$ (5 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), +1, Z(v) - 1) \rightarrow (X(w), 1, Z(v) - 1) \rightarrow$
 $(X(w), 2, Z(v) - 1) \rightarrow (X(w), 2, Z(w))$
- $B \rightarrow S$ (5 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -1, Z(v) - 1) \rightarrow (X(w), -1, Z(v) - 1) \rightarrow$
 $(X(w), -2, Z(v) - 1) \rightarrow (X(w), -2, Z(w))$

- $B \rightarrow E$ (5 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), +1, Z(v) - 1) \rightarrow (X(w) + 2, 1, Z(v) - 1) \rightarrow$
 $(X(w) + 2, 0, Z(v) - 1) \rightarrow (X(w) + 2, 0, Z(w))$
- $B \rightarrow W$ (5 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -1, Z(v) - 1) \rightarrow (X(w) - 2, -1, Z(v) - 1) \rightarrow$
 $(X(w) - 2, 0, Z(v) - 1) \rightarrow (X(w) - 2, 0, Z(w))$
- $B \rightarrow T$ (6 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), +1, Z(v) - 1) \rightarrow (X(w) - 3, 1, Z(v) - 1) \rightarrow$
 $(X(w) - 3, 0, Z(v) - 1) \rightarrow (X(w) - 3, 0, Z(w) + 3) \rightarrow (X(w), 0, Z(w) + 3)$
- $B \rightarrow B$ (4 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -1, Z(v) - 1) \rightarrow (X(w), -1, Z(v) - 1) \rightarrow$
 $(X(w), 0, Z(v) - 1)$

B Appendix: Self-Loop Routings - Staircase Model

- $T \rightarrow W$ (3 bends):
 $(X(v), 0, Z(v) + 3) \rightarrow (X(v) - 2, 0, Z(v) + 3) \rightarrow (X(v) - 2, 0, Z(v))$
- Similarly, $T \rightarrow N, S, E$.
- $W \rightarrow E$ (4 bends):
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, -3, Z(v)) \rightarrow (X(v) + 2, -3, Z(v)) \rightarrow$
 $(X(v) + 2, 0, Z(v))$
- $W \rightarrow S$ (3 bends):
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, -3, Z(v)) \rightarrow (X(v), -3, Z(v))$
- $B \rightarrow W$ (3 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v) - 2, 0, Z(v) - 1) \rightarrow (X(v) - 2, 0, Z(v))$
- Similarly, $B \rightarrow N, S, E$.
- $B \rightarrow T$ (6 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -1, Z(v) - 1) \rightarrow (X(v) - 3, -1, Z(v) - 1) \rightarrow$
 $(X(v) - 3, 0, Z(v) - 1) \rightarrow (X(v) - 3, 0, Z(v) + 3) \rightarrow (X(v), 0, Z(v) + 3)$
- $N \rightarrow E$ (5 bends):
 $(X(v), +1, Z(v)) \rightarrow (X(v), +1, Z(v) + 2) \rightarrow (X(v) + 2, 1, Z(v) + 2) \rightarrow$
 $(X(v) + 2, 0, Z(v) + 2) \rightarrow (X(v) + 2, 0, Z(v))$
- Similarly, $N \rightarrow W$.
- $N \rightarrow S$ (6 bends):
 $(X(v), +1, Z(v)) \rightarrow (X(v), +1, Z(v) + 2) \rightarrow (X(v) + 1, 1, Z(v) + 2) \rightarrow$
 $(X(v) + 1, -2, Z(v) + 2) \rightarrow (X(v), -2, Z(v) + 2) \rightarrow (X(v), -2, Z(v))$

- $S \rightarrow E$ (5 bends):
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(v) + 2, -1, Z(v) + 1) \rightarrow$
 $(X(v) + 2, 0, Z(v) + 1) \rightarrow (X(v) + 2, 0, Z(v))$

C Appendix: Routings - Orthogonal Spiral model

The 36 port-to-port routings from a vertex v to vertex w will now be enumerated. The offsets for each of the 6 layers are denoted as: T', N', S', E', W', B' and have the values 5, 4, 3, 2, 1, -1 respectively. In each case, the initial and terminating points of the route are $(X(v), Y(v), Z(v))$ and $(X(w), Y(w), Z(w))$; the computation of these coordinates is presented in Section 3.1.

- $N \rightarrow S$ (6 bends):
 $(X(v), Y(v) + 1, Z(v)) \rightarrow (X(v), Y(v) + 1, Z(v) + N') \rightarrow$
 $(X(w) + 2, Y(v) + 1, Z(v) + N') \rightarrow (X(w) + 2, Y(w) - 3, Z(v) + N') \rightarrow$
 $(X(w), Y(w) - 3, Z(v) + N') \rightarrow (X(w), Y(w) - 3, Z(w))$
- $N \rightarrow N$ (6 bends):
 $(X(v), Y(v) + 1, Z(v)) \rightarrow (X(v), Y(v) + 1, Z(v) + N') \rightarrow$
 $(X(w) + 2, Y(v) + 1, Z(v) + N') \rightarrow (X(w) + 2, Y(w) + 3, Z(v) + N') \rightarrow$
 $(X(w), Y(w) + 3, Z(v) + N') \rightarrow (X(w), Y(w) + 3, Z(w))$
- $N \rightarrow E$ (6 bends):
 $(X(v), Y(v) + 1, Z(v)) \rightarrow (X(v), Y(v) + 1, Z(v) + N') \rightarrow$
 $(X(w) + 2, Y(v) + 1, Z(v) + N') \rightarrow (X(w) + 2, Y(w), Z(v) + N') \rightarrow$
 $(X(w) + 3, Y(w), Z(v) + N') \rightarrow (X(w) + 3, Y(w), Z(w))$
- $N \rightarrow W$ (6 bends):
 $(X(v), Y(v) + 1, Z(v)) \rightarrow (X(v), Y(v) + 1, Z(v) + N') \rightarrow$
 $(X(w) - 2, Y(v) + 1, Z(v) + N') \rightarrow (X(w) - 2, Y(w), Z(v) + N') \rightarrow$
 $(X(w) - 3, Y(w), Z(v) + N') \rightarrow (X(w) - 3, Y(w), Z(w))$
- $N \rightarrow T$ (7 bends):
 $(X(v), Y(v) + 1, Z(v)) \rightarrow (X(v), Y(v) + 1, Z(v) + N') \rightarrow$
 $(X(w) + 2, Y(v) + 1, Z(v) + N') \rightarrow (X(w) + 2, Y(w) + 4, Z(v) + N') \rightarrow$
 $(X(w), Y(w) + 4, Z(v) + N') \rightarrow (X(w), Y(w) + 4, Z(w) + T') \rightarrow$
 $(X(w), Y(w), Z(w) + T')$
- $N \rightarrow B$ (5 bends):
 $(X(v), Y(v) + 1, Z(v)) \rightarrow (X(v), Y(v) + 1, Z(v) + N') \rightarrow$
 $(X(w) + 2, Y(v) + 1, Z(v) + N') \rightarrow (X(w) + 2, Y(w), Z(v) + N') \rightarrow$
 $(X(w), Y(w), Z(v) + N')$
- $S \rightarrow N$ (6 bends):
 $(X(v), Y(v) - 1, Z(v)) \rightarrow (X(v), Y(v) - 1, Z(v) + S') \rightarrow$
 $(X(w) - 2, Y(v) - 1, Z(v) + S') \rightarrow (X(w) - 2, Y(w) + 3, Z(v) + S') \rightarrow$
 $(X(w), Y(w) + 3, Z(v) + S') \rightarrow (X(w), Y(w) + 3, Z(w))$

- $S \rightarrow S$ (6 bends):
 $(X(v), Y(v) - 1, Z(v)) \rightarrow (X(v), Y(v) - 1, Z(v) + S') \rightarrow$
 $(X(w) - 2, Y(v) - 1, Z(v) + S') \rightarrow (X(w) - 2, Y(w) - 3, Z(v) + S') \rightarrow$
 $(X(w), Y(w) - 3, Z(v) + S') \rightarrow (X(w), Y(w) - 3, Z(w))$
- $S \rightarrow E$ (6 bends):
 $(X(v), Y(v) - 1, Z(v)) \rightarrow (X(v), Y(v) - 1, Z(v) + S') \rightarrow$
 $(X(w) + 2, Y(v) - 1, Z(v) + S') \rightarrow (X(w) + 2, Y(w), Z(v) + S') \rightarrow$
 $(X(w) + 3, Y(w), Z(v) + S') \rightarrow (X(w) + 3, Y(w), Z(w))$
- $S \rightarrow W$ (6 bends):
 $(X(v), Y(v) - 1, Z(v)) \rightarrow (X(v), Y(v) - 1, Z(v) + S') \rightarrow$
 $(X(w) - 2, Y(v) - 1, Z(v) + S') \rightarrow (X(w) - 2, Y(w), Z(v) + S') \rightarrow$
 $(X(w) - 3, Y(w), Z(v) + S') \rightarrow (X(w) - 3, Y(w), Z(w))$
- $S \rightarrow T$ (7 bends):
 $(X(v), Y(v) - 1, Z(v)) \rightarrow (X(v), Y(v) - 1, Z(v) + S') \rightarrow$
 $(X(w) - 2, Y(v) - 1, Z(v) + S') \rightarrow (X(w) - 2, Y(w) + 4, Z(v) + S') \rightarrow$
 $(X(w), Y(w) + 4, Z(v) + S') \rightarrow (X(w), Y(w) + 4, Z(w) + T') \rightarrow$
 $(X(w), Y(w), Z(w) + T')$
- $S \rightarrow B$ (5 bends):
 $(X(v), Y(v) - 1, Z(v)) \rightarrow (X(v), Y(v) - 1, Z(v) + S') \rightarrow$
 $(X(w) - 2, Y(v) - 1, Z(v) + S') \rightarrow (X(w) - 2, Y(w), Z(v) + S') \rightarrow$
 $(X(w), Y(w), Z(v) + S')$
- $E \rightarrow N$ (6 bends):
 $(X(v) + 1, Y(v), Z(v)) \rightarrow (X(v) + 1, Y(v), Z(v) + E') \rightarrow$
 $(X(v) + 1, Y(w) + 2, Z(v) + E') \rightarrow (X(w), Y(w) + 2, Z(v) + E') \rightarrow$
 $(X(w), Y(w) + 3, Z(v) + E') \rightarrow (X(w), Y(w) + 3, Z(w))$
- $E \rightarrow S$ (6 bends):
 $(X(v) + 1, Y(v), Z(v)) \rightarrow (X(v) + 1, Y(v), Z(v) + E') \rightarrow$
 $(X(v) + 1, Y(w) - 2, Z(v) + E') \rightarrow (X(w), Y(w) - 2, Z(v) + E') \rightarrow$
 $(X(w), Y(w) - 3, Z(v) + E') \rightarrow (X(w), Y(w) - 3, Z(w))$
- $E \rightarrow E$ (6 bends):
 $(X(v) + 1, Y(v), Z(v)) \rightarrow (X(v) + 1, Y(v), Z(v) + E') \rightarrow$
 $(X(v) + 1, Y(w) + 2, Z(v) + E') \rightarrow (X(w) + 3, Y(w) + 2, Z(v) + E') \rightarrow$
 $(X(w) + 3, Y(w), Z(v) + E') \rightarrow (X(w) + 3, Y(w), Z(w))$
- $E \rightarrow W$ (6 bends):
 $(X(v) + 1, Y(v), Z(v)) \rightarrow (X(v) + 1, Y(v), Z(v) + E') \rightarrow$
 $(X(v) + 1, Y(w) + 2, Z(v) + E') \rightarrow (X(w) - 3, Y(w) + 2, Z(v) + E') \rightarrow$
 $(X(w) - 3, Y(w), Z(v) + E') \rightarrow (X(w) - 3, Y(w), Z(w))$
- $E \rightarrow T$ (7 bends):
 $(X(v) + 1, Y(v), Z(v)) \rightarrow (X(v) + 1, Y(v), Z(v) + E') \rightarrow$
 $(X(v) + 1, Y(w) + 2, Z(v) + E') \rightarrow (X(w) - 4, Y(w) + 2, Z(v) + E') \rightarrow$

$$(X(w) - 4, Y(w), Z(v) + E') \rightarrow (X(w) - 4, Y(w), Z(w) + T') \rightarrow (X(w), Y(w), Z(w) + T')$$

- $E \rightarrow B$ (5 bends):

$$(X(v) + 1, Y(v), Z(v)) \rightarrow (X(v) + 1, Y(v), Z(v) + E') \rightarrow (X(v) + 1, Y(w) + 2, Z(v) + E') \rightarrow (X(w), Y(w) + 2, Z(v) + E') \rightarrow (X(w), Y(w), Z(v) + E')$$
- $W \rightarrow N$ (6 bends):

$$(X(v) - 1, Y(v), Z(v)) \rightarrow (X(v) - 1, Y(v), Z(v) + W') \rightarrow (X(v) - 1, Y(w) + 2, Z(v) + W') \rightarrow (X(w), Y(w) + 2, Z(v) + W') \rightarrow (X(w), Y(w) + 3, Z(v) + W') \rightarrow (X(w), Y(w) + 3, Z(w))$$
- $W \rightarrow S$ (6 bends):

$$(X(v) - 1, Y(v), Z(v)) \rightarrow (X(v) - 1, Y(v), Z(v) + W') \rightarrow (X(v) - 1, Y(w) - 2, Z(v) + W') \rightarrow (X(w), Y(w) - 2, Z(v) + W') \rightarrow (X(w), Y(w) - 3, Z(v) + W') \rightarrow (X(w), Y(w) - 3, Z(w))$$
- $W \rightarrow W$ (6 bends):

$$(X(v) - 1, Y(v), Z(v)) \rightarrow (X(v) - 1, Y(v), Z(v) + W') \rightarrow (X(v) - 1, Y(w) - 2, Z(v) + W') \rightarrow (X(w) - 3, Y(w) - 2, Z(v) + W') \rightarrow (X(w) - 3, Y(w), Z(v) + W') \rightarrow (X(w) - 3, Y(w), Z(w))$$
- $W \rightarrow E$ (6 bends):

$$(X(v) - 1, Y(v), Z(v)) \rightarrow (X(v) - 1, Y(v), Z(v) + W') \rightarrow (X(v) - 1, Y(w) - 2, Z(v) + W') \rightarrow (X(w) + 3, Y(w) - 2, Z(v) + W') \rightarrow (X(w) + 3, Y(w), Z(v) + W') \rightarrow (X(w) + 3, Y(w), Z(w))$$
- $W \rightarrow T$ (7 bends):

$$(X(v) - 1, Y(v), Z(v)) \rightarrow (X(v) - 1, Y(v), Z(v) + W') \rightarrow (X(v) - 1, Y(w) - 2, Z(v) + W') \rightarrow (X(w) + 4, Y(w) - 2, Z(v) + W') \rightarrow (X(w) + 4, Y(w), Z(v) + W') \rightarrow (X(w) + 4, Y(w), Z(w) + T') \rightarrow (X(w), Y(w), Z(w) + T')$$
- $W \rightarrow B$ (5 bends):

$$(X(v) - 1, Y(v), Z(v)) \rightarrow (X(v) - 1, Y(v), Z(v) + W') \rightarrow (X(v) - 1, Y(w) - 2, Z(v) + W') \rightarrow (X(w), Y(w) - 2, Z(v) + W') \rightarrow (X(w), Y(w), Z(v) + W')$$
- $T \rightarrow N$ (6 bends):

$$(X(v), Y(v), Z(v) + T') \rightarrow (X(v) + 2, Y(v), Z(v) + T') \rightarrow (X(v) + 2, Y(w) + 2, Z(v) + T') \rightarrow (X(w), Y(w) + 2, Z(v) + T') \rightarrow (X(w), Y(w) + 3, Z(v) + T') \rightarrow (X(w), Y(w) + 3, Z(w))$$
- $T \rightarrow S$ (6 bends):

$$(X(v), Y(v), Z(v) + T') \rightarrow (X(v) + 2, Y(v), Z(v) + T') \rightarrow (X(v) + 2, Y(w) - 2, Z(v) + T') \rightarrow (X(w), Y(w) - 2, Z(v) + T') \rightarrow (X(w), Y(w) - 3, Z(v) + T') \rightarrow (X(w), Y(w) - 3, Z(w))$$

- $T \rightarrow E$ (6 bends):
 $(X(v), Y(v), Z(v) + T') \rightarrow (X(v), Y(v) + 2, Z(v) + T') \rightarrow$
 $(X(w) + 2, Y(v) + 2, Z(v) + T') \rightarrow (X(w) + 2, Y(w), Z(v) + T') \rightarrow$
 $(X(w) + 3, Y(w), Z(v) + T') \rightarrow (X(w) + 3, Y(w), Z(w))$
- $T \rightarrow W$ (6 bends):
 $(X(v), Y(v), Z(v) + T') \rightarrow (X(v), Y(v) + 2, Z(v) + T') \rightarrow$
 $(X(w) - 2, Y(v) + 2, Z(v) + T') \rightarrow (X(w) - 2, Y(w), Z(v) + T') \rightarrow$
 $(X(w) - 3, Y(w), Z(v) + T') \rightarrow (X(w) - 3, Y(w), Z(w))$
- $T \rightarrow B$ (5 bends):
 $(X(v), Y(v), Z(v) + T') \rightarrow (X(v), Y(v) + 2, Z(v) + T') \rightarrow$
 $(X(w) - 2, Y(v) + 2, Z(v) + T') \rightarrow (X(w) - 2, Y(w), Z(v) + T') \rightarrow$
 $(X(w), Y(w), Z(v) + T')$
- $T \rightarrow T$ (7 bends):
 $(X(v), Y(v), Z(v) + T') \rightarrow (X(v), Y(v) + 2, Z(v) + T') \rightarrow$
 $(X(w) - 2, Y(v) + 2, Z(v) + T') \rightarrow (X(w) - 2, Y(w) + 4, Z(v) + T') \rightarrow$
 $(X(w), Y(w) + 4, Z(v) + T') \rightarrow (X(w), Y(w) + 4, Z(w) + T') \rightarrow$
 $(X(w), Y(w), Z(w) + T')$
- $B \rightarrow N$ (6 bends):
 $(X(v), Y(v), Z(v) + B') \rightarrow (X(v) + 2, Y(v), Z(v) + B') \rightarrow$
 $(X(v) + 2, Y(w) + 2, Z(v) + B') \rightarrow (X(w), Y(w) + 2, Z(v) + B') \rightarrow$
 $(X(w), Y(w) + 3, Z(v) + B') \rightarrow (X(w), Y(w) + 3, Z(w))$
- $B \rightarrow S$ (6 bends):
 $(X(v), Y(v), Z(v) + B') \rightarrow (X(v) + 2, Y(v), Z(v) + B') \rightarrow$
 $(X(v) + 2, Y(w) - 2, Z(v) + B') \rightarrow (X(w), Y(w) - 2, Z(v) + B') \rightarrow$
 $(X(w), Y(w) - 3, Z(v) + B') \rightarrow (X(w), Y(w) - 3, Z(w))$
- $B \rightarrow E$ (6 bends):
 $(X(v), Y(v), Z(v) + B') \rightarrow (X(v), Y(v) - 2, Z(v) + B') \rightarrow$
 $(X(w) + 2, Y(v) - 2, Z(v) + B') \rightarrow (X(w) + 2, Y(w), Z(v) + B') \rightarrow$
 $(X(w) + 3, Y(w), Z(v) + B') \rightarrow (X(w) + 3, Y(w), Z(w))$
- $B \rightarrow W$ (6 bends):
 $(X(v), Y(v), Z(v) + B') \rightarrow (X(v), Y(v) - 2, Z(v) + B') \rightarrow$
 $(X(w) - 2, Y(v) - 2, Z(v) + B') \rightarrow (X(w) - 2, Y(w), Z(v) + B') \rightarrow$
 $(X(w) - 3, Y(w), Z(v) + B') \rightarrow (X(w) - 3, Y(w), Z(w))$
- $B \rightarrow T$ (7 bends):
 $(X(v), Y(v), Z(v) + B') \rightarrow (X(v), Y(v) - 2, Z(v) + B') \rightarrow$
 $(X(w) - 2, Y(v) - 2, Z(v) + B') \rightarrow (X(w) - 2, Y(w) - 4, Z(v) + B') \rightarrow$
 $(X(w), Y(w) - 4, Z(v) + B') \rightarrow (X(w), Y(w) - 4, Z(w) + T') \rightarrow$
 $(X(w), Y(w), Z(w) + T')$
- $B \rightarrow B$ (5 bends):
 $(X(v), Y(v), Z(v) + B') \rightarrow (X(v), Y(v) - 2, Z(v) + B') \rightarrow$

$$\begin{aligned} & (X(w) - 2, Y(v) - 2, Z(v) + B') \rightarrow (X(w) - 2, Y(w), Z(v) + B') \rightarrow \\ & (X(w), Y(w), Z(v) + B') \end{aligned}$$

D Appendix: Routings - Directed model

There are 72 port-to-port routings from a vertex v to vertex w in a directed drawing. Each of the following specifications represents 2 routings depending on whether v is above or below w on the staircase. In each case, the initial and terminating points of the route are $(X(v), Y(v) = 0, Z(v))$ and $(X(w), Y(w) = 0, Z(w))$ respectively and each edge contains exactly 6 bends.

- $N \rightarrow S$:
 $(X(v), 2, Z(v)) \rightarrow (X(v), 2, Z(v) - 1) \rightarrow (X(w) - 1, 2, Z(v) - 1) \rightarrow$
 $(X(w) - 1, -1, Z(v) - 1) \rightarrow (X(w) - 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $N \rightarrow N$:
 $(X(v), 2, Z(v)) \rightarrow (X(v), 2, Z(v) - 1) \rightarrow (X(w) + 1, 2, Z(v) - 1) \rightarrow$
 $(X(w) + 1, 1, Z(v) - 1) \rightarrow (X(w) + 1, 1, Z(w)) \rightarrow (X(w), 1, Z(w))$
- $N \rightarrow E$:
 $(X(v), 2, Z(v)) \rightarrow (X(v), 2, Z(v) - 1) \rightarrow (X(w) + 2, 2, Z(v) - 1) \rightarrow$
 $(X(w) + 2, -1, Z(v) - 1) \rightarrow (X(w) + 2, -1, Z(w)) \rightarrow (X(w) + 2, 0, Z(w))$
- $N \rightarrow W$:
 $(X(v), 2, Z(v)) \rightarrow (X(v), 2, Z(v) - 1) \rightarrow (X(w) - 2, 2, Z(v) - 1) \rightarrow$
 $(X(w) - 2, 1, Z(v) - 1) \rightarrow (X(w) - 2, 1, Z(w)) \rightarrow (X(w) - 2, 0, Z(w))$
- $N \rightarrow T$:
 $(X(v), 2, Z(v)) \rightarrow (X(v), 2, Z(v) - 1) \rightarrow (X(w) - 3, 2, Z(v) - 1) \rightarrow$
 $(X(w) - 3, 0, Z(v) - 1) \rightarrow (X(w) - 3, 0, Z(w) + 1) \rightarrow (X(w), 0, Z(w) + 1)$
- $N \rightarrow B$:
 $(X(v), 2, Z(v)) \rightarrow (X(v), 2, Z(v) - 1) \rightarrow (X(w) + 3, 2, Z(v) - 1) \rightarrow$
 $(X(w) + 3, 0, Z(v) - 1) \rightarrow (X(w) + 3, 0, Z(w) - 1) \rightarrow (X(w), 0, Z(w) - 1)$
- $S \rightarrow S$:
 $(X(v), -2, Z(v)) \rightarrow (X(v), -2, Z(v) + 1) \rightarrow (X(w) - 1, -2, Z(v) + 1) \rightarrow$
 $(X(w) - 1, -1, Z(v) + 1) \rightarrow (X(w) - 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $S \rightarrow N$:
 $(X(v), -2, Z(v)) \rightarrow (X(v), -2, Z(v) + 1) \rightarrow (X(w) + 1, -2, Z(v) + 1) \rightarrow$
 $(X(w) + 1, 1, Z(v) + 1) \rightarrow (X(w) + 1, 1, Z(w)) \rightarrow (X(w), 1, Z(w))$
- $S \rightarrow E$:
 $(X(v), -2, Z(v)) \rightarrow (X(v), -2, Z(v) + 1) \rightarrow (X(w) + 2, -2, Z(v) + 1) \rightarrow$
 $(X(w) + 2, -1, Z(v) + 1) \rightarrow (X(w) + 2, -1, Z(w)) \rightarrow (X(w) + 2, 0, Z(w))$

- $S \rightarrow W$:
 $(X(v), -2, Z(v)) \rightarrow (X(v), -2, Z(v) + 1) \rightarrow (X(w) - 2, -2, Z(v) + 1) \rightarrow$
 $(X(w) - 2, 1, Z(v) + 1) \rightarrow (X(w) - 2, 1, Z(w)) \rightarrow (X(w) - 2, 0, Z(w))$
- $S \rightarrow T$:
 $(X(v), -2, Z(v)) \rightarrow (X(v), -2, Z(v) + 1) \rightarrow (X(w) - 3, -2, Z(v) + 1) \rightarrow$
 $(X(w) - 3, 0, Z(v) + 1) \rightarrow (X(w) - 3, 0, Z(w) + 1) \rightarrow (X(w), 0, Z(w) + 1)$
- $S \rightarrow B$:
 $(X(v), -2, Z(v)) \rightarrow (X(v), -2, Z(v) + 1) \rightarrow (X(w) + 3, -2, Z(v) + 1) \rightarrow$
 $(X(w) + 3, 0, Z(v) + 1) \rightarrow (X(w) + 3, 0, Z(w) - 1) \rightarrow (X(w), 0, Z(w) - 1)$
- $E \rightarrow N$:
 $(X(v) + 2, 0, Z(v)) \rightarrow (X(v) + 2, -3, Z(v)) \rightarrow (X(w) + 1, -3, Z(v)) \rightarrow$
 $(X(w) + 1, 1, Z(v)) \rightarrow (X(w) + 1, 1, Z(w)) \rightarrow (X(w), 1, Z(w))$
- $E \rightarrow S$:
 $(X(v) + 2, 0, Z(v)) \rightarrow (X(v) + 2, -3, Z(v)) \rightarrow (X(w) - 1, -3, Z(v)) \rightarrow$
 $(X(w) - 1, -1, Z(v)) \rightarrow (X(w) - 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $E \rightarrow E$:
 $(X(v) + 2, 0, Z(v)) \rightarrow (X(v) + 2, -3, Z(v)) \rightarrow (X(w) + 2, -3, Z(v)) \rightarrow$
 $(X(w) + 2, -1, Z(v)) \rightarrow (X(w) + 2, -1, Z(w)) \rightarrow (X(w) + 2, 0, Z(w))$
- $E \rightarrow W$:
 $(X(v) + 2, 0, Z(v)) \rightarrow (X(v) + 2, -3, Z(v)) \rightarrow (X(w) - 2, -3, Z(v)) \rightarrow$
 $(X(w) - 2, 1, Z(v)) \rightarrow (X(w) - 2, 1, Z(w)) \rightarrow (X(w) - 2, 0, Z(w))$
- $E \rightarrow T$:
 $(X(v) + 2, 0, Z(v)) \rightarrow (X(v) + 2, -3, Z(v)) \rightarrow (X(w) - 3, -3, Z(v)) \rightarrow$
 $(X(w) - 3, 0, Z(v)) \rightarrow (X(w) - 3, 0, Z(w) + 1) \rightarrow (X(w), 0, Z(w) + 1)$
- $E \rightarrow B$:
 $(X(v) + 2, 0, Z(v)) \rightarrow (X(v) + 2, -3, Z(v)) \rightarrow (X(w) + 3, -3, Z(v)) \rightarrow$
 $(X(w) + 3, 0, Z(v)) \rightarrow (X(w) + 3, 0, Z(w) - 1) \rightarrow (X(w), 0, Z(w) - 1)$
- $W \rightarrow N$:
 $(X(v) - 2, 0, Z(v)) \rightarrow (X(v) - 2, 3, Z(v)) \rightarrow (X(w) + 1, 3, Z(v)) \rightarrow$
 $(X(w) + 1, 1, Z(v)) \rightarrow (X(w) + 1, 1, Z(w)) \rightarrow (X(w), 1, Z(w))$
- $W \rightarrow S$:
 $(X(v) - 2, 0, Z(v)) \rightarrow (X(v) - 2, 3, Z(v)) \rightarrow (X(w) - 1, 3, Z(v)) \rightarrow$
 $(X(w) - 1, -1, Z(v)) \rightarrow (X(w) - 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $W \rightarrow E$:
 $(X(v) - 2, 0, Z(v)) \rightarrow (X(v) - 2, 3, Z(v)) \rightarrow (X(w) + 2, 3, Z(v)) \rightarrow$
 $(X(w) + 2, -1, Z(v)) \rightarrow (X(w) + 2, -1, Z(w)) \rightarrow (X(w) + 2, 0, Z(w))$
- $W \rightarrow W$:
 $(X(v) - 2, 0, Z(v)) \rightarrow (X(v) - 2, 3, Z(v)) \rightarrow (X(w) - 2, 3, Z(v)) \rightarrow$
 $(X(w) - 2, 1, Z(v)) \rightarrow (X(w) - 2, 1, Z(w)) \rightarrow (X(w) - 2, 0, Z(w))$

- $W \rightarrow T$:
 $(X(v) - 2, 0, Z(v)) \rightarrow (X(v) - 2, 3, Z(v)) \rightarrow (X(w) - 3, 3, Z(v)) \rightarrow$
 $(X(w) - 3, 0, Z(v)) \rightarrow (X(w) - 3, 0, Z(w) + 1) \rightarrow (X(w), 0, Z(w) + 1)$
- $W \rightarrow B$:
 $(X(v) - 2, 0, Z(v)) \rightarrow (X(v) - 2, 3, Z(v)) \rightarrow (X(w) + 3, 3, Z(v)) \rightarrow$
 $(X(w) + 3, 0, Z(v)) \rightarrow (X(w) + 3, 0, Z(w) - 1) \rightarrow (X(w), 0, Z(w) - 1)$
- $T \rightarrow N$:
 $(X(v), 0, Z(v) + 1) \rightarrow (X(v), +2, Z(v) + 1) \rightarrow (X(w) + 1, 2, Z(v) + 1) \rightarrow$
 $(X(w) + 1, 1, Z(v) + 1) \rightarrow (X(w) + 1, 1, Z(w)) \rightarrow (X(w), 1, Z(w))$
- $T \rightarrow S$:
 $(X(v), 0, Z(v) + 1) \rightarrow (X(v), 2, Z(v) + 1) \rightarrow (X(w) - 1, 2, Z(v) + 1) \rightarrow$
 $(X(w) - 1, -1, Z(v) + 1) \rightarrow (X(w) - 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $T \rightarrow E$:
 $(X(v), 0, Z(v) + 1) \rightarrow (X(v), 2, Z(v) + 1) \rightarrow (X(w) + 2, 2, Z(v) + 1) \rightarrow$
 $(X(w) + 2, -1, Z(v) + 1) \rightarrow (X(w) + 2, -1, Z(w)) \rightarrow (X(w) + 2, 0, Z(w))$
- $T \rightarrow W$:
 $(X(v), 0, Z(v) + 1) \rightarrow (X(v), 2, Z(v) + 1) \rightarrow (X(w) - 2, 2, Z(v) + 1) \rightarrow$
 $(X(w) - 2, -1, Z(v) + 1) \rightarrow (X(w) - 2, -1, Z(w)) \rightarrow (X(w) - 2, 0, Z(w))$
- $T \rightarrow B$:
 $(X(v), 0, Z(v) + 1) \rightarrow (X(v), 2, Z(v) + 1) \rightarrow (X(w) + 3, 2, Z(v) + 1) \rightarrow$
 $(X(w) + 3, 0, Z(v) + 1) \rightarrow (X(w) + 3, 0, Z(w) - 1) \rightarrow (X(w), 0, Z(w) - 1)$
- $T \rightarrow T$:
 $(X(v), 0, Z(v) + 1) \rightarrow (X(v), 2, Z(v) + 1) \rightarrow (X(w) - 3, 2, Z(v) + 1) \rightarrow$
 $(X(w) - 3, 0, Z(v) + 1) \rightarrow (X(w) - 3, 0, Z(w) + 1) \rightarrow (X(w), 0, Z(w) + 1)$
- $B \rightarrow N$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -2, Z(v) - 1) \rightarrow (X(w) + 1, -2, Z(v) - 1) \rightarrow$
 $(X(w) + 1, 1, Z(v) - 1) \rightarrow (X(w) + 1, 1, Z(w)) \rightarrow (X(w), 1, Z(w))$
- $B \rightarrow S$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -2, Z(v) - 1) \rightarrow (X(w) - 1, -2, Z(v) - 1) \rightarrow$
 $(X(w) - 1, -1, Z(v) - 1) \rightarrow (X(w) - 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $B \rightarrow E$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -2, Z(v) - 1) \rightarrow (X(w) + 2, -2, Z(v) - 1) \rightarrow$
 $(X(w) + 2, -1, Z(v) - 1) \rightarrow (X(w) + 2, -1, Z(w)) \rightarrow (X(w) + 2, 0, Z(w))$
- $B \rightarrow W$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -2, Z(v) - 1) \rightarrow (X(w) - 2, -2, Z(v) - 1) \rightarrow$
 $(X(w) - 2, -1, Z(v) - 1) \rightarrow (X(w) - 2, -1, Z(w)) \rightarrow (X(w) - 2, 0, Z(w))$
- $B \rightarrow B$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -2, Z(v) - 1) \rightarrow (X(w) + 3, -2, Z(v) - 1) \rightarrow$
 $(X(w) + 3, 0, Z(v) - 1) \rightarrow (X(w) + 3, 0, Z(w) - 1) \rightarrow (X(w), 0, Z(w) - 1)$

- $B \rightarrow T$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), -2, Z(v) - 1) \rightarrow (X(w) - 3, -2, Z(v) - 1) \rightarrow$
 $(X(w) - 3, 0, Z(v) - 1) \rightarrow (X(w) - 3, 0, Z(w) + 1) \rightarrow (X(w), 0, Z(w) + 1)$

E Appendix: Routings - 5 Bend Staircase model

The 36 possible port-to-port routings from a vertex v to vertex w will now be enumerated. In each case, the initial and terminating points of the route are $(X(v), Y(v) = 0, Z(v))$ and $(X(w), Y(w) = 0, Z(w))$ respectively. We use $K(v)$ and $L(v)$ to denote the underlayers associated with vertex v : if v is the i th vertex, then $K(v) = -2(n - i) - 2$ and $L(v) = -2(n - i) - 3$, where n is the number of vertices in the staircase. All edge routes contain 5 bends except Top to Bottom, East to Bottom, North to Bottom, South to Bottom and Bottom to Bottom which have 4 bends.

- $N \rightarrow S$:
 $(X(v), 2, Z(v)) \rightarrow (X(v) + 1, 2, Z(v)) \rightarrow (X(v) + 1, 2, Z(w)) \rightarrow$
 $(X(v) + 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $N \rightarrow N$:
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 1) \rightarrow (X(w), 1, Z(v) + 1) \rightarrow$
 $(X(w), +2, Z(v) + 1) \rightarrow (X(w), +2, Z(w))$
- $N \rightarrow E$:
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 1) \rightarrow (X(w) + 2, 1, Z(v) + 1) \rightarrow$
 $(X(w) + 2, 0, Z(v) + 1) \rightarrow (X(w) + 2, 0, Z(w))$
- $N \rightarrow W$:
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 1) \rightarrow (X(w) - 1, 1, Z(v) + 1) \rightarrow$
 $(X(w) - 1, 0, Z(v) + 1) \rightarrow (X(w) - 1, 0, Z(w))$
- $N \rightarrow T$:
 $(X(v), 2, Z(v)) \rightarrow (X(v) + 1, 2, Z(v)) \rightarrow (X(v) + 1, 2, Z(w) + 2) \rightarrow$
 $(X(v) + 1, 0, Z(w) + 2) \rightarrow (X(w), 0, Z(w) + 2)$
- $N \rightarrow B$ (4 bends):
 $(X(v), 1, Z(v)) \rightarrow (X(v), 1, Z(v) + 1) \rightarrow (X(w), 1, Z(v) + 1) \rightarrow$
 $(X(w), 0, Z(v) + 1)$
- $S \rightarrow N$:
 $(X(v), -2, Z(v)) \rightarrow (X(v), -2, K(v)) \rightarrow (X(w), -2, K(v)) \rightarrow$
 $(X(w), +2, K(v)) \rightarrow (X(w), 2, Z(w))$
- $S \rightarrow S$:
 $(X(v), -2, Z(v)) \rightarrow (X(v) - 1, -2, Z(v)) \rightarrow (X(v) - 1, -2, Z(w)) \rightarrow$
 $(X(v) - 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$

- $S \rightarrow E$:
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w) + 2, -1, Z(v) + 1) \rightarrow$
 $(X(w) + 2, 0, Z(v) + 1) \rightarrow (X(w) + 2, 0, Z(w))$
- $S \rightarrow W$:
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w) - 1, -1, Z(v) + 1) \rightarrow$
 $(X(w) - 1, 0, Z(v) + 1) \rightarrow (X(w) - 1, 0, Z(w))$
- $S \rightarrow T$:
 $(X(v), -2, Z(v)) \rightarrow (X(v) - 1, -2, Z(v)) \rightarrow (X(v) - 1, -2, Z(w) + 2) \rightarrow$
 $(X(v) - 1, 0, Z(w) + 2) \rightarrow (X(w), 0, Z(w) + 2)$
- $S \rightarrow B$ (4 bends):
 $(X(v), -1, Z(v)) \rightarrow (X(v), -1, Z(v) + 1) \rightarrow (X(w), -1, Z(v) + 1) \rightarrow$
 $(X(w), 0, Z(v) + 1)$
- $E \rightarrow N$:
 $(X(v) + 2, 0, Z(v)) \rightarrow (X(v) + 2, 0, L(v)) \rightarrow (X(w), 0, L(v)) \rightarrow$
 $(X(w), 2, L(v)) \rightarrow (X(w), 2, Z(w))$
- $E \rightarrow S$:
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, -2, Z(v)) \rightarrow (X(v) + 1, -2, Z(w)) \rightarrow$
 $(X(v) + 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $E \rightarrow E$:
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, -1, Z(v)) \rightarrow (X(w) + 2, -1, Z(v)) \rightarrow$
 $(X(w) + 2, 0, Z(v)) \rightarrow (X(w) + 2, 0, Z(w))$
- $E \rightarrow W$:
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, -1, Z(v)) \rightarrow (X(w) - 1, -1, Z(v)) \rightarrow$
 $(X(w) - 1, 0, Z(v)) \rightarrow (X(w) - 1, 0, Z(w))$
- $E \rightarrow T$:
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, -2, Z(v)) \rightarrow (X(v) + 1, -2, Z(w) + 2) \rightarrow$
 $(X(v) + 1, 0, Z(w) + 2) \rightarrow (X(w), 0, Z(w) + 2)$
- $E \rightarrow B$ (4 bends):
 $(X(v) + 1, 0, Z(v)) \rightarrow (X(v) + 1, -1, Z(v)) \rightarrow (X(w), -1, Z(v)) \rightarrow$
 $(X(w), 0, Z(v))$
- $W \rightarrow N$:
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, 3, Z(v)) \rightarrow (X(w), 3, Z(v)) \rightarrow$
 $(X(w), 2, Z(v)) \rightarrow (X(w), 2, Z(w))$
- $W \rightarrow S$:
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, 2, Z(v)) \rightarrow (X(v) - 1, 2, Z(w)) \rightarrow$
 $(X(v) - 1, -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $W \rightarrow W$:
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, 3, Z(v)) \rightarrow (X(w) - 1, 3, Z(v)) \rightarrow$
 $(X(w) - 1, 0, Z(v)) \rightarrow (X(w) - 1, 0, Z(w))$

- $W \rightarrow E$:
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, 3, Z(v)) \rightarrow (X(w) + 2, 3, Z(v)) \rightarrow$
 $(X(w) + 2, 0, Z(v)) \rightarrow (X(w) + 2, 0, Z(w))$
- $W \rightarrow T$:
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, 2, Z(v)) \rightarrow (X(v) - 1, 2, Z(w) + 2) \rightarrow$
 $(X(v) - 1, 0, Z(w) + 2) \rightarrow (X(w), 0, Z(w) + 2)$
- $W \rightarrow B$:
 $(X(v) - 1, 0, Z(v)) \rightarrow (X(v) - 1, 3, Z(v)) \rightarrow (X(w) + 1, 3, Z(v)) \rightarrow$
 $(X(w) + 1, 0, Z(v)) \rightarrow (X(w), 0, Z(v))$
- $T \rightarrow N$:
 $(X(v), 0, Z(v) + 2) \rightarrow (X(v), +1, Z(v) + 2) \rightarrow (X(w), 1, Z(v) + 2) \rightarrow$
 $(X(w), 2, Z(v) + 2) \rightarrow (X(w), 2, Z(w))$
- $T \rightarrow S$:
 $(X(v), 0, Z(v) + 2) \rightarrow (X(v), 4, Z(v) + 2) \rightarrow (X(v), 4, Z(w)) \rightarrow$
 $(X(v), -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $T \rightarrow E$:
 $(X(v), 0, Z(v) + 2) \rightarrow (X(v), +1, Z(v) + 2) \rightarrow (X(w) + 2, 1, Z(v) + 2) \rightarrow$
 $(X(w) + 2, 0, Z(v) + 2) \rightarrow (X(w) + 2, 0, Z(w))$
- $T \rightarrow W$:
 $(X(v), 0, Z(v) + 2) \rightarrow (X(v), 1, Z(v) + 2) \rightarrow (X(w) - 1, 1, Z(v) + 2) \rightarrow$
 $(X(w) - 1, 0, Z(v) + 2) \rightarrow (X(w) - 1, 0, Z(w))$
- $T \rightarrow B$ (4 bends):
 $(X(v), 0, Z(v) + 2) \rightarrow (X(v), 1, Z(v) + 2) \rightarrow (X(w), 1, Z(v) + 2) \rightarrow$
 $(X(w), 0, Z(v) + 3)$
- $T \rightarrow T$:
 $(X(v), 0, Z(v) + 2) \rightarrow (X(v), +4, Z(v) + 2) \rightarrow (X(v), 4, Z(w) + 2) \rightarrow$
 $(X(v), 0, Z(w) + 2) \rightarrow (X(w), 0, Z(w) + 2)$
- $B \rightarrow N$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), +1, Z(v) - 1) \rightarrow (X(w), 1, Z(v) - 1) \rightarrow$
 $(X(w), 2, Z(v) - 1) \rightarrow (X(w), 2, Z(w))$
- $B \rightarrow S$:
 $(X(v), 0, L(v)) \rightarrow (X(v), -3, L(v)) \rightarrow (X(v), -3, Z(w)) \rightarrow$
 $(X(v), -1, Z(w)) \rightarrow (X(w), -1, Z(w))$
- $B \rightarrow E$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), +1, Z(v) - 1) \rightarrow (X(w) + 2, 1, Z(v) - 1) \rightarrow$
 $(X(w) + 2, 0, Z(v) - 1) \rightarrow (X(w) + 2, 0, Z(w))$
- $B \rightarrow W$:
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), 1, Z(v) - 1) \rightarrow (X(w) - 1, 1, Z(v) - 1) \rightarrow$
 $(X(w) - 1, 0, Z(v) - 1) \rightarrow (X(w) - 1, 0, Z(w))$

- $B \rightarrow T$:
 $(X(v), 0, L(v)) \rightarrow (X(v), -3, L(v)) \rightarrow (X(v), -3, Z(w) + 2) \rightarrow$
 $(X(v), 0, Z(w) + 2) \rightarrow (X(w), 0, Z(w) + 2)$
- $B \rightarrow B$ (4 bends):
 $(X(v), 0, Z(v) - 1) \rightarrow (X(v), 1, Z(v) - 1) \rightarrow (X(w), 1, Z(v) - 1) \rightarrow$
 $(X(w), 0, Z(v) - 1)$