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Triangle-Free Planar Graphs as Segment Intersection Graphs

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Abstract

We prove that every triangle-free planar graph is the intersection graph of a set of segments in the plane. Moreover, the segments can be chosen in only three directions (horizontal, vertical and oblique) and in such a way that no two segments cross, i.e., intersect in a common interior point. This particular class of intersection graphs is also known as *contact graphs*.

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1 Introduction

Given a set S of segments in the plane, its *intersection graph* has a vertex for every segment and two vertices are adjacent if the corresponding segments intersect. Intersection graphs of segments and other geometrical objects have been widely studied in the past. For instance, if the segments are contained in a straight line then we have the *interval graphs* [5], a well-known family of perfect graphs. If the segments are chords of a circle then the intersection graph is called a *circle graph*, see for instance [8, 10].

An interesting result, due to de Fraysseix, Osona de Mendez and Pach [4], and independently to Ben-Arroyo Hartman, Newman and Ziv [2], says that every planar bipartite graph is the contact graph (a particular case of intersection graph, where no two segments cross) of a set of horizontal and vertical segments and, in a different formulation, Tamassia and Tollis proved in [11] almost the same result. On the other hand, it is known that the recognition of such graphs is an NP-complete problem (see [6], [7] and recently [3]).

This result provides a partial answer to a question of Scheinerman [9]: *Is every planar graph the intersection graph of a set of segments in the plane?*

The main result in this paper, which is a significant extension of [4], is that every triangle-free planar graph is the intersection graph of a family of segments. Moreover, the segments can be drawn in only three directions and in such a way that they do not cross, i.e. the segments do not have any interior point in common. This particular class of intersection graphs is also known as *contact* graphs. We call such a representation *a segment representation* of the graph.

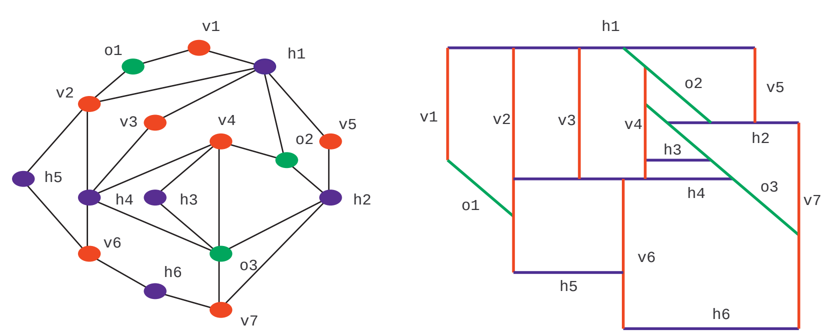

Figure 1: A segment representation of a planar graph.

Our technique is similar in spirit to the one used in [2] for bipartite planar graphs with some transformations. A key point in our proof is Grötzsch's Theorem [12], which guarantees that every planar triangle-free graph is 3-colorable.

The sketch of the proof is as follows. Fixing an embedding of a given triangle-free planar graph G, adding new vertices and (induced) paths between the vertices of the graph we can obtain a new triangle-free graph embedded in the plane which is a subdivision of a 3-connected graph. Starting with a 3-coloring, a segment representation in three directions is obtained for this new graph using several technical lemmas. Finally, removing the dummy vertices and paths, we obtain a segment representation of the graph G. The three directions considered are horizontal, vertical and oblique (parallel to the bisector of the second quadrant of the plane).

2 Pseudo-convex faces with three directions

In this section we present some preliminaries and we indicate the basic techniques that are used.

Let G be a planar graph, and consider a fixed embedding of G in the plane. The embedding divides the plane into a number of regions that we call faces of the graph. Let I_G be a segment representation of G. We assume throughout the paper that no segment continues beyond its last intersection with a segment in another direction. The segment representation also partitions the plane into regions which we call *faces* of the representation, whose are simple n-polygons (if $n \geq 4$, there is no pair of non-consecutive edges sharing a point).

The proof of our main result is based on adapting the concept of "convex polygon" for segments in three directions.

A *segment path* of length n is a sequence of horizontal, vertical and oblique segments $P = \{s_1, \ldots, s_n\}$ such that segment s_i intersects s_{i-1} and s_{i+1}, for $1 < i < n$. A segment path $P = \{s_1, \ldots, s_n\}$ is said to be *monotone* with respect to a straight line l if any line orthogonal to l intersects P in exactly one point.

We consider monotone paths of the segment representation with respect to the following two lines: the bisector of the second quadrant of the plane ($l_1 = \{x + y = 0\}$) and the line ($l_2 = \{x - 2y = 0\}$).

Since we can draw a monotone path rightward or leftward, we distinguish four different kinds of monotone paths:

- A path P is an *increasing monotone path to the right* (IMPR) if its horizontal segments are drawn rightward, its vertical segments are drawn upward, and its oblique segments are drawn rightward and downward.
- A path P is a *decreasing monotone path to the right* (DMPR) if its horizontal segments are drawn rightward, its vertical segments are drawn downward, and its oblique segments are drawn rightward and downward.
- A path P is a *decreasing monotone path to the left* (DMPL) if its horizontal segments are drawn leftward, its vertical segments are drawn downward, and its oblique segments are drawn leftward and upward.

- A path P is an *increasing monotone path to the left* (IMPL) if its horizontal segments are drawn leftward, its vertical segments are drawn upward, and its oblique segments are drawn leftward and upward.

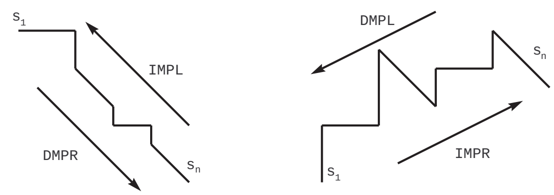

Figure 2: Monotone paths.

Let I_G be a segment representation of a plane triangle-free graph G. Recall that the face boundaries of the representation correspond to closed walks in G. A face F of a segment representation is *pseudo-convex* if its boundary can be divided into an IMPR, a DMPR, a DMPL and an IMPL (clockwise). It can be seen that the partition of a pseudo-convex face into monotone paths is not unique, but there are only a few possibilities and this does not modify the proofs of the theorems. From now on, we consider the boundary of a pseudo-convex face clockwise.

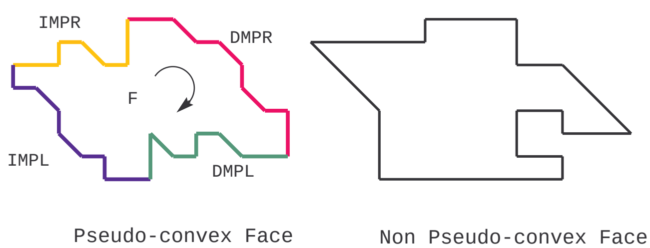

Figure 3: The four paths of a pseudo-convex face.

Given a pseudo-convex face, we define the *upper subpath* as the union of its IMPR and its DMPR and, the *lower subpath* as the union of its DMPL and IMPL subpath. Analogously, we define the *right subpath* as the union of the DMPR and DMPL, and the *left subpath* as the union of the IMPL and IMPR. In this way the boundary of a pseudo-convex face can be considered as the union of the upper and lower subpaths, and also as the union of the right and left subpaths. We say that the upper and lower subpaths are *opposite subpaths* to each other (analogously with the right and the left subpaths).

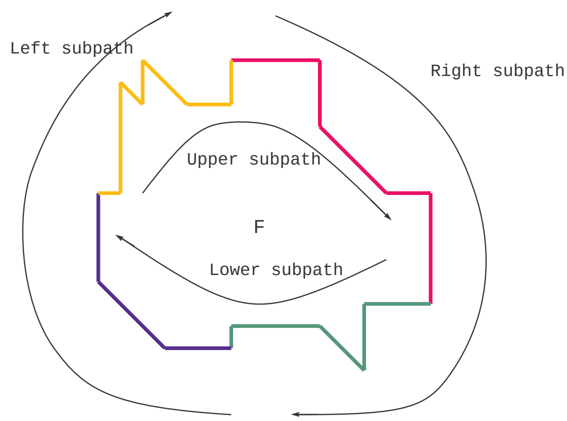

Figure 4: The upper, lower, right and left subpaths of a pseudo-convex face.

Note that if F is a pseudo-convex face, and P is a monotone path inside F connecting two segments in its boundary, then P partitions F into two new pseudo-convex faces. Moreover, if two segments on the boundary of F can be extended inside F until they cross, those extended segments also would split F into two new pseudo-convex faces.

In order to build a segment representation of a plane graph G, we proceed by representing the boundary of the outerface of G by contact segments, using a pseudo-convex representation. If two vertices of the outerface are joined in G by a path, we represent this path as a monotone segment path inside the representation of the outerface, obtaining two pseudo-convex faces. Recursively we represent monotone paths of G until the representation of the graph is completed.

We see then that in order to represent a graph by contact segments it is necessary to join segments by segment paths inside the faces of the representation. We need to extend segments of the boundary of the faces inside them to contact the segments. However it is not always possible to extend segments inside a pseudo-convex face. A segment of a pseudo-convex is extensible along one of its ends if and only if its angle inside the face is concave. Thus, a segment of a pseudo-convex face is extensible by one of its ends, or by both or by none, depending on the adjacent segments (see Figure 5).

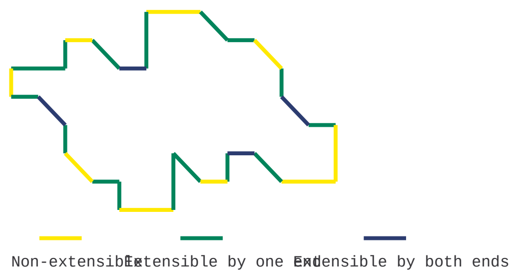

Figure 5: Extensible segments.

It is obvious that in order to join two segments inside a pseudo-convex face, one of them must be extended, but this is not sufficient in general as we can see in Figure 6: segment r_2 can be extended inside F, but we can not join it with r_1 because r_2 intersects other segments of the face. The segment r_3 can be also extended inside the face, but we cannot join r_1 with r_3 because they never contact.

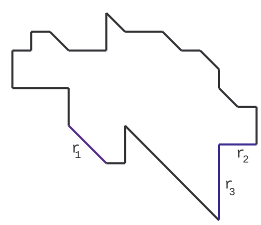

Figure 6: Extension of segments.

According to the above remark, we have two different problems when we try to join segments inside a pseudo-convex face; when the segments can be extended inside the face, but they find some "obstacle" before the intersection (other segments of the face), and on the other hand when the segments cannot be extended or they never contact.

To solve these problems we define two basic operations in the segment representation. The first operation is *enlarging a segment* of the representation, and the second one is *changing the sense of drawing* of some segments. Although we are interested in some specific segments, these operations will transform other segments of the representation.

Enlarging a segment means to increase its length, and obviously we also need to increase the length of other segments of the representation and make a translation of other segments without modifying their length. With this operation we can keep away the obstacles and to overlap parallel segments, if we want to join them by a segment path. The other operation allows us to extend a segment by one of its ends, if it was not possible before. It changes the sense of drawing of its adjacent segment and the other segments of its monotone path, to preserve the pseudo-convexity of the face.

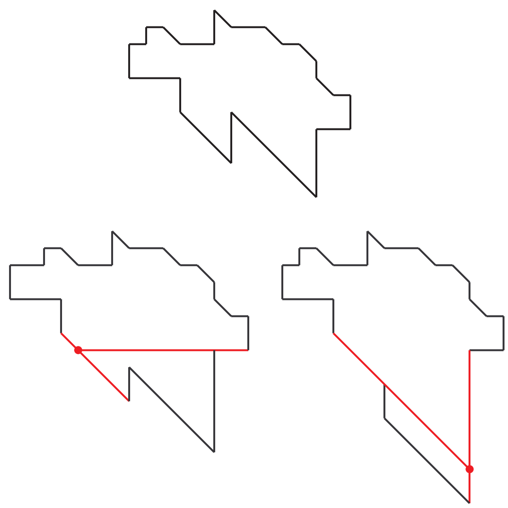

Figure 7: Transforming the segment representation.

The following technical lemmas allow us to construct a topological equivalent representation where we modify some segments, allowing to join the segments by a path of any length.

Lemma 1 *Let I_G be a segment representation of a subdivision of a 3-connected plane graph G such that all its faces are pseudo-convex, let k be a positive real number and let s be a segment on the boundary of a face F of I_G. Then I_G can be transformed into another segment representation $\overline{I_G}$ satisfying:*

1. *$\overline{I_G}$ is topologically equivalent to I_G (that is, the faces of I_G and $\overline{I_G}$ have the same facial walks), and the faces of $\overline{I_G}$ are pseudo-convex;*
2. *it is possible to enlarge s (and at most two segments on the opposite subpath of F), such that the length of $\overline{s}$ in $\overline{I_G}$ is $k + l$.*

Proof:

The proof is by induction on the number of faces of the graph G. Firstly, we suppose that F is just the boundary of I_G, and let s be the segment of F that we want to enlarge.

We can distinguish three cases depending on the direction of s.

Case 1 Let s be a horizontal segment.

If s is a horizontal segment in the upper subpath of F (analogously if s is in the lower subpath) we look for a horizontal segment s' in the lower subpath of F. If there exists such segment we enlarge s and s' by the same amount and we make a translation of all the segments between s and s', transforming F into another pseudo-convex face (see Figure 8 (a)).

If there does not exist a horizontal segment s' in the lower subpath of F, there must exist a vertical segment s' and an oblique segment s'' which are adjacent in the lower subpath of F, because the boundary of F is a closed path. In this case, we increase the length of s, s' and s'' and we make a translation of the rest of the segments (see Figure 8 (b)).

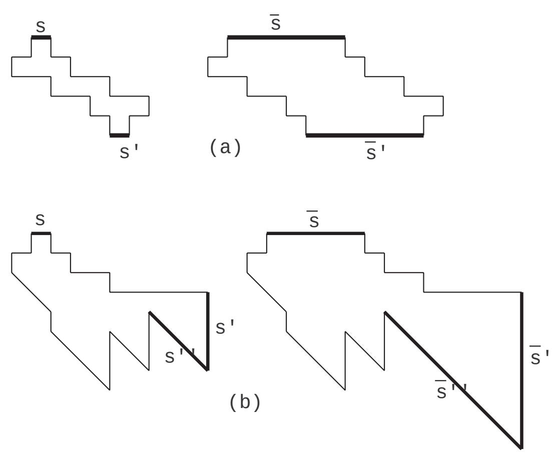

Figure 8: How to enlarge a segment of a pseudo-convex face by any amount, using parallel segments (a), using three segments (b).

Case 2 If s is a vertical segment, the proof is as Case 1, considering the boundary of F as the union of the right and left subpaths.

Case 3 If s is an oblique segment, the proof is similar to the other cases, considering the boundary of F as the union of the upper and lower subpaths.

Suppose that G has two bounded faces and let us denote by $u_1, \ldots, u_n$ the vertices of G which are in both faces. Let $\{s_1, \ldots, s_n\}$ be the segments of I_G corresponding to $u_1, \ldots, u_n$, respectively, and let us denote by F_1 and F_2 the boundary of the bounded faces in I_G.

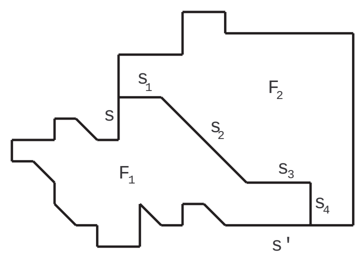

Figure 9: The adjacent segments are $s_1, \ldots, s_4$.

Let s be the segment of F_1 we want to enlarge. By considerations given above, this transformation will enlarge other segments of F_1, but if none of these segments belong to F_2, we can reduce to Case 1 because the transformations does not modify F_2.

If we have selected two (or three) segments in F_1 to enlarge them, they will modify F_2 if and only if F_1 and F_2 share more than one point of those segments. Thus, we can suppose that $F_1 \cap F_2 \cap s_1$ is not just one point, and neither is $F_1 \cap F_2 \cap s_n$. Otherwise, we do not consider the segments s_1 and s_n in the subpath $\{s_1, \ldots, s_n\}$. According to that, the path $\{s_1, \ldots, s_n\}$ is monotone because the faces of I_G are pseudo-convex.

Suppose that s is in F_1. As in the case where G had only one bounded face, in the opposite subpath of F_1 we look for a parallel segment or two contiguous segments non-parallel to s. If these segments are not in F_2 we only modify F_1 according to the first case.

Let us suppose that s belongs to F_1 and we have found a parallel segment s' in the opposite part of F_1. Since $F_1 \cap F_2$ is a monotone path and the segments are in opposite parts of F_1, just one of them belongs to F_2. Now, the task is to find a parallel segment in the opposite part of F_2, and we are sure that this new segment does not belongs to F_1, so enlarging all selected segments, a new representation $I_{G'}$ equivalent to I_G is obtained.

The same proof remains valid if we choose two contiguous segments s' and s'' in F_1 and they belong also to F_2.

Suppose now that I_G has n bounded pseudo-convex faces with boundaries $F_1, \ldots, F_n$, and let s be a segment in the boundary of a face F. Since G is a subdivision of a 3-connected graph, there exists a face (for example F_n) adjacent to the unbounded face (i.e., the boundary of the face in G corresponding to F_n has an edge in the unbounded face H of G) and such that s is not in $F_n \cap H$. Thus, if we construct the graph G_1 as the graph obtained from G by deleting all the edges of the face of G corresponding to F_n which are contained in the outerface, then we can assure that s belongs to G_1.

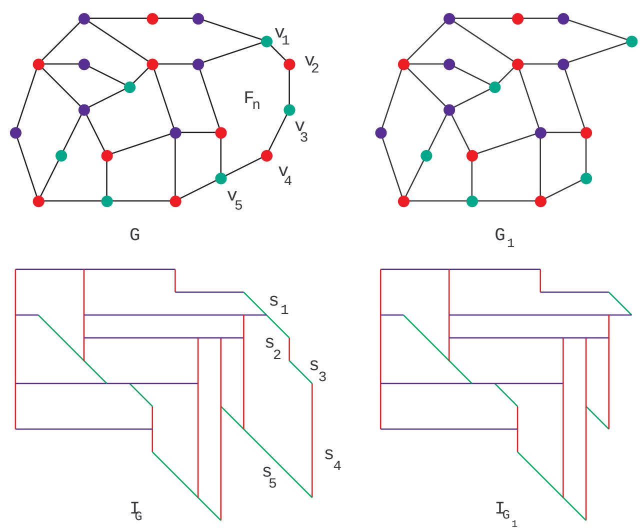

Figure 10: Induction.

Let $v_1, \ldots, v_k$ be the vertices of G, corresponding to the segments $s_1, \ldots, s_k$ of F_n, which are also on the outerface, where $v_2, \ldots, v_{k-1}$ are of degree two. By removing the segments of I_G corresponding to the vertices $v_2, \ldots, v_{k-1}$, we obtain a segment representation I_{G_1} of G_1.

Since I_{G_1} has $n-1$ faces, enlarging in any amount some of the segments, we can transform its segment representation into another one which has the same convex faces. To obtain this new representation, we have to enlarge other segments in I_{G_1} besides s, but we have preserved the structure of the segments in the representation. So, we are able to represent again the segments corresponding to $v_2, \ldots, v_{k-1}$ enlarging the length of one (or two) of them, if it is necessary, to obtain a segment representation of G. □

We can see, using Lemma 1, that by enlarging some segments, a segment representation with pseudo-convex faces can be transformed into another one preserving the topology of the embedding. But this is not sufficient to join segments inside a face, so we need another kind of transformation.

Changing the sense of drawing of the segments of a monotone path of a pseudo-convex face F (for instance, the segments which were drawn to the left are drawn to the right) it is possible to draw a new pseudo-convex face $\overline{F}$ where these segments belong now to another monotone path. The remaining paths of $\overline{F}$ only change in the length of some of the segments (see Figure 11).

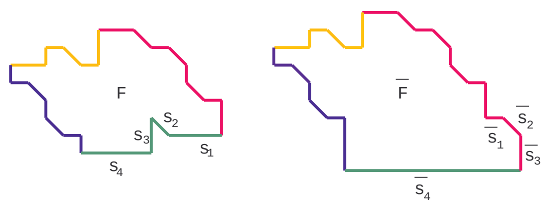

Figure 11: How to restructure a face through a path.

In the segment representation, the change of the face F to $\overline{F}$ produces some changes in the faces adjacent to F along the monotone path, and we obtain another segment representation with pseudo-convex faces as in the proof of Lemma 1 (see Figure 12). When we change the drawing in the manner described above, we say that the face F is *restructured* through the monotone path.

Lemma 2 *Let I_G be a segment representation of a subdivision of a 3-connected plane graph G such that all its faces are pseudo-convex. Let F be a face of I_G. Then, I_G can be transformed into another segment representation $\overline{I_G}$ satisfying:*

1. *$\overline{I_G}$ is topologically equivalent to I_G (that is to say, the faces of I_G and $\overline{I_G}$ have the same facial walks), and the faces of $\overline{I_G}$ are pseudo-convex;*

2. *the face $\overline{F}$ corresponding to F has been restructured through a monotone path.*

Proof: We first suppose that F is just the boundary of I_G. It is easily seen that we can change the drawing of the segments so that they belong to the previous or the following monotone path.

Suppose now that I_G has n bounded pseudo-convex faces. Let F be the face that we want to restructure through the monotone path $s_1, \ldots, s_n$. It is necessary to change the sense of drawing of all the segments of the restructured path, thus the other faces that contain in their boundaries the segments $s_1, \ldots, s_n$ will be also restructured, but they will preserve their pseudo-convexity.

The main idea is that no other face of the representation will be modified, except for the length of some segments, but by Lemma 1 it is possible to increase the length of the segments of the representation preserving the pseudo-convexity.

In fact, we can consider the faces adjacent to F. The union of these faces determines a region of the plane, R, whose boundary is also a pseudo-convex face. Since restructuring F only modifies the segments in the interior of R but never in its boundary, the faces of the representation that are not adjacent to F will keep their structure.

For instance, in Figure 12, we have restructured the face F and this transformation forces restructuring the shaded face. Other faces of the segment representation have been transformed in the length of some segments of their boundaries, but by Lemma 1 they preserve their pseudo-convexity.

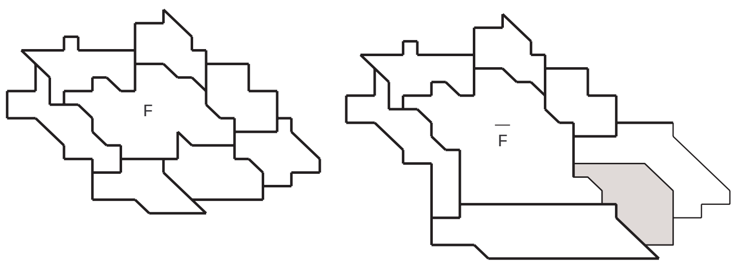

Figure 12: How to restructure the faces adjacent to the restructured face F.

□

3 Triangle-free graphs

We need the following result proved by Barnette [1]:

Theorem 1 *If G is a subdivision of a 3-connected graph, there exists a family $Q_1, \ldots, Q_{n-1}$ of paths in G such that the family of subgraphs defined as $G_1 = G$, $G_2 = G_1 - Q_1, \ldots, G_n = G_{n-1} - Q_{n-1}$ satisfies the properties*

1. *Each G_k is a subdivision of a 3-connected graph.*

2. *G_n is a subdivision of K_4.*

Moreover, if we fix a subgraph of G which is a subdivision of K_4, G can be reduced, by deleting edges and paths, to this subgraph.

An easy consequence is that every planar triangle-free graph which is a subdivision of a 3-connected graph can be reduced, by deleting edges and paths with internal vertices of degree two, to a subdivision of a fixed complete subgraph K_4 in such a way that in each step we have a subdivision of a 3-connected graph.

This result will allow us to build a segment representation in three directions of subdivisions of 3-connected graphs as follows. Firstly we build a segment representation of a subdivision of K_4 and then insert, in reverse order, the edges and paths that were removed to obtain from G the subdivision of K_4.

In order to do this, we need two basic operations: (1) insert a segment path between two segments of the same face and (2) join two segments of the same face. The second operation is, at least, as difficult as the first one, so we will concentrate only in the second operation.

As we see in Figure 6, we can find some problems in the segment representation to join two segments inside a face. Most of the cases can be solved by changing the drawing of the segments of the representation with the basic operations "enlarge segments" and "restructure faces", but there are four rare cases that need an special operation.

We need at this point some new definitions. Given a 3-colored plane graph G with colors $\{h, o, v\}$, a path $P = \{u_1, \ldots, u_n\}$ of G is *rare* if it verifies one of the following conditions:

Case 1. u_1 is colored as h, u_2 as o, and u_n as v.

Case 2. u_1 is colored as v, u_{n-1} as h, and u_n as o.

Case 3. u_1 is colored as h, u_2 as o, u_{n-1} as h, and u_n as o.

Case 4. u_1 is colored as o, u_2 as v, u_{n-1} as v, and u_n as h.

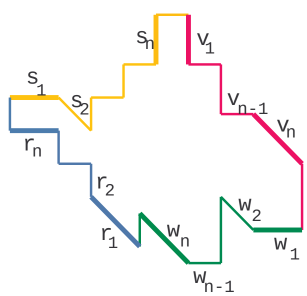

Figure 13: Rare face (four rare paths).

A face of a graph is called *rare* if there exists a monotone path containing a rare subpath (see Figure 13). We say that a rare face is *repaired* if we subdivide its rare path with a new vertex colored as v between u_1 and u_2 in the first and the third case, and between u_{n-1} and u_n in the second case, or a new vertex colored as h between u_1 and u_2 in the fourth case.

In the following lemma, when we say that two segments s and t can be joined by a segment path of length k, we mean a path $\{s, s_1, s_2, \ldots, s_k, t\}$.

Lemma 3 *Let I_G be a segment representation of a plane graph G and let s_1 and s_2 be two distinct non adjacent segments in a pseudo-convex non-rare face F. Then I_G can be transformed into another segment representation $\overline{I_G}$ satisfying*

1. *I_G and $\overline{I_G}$ have the same faces and face boundaries as G, and the faces of $\overline{I_G}$ are pseudo-convex;*
2. *s_1 can be joined to s_2 by a segment path of length k, for all $k > 0$, or directly if s_1 and s_2 have not the same direction inside the pseudo-convex face $\overline{F}$, corresponding to F.*

Proof: The proof falls naturally into two cases: the segments have the same direction, or they do not.

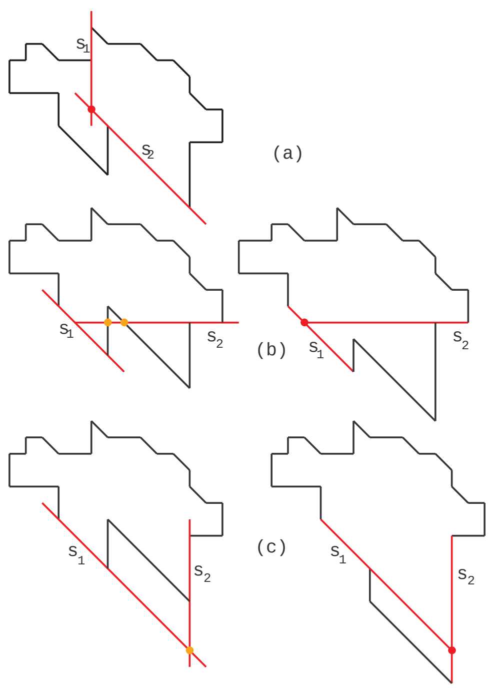

Figure 14: Joining segments

Case 1: s_1 and s_2 do not have the same direction.
Suppose the segments cannot be joined inside F directly. The lines containing

s_1 and s_2 indicate us which end has to be extended for the segments to intersect. Let us mark these ends.

1. At least one of the segments is extensible by the marked end but it intersects other segments of F (Figure 14 (b)). Then we can enlarge some segments of F using Lemma 1 and save the obstacle.

2. No segment is extensible by the marked end (Figure 14 (c)). Then we have to restructure the face through the path containing s_1 (or s_2) to make it extensible. Note that there exists exactly four configurations where the segments are not extensible and restructuring the face does not solve the problem, but these are the rare paths which are excluded.

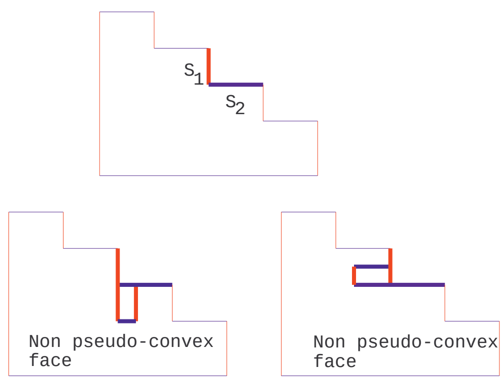

Figure 15: Segments s_1 and s_2 cannot be adjacent.

We need that s_1 and s_2 are non adjacent segments because we cannot join them by a segment path inside F if their angle inside F is concave. Note that in this case F will split into two faces, but one of them will not be pseudo-convex (see Figure 15).

Case 2: s_1 and s_2 have the same direction.
In this case, it suffices to join the segments by a segment path of length one, because this path can be substituted by any path of length greater than one.

Again there are two subcases:

1. If the segments belong to opposite subpaths (they are drawn in opposite sense). By Lemma 1, we can enlarge s_1 and s_2 until there exists a perpendicular line that crosses both segments, and a segment path of any length can be drawn inside the face.

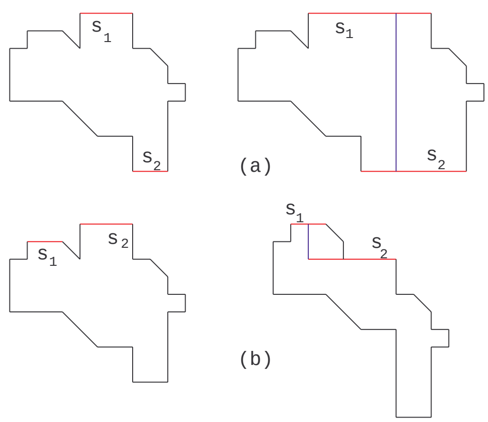

Figure 16: Joining segments.

2. If the segments are drawn in the same sense and they cannot be extended, we have to restructure the face through the path containing s_1 (or s_2) to make it extensible.

□

The interest of these Lemmas is that they allow us to proof the following.

Lemma 4 *Any triangle-free plane graph which is a subdivision of a 3-connected graph can be represented by segments in three directions.*

Proof:

As G is a triangle-free plane graph, in virtue of Grötzsch's Theorem [12], G admits a 3-coloring with the colors h, v and o (horizontal, vertical and oblique, respectively). Since G is a subdivision of a 3-connected graph, we can find a vertex, not in the outerface of G, connected by three disjoint paths to three vertices in the outerface. These paths and the outerface determine a graph K, which is a subdivision of K_4.

Using Theorem 1 [1] it is possible to build a sequence of graphs $G_1, \ldots, G_n$ and a sequence of paths $Q_1, \ldots, Q_{n-1}$ such that $G_1 = G$, G_i is a subdivision of a 3-connected plane graph, G_i is obtained from G_{i-1} by deleting the path Q_{i-1}, and the graph obtained from G_{n-1} deleting Q_{n-1} is K.

When the path Q_i is deleted, a new face F appears in G_{i+1} as the union of two faces in G_i. The boundary of F can be divided into two paths; on the one hand the path beginning in the first vertex of Q_i and ending in the last one and, on the other hand, the rest of the boundary. If one of them is rare, we must repair F by subdividing an edge with a new vertex (that we label as a *repaired* vertex), to make the face F non-rare. So, we can construct another sequence $G'_1, \ldots, G'_n$ where G'_i is G_i with a new repaired vertex, if necessary.

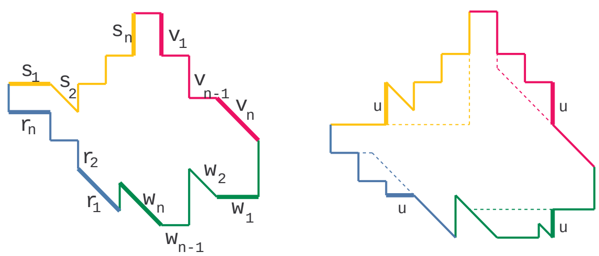

Figure 17: Repairing rare paths.

It is easy to give a segment representation of K' with all its faces pseudo-convex. It suffices to represent pseudo-convexly the outerface of K' and proceed according to the above remark. Using Lemma 3 it is possible to add the path Q_i to the segment representation of G'_{i+1} to obtain a pseudo-convex face representation of G'_i. Since every subgraph is a subdivision of a 3-connected graph, the path Q_i cannot join adjacent vertices and so we can apply Lemma 3.

In order to obtain a segment representation of the graph G, we must remove the repaired vertices. These segments cannot be removed directly, because we must change the drawing of one of the segments adjacent to the repaired vertex, so it is necessary to modify the representation. Since this is the last operation of the representation, and the repaired vertex is not adjacent to any other segment, it is possible to remove it, as illustrated in Figure 18.

Notice that we could not remove the repaired vertices if the rare path would have exactly three segments u_1, u_2 and u_3, but this case is not possible because joining u_1 with u_3 it will be form a triangle, and G was a triangle-free graph. □

We can now formulate our main result as follows.

Theorem 2 *Every triangle-free planar graph is the contact graph of a set of segments in three directions.*

Proof: Let G be a triangle-free plane graph. Since G has no triangles, we can obtain a 3-coloring of G using Grötzsch's Theorem [12]. The colors will be labeled as h, v and o (horizontal, vertical and oblique, respectively). We can build a new triangle-free plane graph G_1, subdivision of a 3-connected graph, which contains G as a subgraph, by adding new vertices and edges joining the

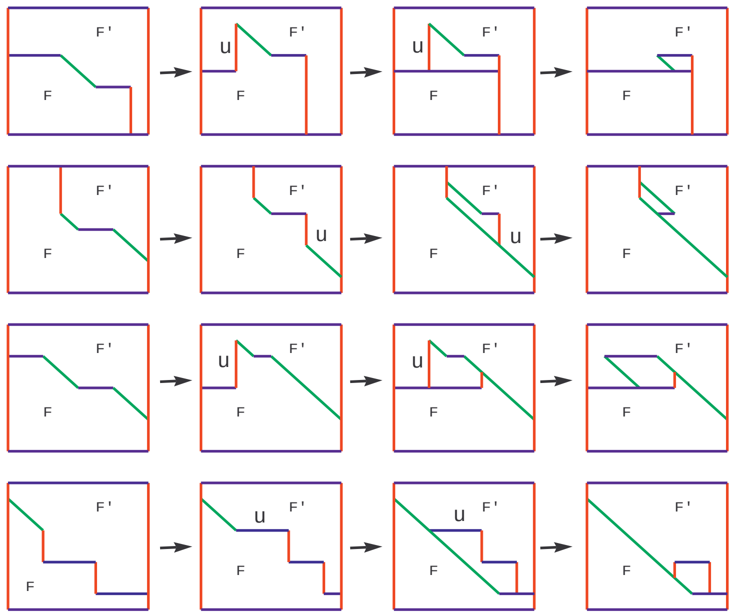

Figure 18: How to remove the repaired vertex u in the four cases.

blocks of G. If this is the case, these vertices are labeled as *dummy* vertices. When an added edge produces a triangle, we subdivide it, and when a new edge joins two vertices with the same color, we subdivide it too. We call these new vertices and edges *virtual*. All the new vertices (dummy and virtual) can be colored so that the 3-coloring is preserved.

By Lemma 4, G_1 admits a segment representation. Out of this segment representation we must remove all the vertices and edges added.

A virtual edge (or a path built with virtual edges and vertices) in the segment representation of G_1 is an adjacency between two segments (or a virtual path joining two segments). It suffices to break this adjacency and to shorten these segments (note that the segments do not cross, they only contact). The dummy vertices are removed as the repaired vertices in Lemma 4 (see Figure 18). □

4 Concluding remarks

The hypothesis that the graph has no triangles can be relaxed in some cases.

No doubt there exist planar graphs with triangles that admit segment representations (see Figure 1). The problem lies in the fact that we have followed a constructive proof to obtain a segment representation using the pseudo-convex faces representation of a subdivision of a 3-connected graph. This construction would not be possible in general if the graph contains triangles, because if we fix the three directions of the segments we can observe that the graph in Figure 19 cannot be represented by non-crossing segments.

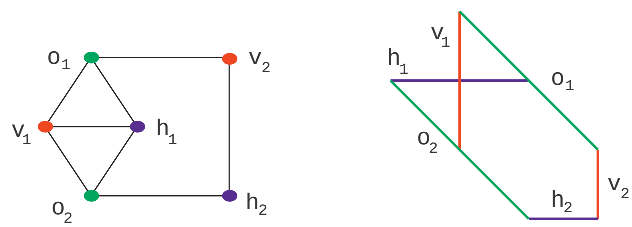

Figure 19: Two segments must cross.

In this example we see the intersection graph of a family of segments, but this representation by segments of the graph contains regions that do not correspond to faces of the embedding of the graph. So, the topology of the embedding of the plane graph cannot be preserved and inductive arguments are no longer possible. This problem admits a partial solution as follows. A 3-coloring of a plane graph G with colors $\{h, v, o\}$ is *good* if all the triangles of G are colored (clockwise), as $h - v - o$.

The proof of Theorem 2 can be adapted yielding the following result:

Theorem 3 *Let G be a 3-colored plane graph. The coloring is good if and only if there exists a segment representation I_G verifying that the faces of G correspond to faces of I_G, and the boundaries of the faces are preserved.*

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