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Confluent Drawings: Visualizing Non-planar Diagrams in a Planar Way

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Abstract

We introduce a new approach for drawing diagrams. Our approach is to use a technique we call *confluent drawing* for visualizing non-planar graphs in a planar way. This approach allows us to draw, in a crossing-free manner, graphs—such as software interaction diagrams—that would normally have many crossings. The main idea of this approach is quite simple: we allow groups of edges to be merged together and drawn as "tracks" (similar to train tracks). Producing such confluent drawings automatically from a graph with many crossings is quite challenging, however, we offer a heuristic algorithm (one version for undirected graphs and one version for directed ones) to test if a non-planar graph can be drawn efficiently in a confluent way. In addition, we identify several large classes of graphs that can be completely categorized as being either confluently drawable or confluently non-drawable.

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1 Introduction

The visualization of graphs have applications to many areas. Examples include method-call graphs [22, 24, 45], data flow diagrams [2], object-oriented class hierarchies [4, 41], as well as the classic application of flowcharts [30] (see also [1, 38, 40, 43]). Moreover, these examples include both directed and undirected diagrams. In most graph visualization applications, graphs are often drawn in a standard way: the vertices of a graph are drawn as simple shapes, such as circles or boxes, and the edges are drawn as individual curves connecting pairs of these shapes (e.g., see [14, 16, 29]).

In addition, it is quite common for drawings of graphs to be constructed automatically rather than being hand-crafted. Thus, there is a need for efficient algorithms that produce aesthetically-pleasing diagrams.

1.1 Related Prior Work

There are several aesthetic criteria that have been explored algorithmically in the area of graph drawing (e.g., see [14, 16, 29]). Examples of aesthetic goals designed to facilitate readability include minimizing edge crossings, minimizing a drawing's area, minimizing bends, and achieving good separation of vertices, edges, and angles. Of all of these criteria, however, the arguably most important is to minimize edge crossings, since crossing edges tend to confuse the eye when one is viewing adjacency relationships. Indeed, an experimental analysis by Purchase [39] suggests that edge-crossing minimization [26, 27, 32] is the most important aesthetic criteria for visualizing graphs. Ideally, we would like drawings that have no edge crossings at all.

Graphs that can be drawn in the standard way in the plane without edge crossings are called *planar graphs* [35], and there are a number of existing efficient algorithms for producing crossing-free drawings of planar graphs (e.g., see [6, 8, 11, 13, 28, 42, 44] and [9, 10, 15, 21, 23]). Unfortunately, most graphs are not planar; hence, most graphs cannot be drawn in the standard way without edge crossings, and such non-planar graphs seem to be common in many applications. There are some heuristic algorithms for minimizing edge crossings of non-planar graphs (e.g., see [26, 27, 32, 33]), but the general problem of drawing a non-planar graph in a standard way that minimizes edge-crossings is NP-hard [20]. Thus, we cannot expect an efficient algorithm for drawing non-planar graphs so as to minimize edge crossings.

The technique of replacing complete bipartite subgraphs (bicliques) with star-like structures is used as *Edge Concentration* in [34] and *Factoring* in [5], both to reduce the number of edges in the original graphs. This technique has the desired side effect of reducing the number of crossings, however, its primary goal is to minimize the total number of edges, not to minimize the number of crossings. Furthermore, the time complexity of the approximation algorithm given in [34] is not desirable. Recently Lin [31] proves that the optimization problem of edge concentration is NP-hard. A similar idea [19] is used for weighted graph compressions, where cliques and bicliques are replaced with stars. It is shown that the general unit weight problem is essentially as hard to approximate as graph coloring and maximum clique. Again, the authors do not directly address the minimization of the number of crossings. Ostry also describes a *vertex concentration* method in his thesis [36, Chap. 2 Vertex Concentration]. In this method, the vertices of a graph are partitioned into clique subgraph, then each clique is visualized as a compound vertex or icon.

1.2 Our Results

Given the difficulty of edge-crossing minimization and the ubiquity of non-planar graphs, we explore in this paper a diagram visualization approach, called *confluent drawing*, that attempts to achieve the best of both worlds—it draws non-planar graphs in a planar way. Moreover, we provide two heuristic algorithms for producing confluent drawings for directed and undirected graphs, respectively, focusing on graphs with bounded arboricity.

The main idea of the confluent drawing approach for visualizing non-planar graphs in a planar way is quite simple—we merge edges into "tracks" so as to turn edge crossings into overlapping paths. (See Figure 1.)

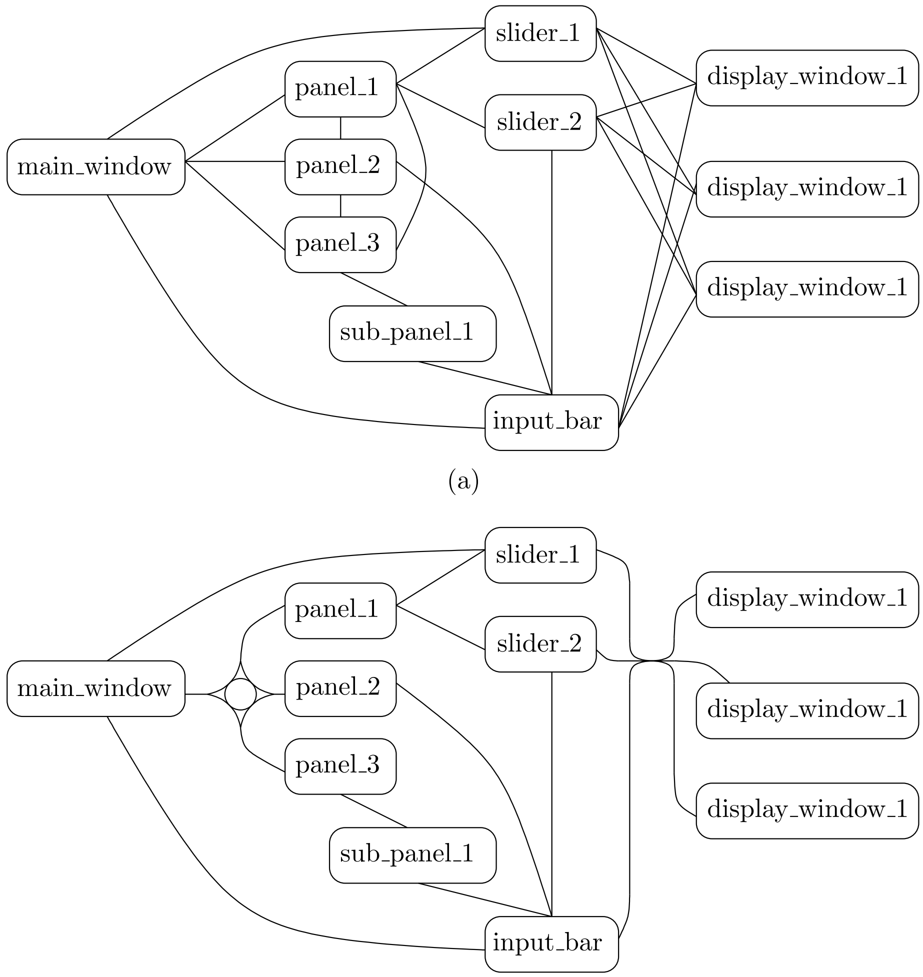
The resulting graphs are easy to read and comprehend, while also encapsulating a high degree of connectivity information. Although we are not familiar with any prior work on the automatic display of graphs using this confluent diagram approach, we have observed that some airlines use hand-crafted confluent diagrams to display their route maps. Diagrams similar to our confluent drawings have also been used by Penner and Harer [37] to study the topology of surfaces.

In addition to providing heuristic algorithms for recognizing and drawing confluent diagrams, we also show that there are large classes of non-planar graphs that can be drawn in a planar way using our confluent diagram approach. For example, any interval graph or the complement of any tree can be visualized with a (planar) confluent diagram. Even so, we also show that there are unfortunately some graphs that cannot be drawn in a confluent way, including 4-dimensional hypercubes and a certain subgraph of the Petersen graph.

This paper is organized as follows. We give a formal definition of directed and undirected confluent diagrams in Section 2. We describe heuristic algorithms for recognizing and drawing directed and undirected confluent diagrams in Section 3. We show several special classes of confluently drawable graphs in Section 4, and in Section 5 we demonstrate several classes of graphs that cannot be drawn in a confluent way.

2 Confluent Drawings

It is well-known that every non-planar graph contains a subgraph homeomorphic to the complete graph on five vertices, K_5, or the complete bipartite graph between two sets of three vertices, $K_{3,3}$ (e.g., see [3]). On the other hand, confluent drawings, with their ability to merge crossing edges into single tracks,



(a)

(b)

Figure 1: An example of confluent drawing of an object-interaction diagram. Nodes here denote components in a GUI program and edges indicate that the adjacent components send messages to each other. We show a standard drawing in (a) and a confluent drawing in (b).

can easily draw any $K_{m,n}$ or K_n in a planar way. Figure 2 shows confluent drawings of $K_{3,3}$ and K_5.

A curve is *locally-monotone* if it contains no self intersections and no sharp turns, that is, it contains no points with left and right tangents that form an angle less than or equal to 90 degrees. Intuitively, a locally-monotone curve is like a single train track, which can make no sharp turns. Confluent drawings

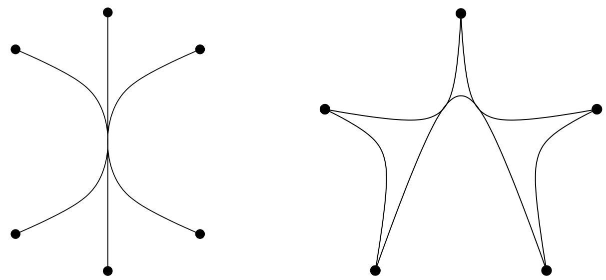


Figure 2: Confluent drawings of $K_{3,3}$ and K_5.

are a way to draw graphs in a planar manner by merging edges together into *tracks*, which are the unions of locally-monotone curves.

An undirected graph G is *confluent* if and only if there exists a drawing A such that:

- There is a one-to-one mapping between the vertices in G and A, so that, for each vertex $v \in V(G)$, there is a corresponding vertex $v' \in A$, which has a unique point placement in the plane.
- There is an edge (v_i, v_j) in $E(G)$ if and only if there is a locally-monotone curve e' connecting v'_i and v'_j in A.
- A is planar. That is, while locally-monotone curves in A can share overlapping portions, no two can cross.

Our definition does not allow for confluent graphs to contain self loops or parallel edges, although we do allow for tracks to contain cycles and even multiple ways of realizing the same edge. Moreover, our definition implies that tracks in a confluent drawing have a “diode” property that does not allow one to double-back or make sharp turns after one has started going along a track in a certain direction.

Directed confluent drawings are defined similarly, except that in such drawings the locally-monotone curves are directed and the tracks formed by unions of curves must be oriented consistently.

Formally, a directed graph D is *confluent* if and only if there exists a drawing B such that:

- There is a one-to-one mapping between the vertices in D and B, so that, for each vertex $v \in V(D)$, there is a corresponding vertex $v' \in B$, which has a unique point placement in the plane.
- There is an edge $(v_i, v_j) \in E(D)$ if and only if there is a locally-monotone curve e' connecting v'_i and v'_j in B.
- Locally-monotone curves in B may share some overlapping portions, but the edges sharing the same portion of a track must all have the same direction along that portion.

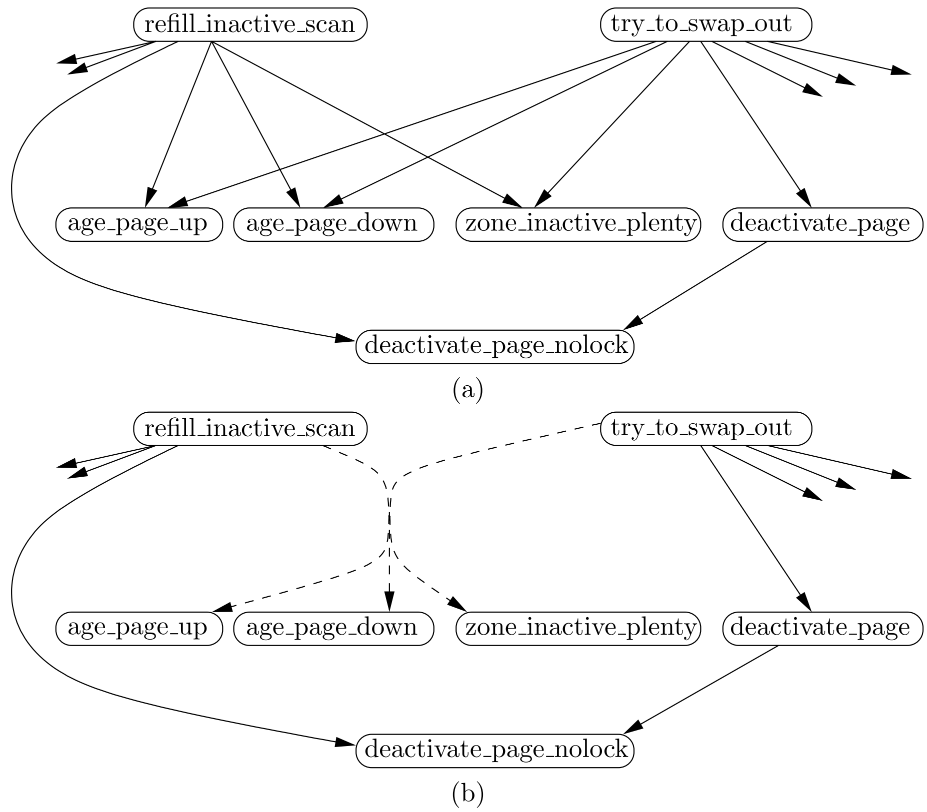


Figure 3: A call graph (a) and its confluent drawing (b), with the dashed part showing the confluence.

- B is directed and planar.

Figure 3(a) shows a part of the call graph of a Linux memory management module and its corresponding confluent drawing. We choose this non-planar drawing to illustrate how confluent drawing works, and the hierarchy of the drawing is still preserved. In Figure 3(b) we can easily tell the three functions have two common callers, while in the original graph, it is a little more difficult to explore that information. One can imagine that confluent drawings can make complicated graphs more readable.

Confluent drawings remove crossings present in non-planar graphs, making the graphs' structure easier to be understood. We feel that such drawings may also be helpful in discovering certain characteristic of the graphs. For example, given a confluent drawing, we can easily find the common source vertices and destination vertices of merged edges. Such common structures could indicate in a method-call diagram, say, separate methods that can be joined together for the sake of efficiency. Likewise, structures in which many sources all communicate with many destinations could indicate a need for refactoring or lead to other useful insights about a software design.

3 Heuristic Algorithms

Though the planarity of a graph can be tested in linear time, it appears difficult to quickly determine whether or not a graph can be drawn confluently. If a graph G contains a non-planar subgraph, then G itself is non-planar too. But similar closure properties are not true for confluent graphs. Adding vertices and edges to a non-confluent graph increases the chances of edges crossing each other, but it also increases the chances of edges merging. Currently, the best method we know for determining conclusively in the worst case whether a graph is confluent or not is a brute force one of exhaustively listing all possible ways of edge merging and checking the merged graphs for planarity. Therefore, it is of interest to develop heuristics that can find confluent drawings in many cases.

Figure 4 shows confluent drawings using a "traffic circle" structure for complete subgraphs (cliques) and complete bipartite subgraphs (bicliques). At a high level, our heuristic drawing algorithm iteratively finds clique subgraphs and biclique subgraphs and replaces them with traffic-circle subdrawings.

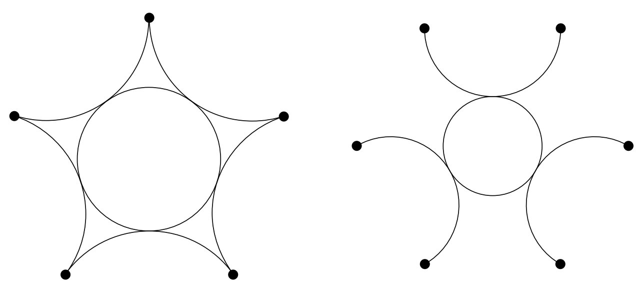


Figure 4: Confluent drawings of K_5 and $K_{3,3}$ using "traffic circle" structures.

In our heuristic algorithm for undirected graphs, we will use the clique subgraphs listing and the biclique subgraphs listing algorithms as our subroutines.

In step 3 of the algorithm in Figure 5, the cliques are given higher priority over bicliques, otherwise a clique would be partially covered by a biclique. Cliques of three or fewer vertices, and bicliques with one side consisting of only one vertex, are not replaced because the replacement cannot change the planarity of the graph. We now discuss the time performance of this heuristic.

The *arboricity* $a(G)$ for a graph G is the minimum number of forests into which the edges of G can be partitioned.

Theorem 1 *In graphs of bounded arboricity, algorithm* HeuristicDrawUndirected *can be made to run in time $O(n)$, assuming hash tables with constant time per operation.*

HeuristicDrawUndirected(G)
Input. A undirected sparse graph G.
Output. Confluent drawing of G if succeed, fail otherwise.
1. If G is planar
2. draw G using any known planar graph drawing algorithm
3. else if G contains a large clique or biclique subgraph C
4. create a new vertex v
5. obtain a new graph G' by removing edges of C and connecting each vertex of C to v
6. HeuristicDrawUndirected(G')
7. replace v by a small "traffic circle" to get a confluent drawing of G
8. else fail

Figure 5: The heuristic for undirected graphs with bounded $a(G)$.

Proof: Chiba and Nishizeki [7] discuss the problem of listing complete subgraphs for graphs of bounded arboricity. The listing algorithm is applicable for such graphs. Chiba and Nishizeki show that there can be at most $O(n)$ cliques of a given size in such graphs and give a linear time algorithm for listing these clique subgraphs. Eppstein [17] gives a linear time algorithm for listing maximal complete bipartite subgraphs in a graph with bounded arboricity. He shows that in any graph of bounded arboricity, there are at most $O(n)$ maximal complete bipartite subgraphs, and these subgraphs have a total of $O(n)$ vertices and $O(n)$ edges. And again they can be listed in linear time.

We store a bit per edge of the original graph so we can quickly look up whether it is still part of our replacement. We begin the heuristic by looking for cliques, since we want to give them priority over bicliques. List all the complete subgraphs in the graph with four or more vertices, and sort them by size (the size of the complete subgraph is bounded too in graphs with bounded arboricity). Then, for each complete subgraph X in sorted order, we check whether X is still a clique of the modified graph, and if so perform a replacement of X. It is not hard to see that the new vertex v of the replacement cannot belong to any clique, so this algorithm correctly finds a maximal sequence of cliques to replace.

Next, we need to similarly dynamize the search for bicliques. This is more difficult, because a biclique may have non-constant size and because the replacement vertex v may belong to additional bicliques. We perform this step by dynamizing the algorithm of Eppstein [17] for listing all bicliques. This algorithm uses the idea of a *d-bounded acyclic orientation*: that is, an orientation of the edges of the graph, such that the oriented graph is acyclic and the vertices have maximum outdegree d. For graphs of arboricity a, a $(2a-1)$-bounded acyclic orientation may easily be found in linear time. For such an orientation, define a *tuple* to be a subset of the outgoing neighbors of any vertex, and let v

be a *tuple creator* of tuple T if all vertices of T are outgoing neighbors of v. For graphs of bounded arboricity, there are at most linearly many distinct tuples. For each maximal biclique, one of the two sides of the bipartition must be a tuple, T [17]. The other side consists of two types of vertices: tuple creators of T, and outgoing neighbors of vertices of T.

Our algorithm stores a hash table indexed by the set of all tuples in the modified graph. The hash table entry for tuple T stores the number of tuple creators of T, and a list of outgoing neighbors of vertices of T that are adjacent to all tuple members. For each edge (u, v) in the graph, oriented from u to v, we store a list of the tuples T containing v for which u is listed as an outgoing neighbor. We also store a priority queue of the maximal bicliques generated by each tuple, prioritized by size; it will suffice for our purposes if the time to find the largest biclique is proportional to the biclique size, and it is easy to implement a priority queue with such a time bound. With these structures, we may easily look up each successive biclique replacement to perform in algorithm HeuristicDrawUndirected. Each replacement takes time proportional to the number of edges removed from the graph, so the total time for performing replacements is linear.

It remains to show how to update these data structures when we perform a biclique replacement. To update the acyclic orientation, orient each edge from C to v, except for those edges from vertices of C that have no outgoing edges in C. It can be seen that this orientation preserves d-boundedness and acyclicity. When a new vertex v is created by a replacement, create the appropriate hash table entries for tuples containing v; the number of tuples created by a replacement is proportional to the number of edges removed in the same replacement, so the total number of tuples created over the course of the algorithm is linear. Whenever a replacement causes edges from a vertex x to change, update the hash entries for all tuples for which x is a creator; this step takes $O(1)$ time per change. Also, update the hash entries for all tuples to which x belongs, to remove vertices that are no longer outgoing neighbors of x; this step takes time $O(1)$ per changed tuple, and each tuple changes $O(1)$ times over the course of the algorithm. Whenever a change removes incoming edges of x, we must remove the other endpoints of those edges from the lists of outgoing neighbors of tuples to which x belongs; using the lists associated with each incoming edge, this takes constant time per removal. Therefore, all steps can be performed in linear total time. □

An example of the input for algorithm HeuristicDrawUndirected and the output drawing produced by this heuristic is shown in Figure 6. In the drawing of a complete bipartite subgraph, tracks can be partitioned into two subsets according to the directions in which they are merged with the circle. It is easy to see that if tracks of these two sets are not interleaved around the circle, the circle becomes redundant thus can be removed. If needed we can post-process the output of the above algorithm to remove redundant circles.

For directed graphs, the algorithm is slightly different. Because the tracks in directed confluent drawings are required to have directions, the "traffic circle"

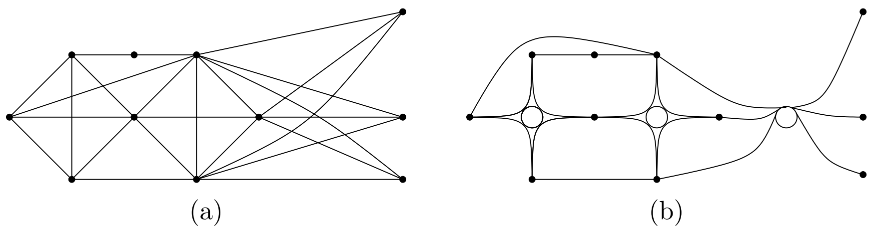


Figure 6: An example of running the undirected heuristic algorithm. The input graph is shown in (a) and the output drawing is shown in (b).

structure will not work for directed cliques. Thus we only look for maximal directed bicliques in step 3 in the directed version of the heuristic algorithm. Next we discuss how to find maximal directed bicliques. Maximal directed complete bipartite subgraphs in a sparse directed graph G can be found by first listing maximal undirected complete bipartite subgraphs in the underlying undirected graph of G. Then for each of these subgraphs examine the corresponding directed subgraph. We choose the side of the bipartition with larger size and partition it according to how their edges are oriented to the other side of the bipartition. (In Figure 7, the right directed $K_{3,4}$ is obtained from the left graph).

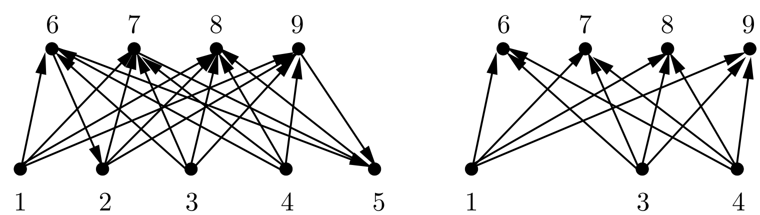


Figure 7: Maximal directed complete bipartite subgraphs.

4 Some Confluent Graphs

The heuristic algorithms presented in the previous section are most applicable to sparse graphs, because sparseness is needed for the linear time bound of the maximal bipartite subgraph listing subroutine. However, there are also several denser classes of graphs that we can show to be confluent.

4.1 Cographs

A *complement reducible graph* (also called a *cograph*) is defined recursively as follows [12]:

- A graph on a single vertex is a cograph.

- If G_1, G_2, …, G_k are cographs, then so is their union $G_1 \cup G_2 \cup \cdots \cup G_k$.
- If G is a cograph, then so is its complement $\overline{G}$.

Cographs can be obtained from single node graphs by performing a finite number of unions and complementations.

Theorem 2 *Cographs are confluent.*

Proof: If cographs A and B are confluent, we can show $A \cup B$ and $\overline{A \cup B}$ are confluent too. First we draw A confluently inside a disk and attach a "tail" to the boundary of the disk. Connect the attachment point to each vertex in the disk. B is drawn in the same way. Then $A \cup B$ is formed by joining the two "tails" together so that they don't connect to each other. $\overline{A \cup B}$ is formed by joining the two "tails" of $\overline{A}$ and $\overline{B}$ together so that they connect to each other. (See Figure 8.) By the definition of cographs and induction it is easy to see that cographs are confluent. □

Figure 8: Confluent $A \cup B$ and $\overline{A \cup B}$.

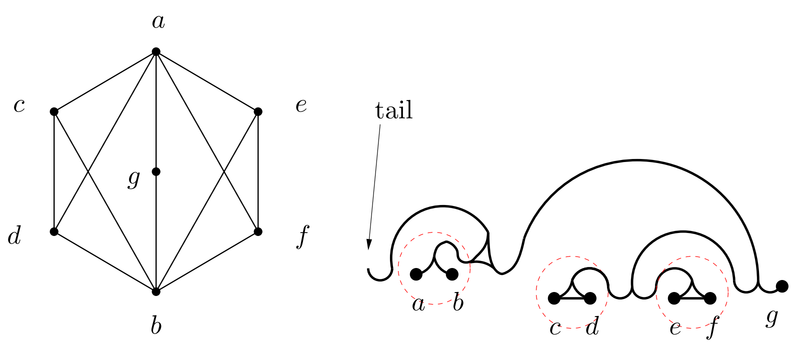


Figure 9: Confluent drawing of a cograph $\overline{\cup}(\overline{\cup}(a, b), \overline{\cup}(\overline{\cup}(c, d), \overline{\cup}(e, f), g))$. Imaginary disks are drawn in dashed circle.

4.2 Complements of trees

The complements of trees (graphs formed by connecting all pairs of vertices that are not connected in some tree) are also called *cotrees*. In general, cotrees are highly non-planar and dense, since a cotree with n vertices has $n(n-1)/2-n+1$ edges. Nevertheless, we have the following interesting fact.

Theorem 3 *The complement of a tree is confluent.*

Proof: We prove the claim by recursive construction, using a single track for the entire graph. Assign a bounding rectangle for the tree and a bounding rectangle for every subtree in that tree. Place the complement of the tree into the bounding rectangles such that nodes of every subtree is within its bounding rectangle and the bounding rectangles of subtrees are contained in their parent's bounding rectangle. In addition, place a connector at the Northeastern corner of every bounding box. This connector is an imaginary point at which the single track for the entire graph will connect into this portion of the cotree. (See Figure 10.) Connect the root node in each subtree to every connector of its children. Connect every node to the connector of its parent. Also connect every node to its siblings and the connectors of its siblings, as shown in the figure. The obtained drawing is the confluent drawing of the complement of the given tree. □

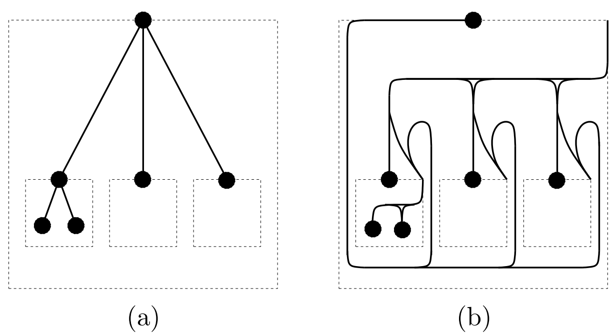


Figure 10: Illustrating a confluent way to draw the complement of a tree: (a) a node and its children in the tree; (b) the corresponding portion of a track in the confluent drawing of the complement.

Paths are very special cases of trees. Every vertex in a path has a degree of 2 except its two endpoints, each of which has a degree of 1. The complement of a path can be drawn using the cotree method in the above proof. We show a nice confluent drawing of the complement of a path in Figure 11.

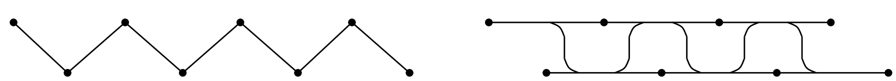


Figure 11: A path of 8 vertices and the confluent drawing of its complement.

4.3 Complements of n-cycles

An n-cycle is a cycle with n vertices.

Theorem 4 *The complement of an n-cycle is confluent.*

Proof: First remove one vertex from the n-cycle and draw the confluent graph for the complement of the obtained path with $n-1$ vertices. Then add the vertex back and connect it with all vertices in the path except for its two neighbors. The obtained drawing is a confluent drawing (see Figure 12 for an example.) □

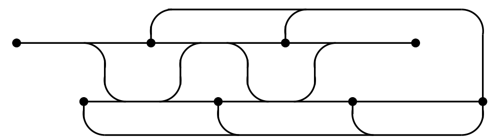


Figure 12: A confluent drawing of the complement of a cycle of 8 vertices.

4.4 Interval graphs

An *interval graph* is formed by a set of closed intervals $S = \{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. The interval graph is defined to have the intervals in S as its vertices and two vertices $[a_i, b_i]$ and $[a_j, b_j]$ are connected by an edge if and only if these two intervals have a non-empty intersection. Such graphs are typically non-planar, but we can draw them in a planar way using a confluent drawing[1].

Theorem 5 *Every interval graph is confluent.*

Proof: The proof is by construction. We number the interval endpoints by rank, $X = \{0, 1, \ldots, n-1\}$, and place these endpoints along the x-axis. We then build a two-dimensional lattice on top of these points in a fashion similar to Pascal's triangle, using a connector similar to an upside-down "V". These connectors stack on top of one another so that the apex of each is associated with a unique interval on X. We place each point from our set S of intervals just under its corresponding apex and connect it into the (single) track so that it can reach everything directly dominated by this apex in the lattice. At the bottom level, we connect the upside-down V's with rounded connectors. By this construction, we create a single track that allows each pair of vertices connected in the interval graph to have a locally-monotone path connecting them. (See Figure 13.) □

5 Some Non-confluent Graphs

In this section, we show that some graphs cannot be drawn confluently. These graphs include the Petersen graph P, the graph $P - v$ formed by removing one vertex from the Petersen graph, and the 4-dimensional hypercube.

[1]A similar construction works for circular-arc graphs and is left as an exercise for the interested reader.

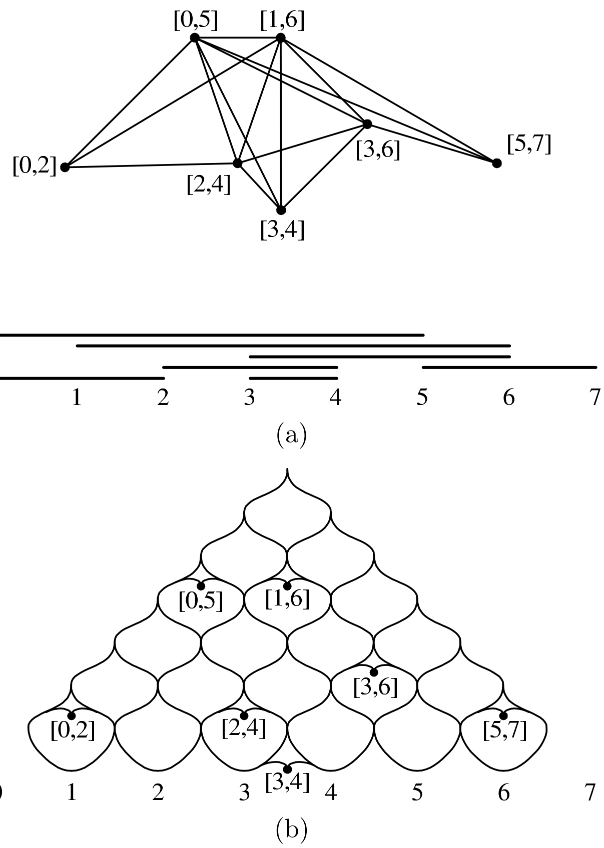


Figure 13: Illustrating a confluent way to draw a non-planar interval graph: (a) an interval graph and its defining intervals; (b) its corresponding confluent drawing.

5.1 The Petersen graph

By removing one vertex and its incident edges from the Petersen graph we obtain a graph homeomorphic to $K_{3,3}$. It contains no $K_{2,2}$ as a subgraph. Moreover, note that $K_{2,2}$ is the most basic structure that allows for edge merging into tracks. Thus the resulting graph is non-confluent. This graph is the smallest non-confluent graph we know of.

The Petersen graph itself is also non-confluent, as adding the vertex and edges back to its non-confluent subgraph does not create any four-cycles that could be used for confluent tracks.

5.2 4-dimensional hypercube

The 4-dimensional hypercube in Figure 15 is non-confluent.

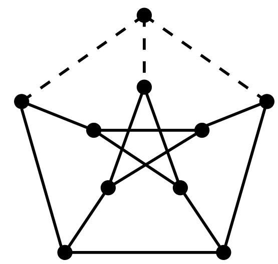


Figure 14: The Petersen graph. The edges incident on one of the vertices are shown dashed.

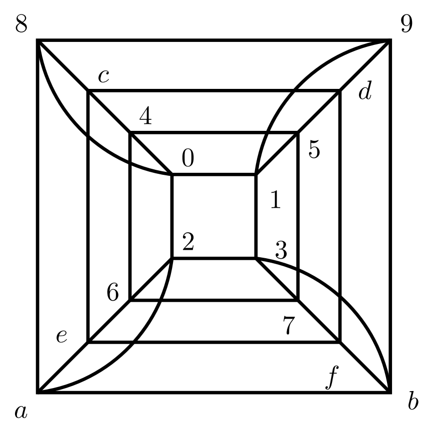


Figure 15: The 4-dimensional hypercube.

The hypercube contains many subgraphs isomorphic to 3-dimensional cubes. Cubes are planar graphs, but in order to show non-confluence for the hypercube, we analyze more carefully the possible drawings of the cubes. Observe that, because there are no $K_{2,3}$ subgraphs in cubes or hypercubes, the only possible confluent tracks are $K_{2,2}$'s formed from the vertices of a single cube face.

Lemma 1 *A cube has only the four confluent drawings (three of them are shown in Figure 16), or combinatorially equivalent rearrangements of these drawings in which we choose a different face as the outer one.*

Proof: For convenience, we consider the drawings to be on a sphere instead of in the plane, so the outer face is not distinguished. Every cube face can be drawn either as a quadrilateral or as a track in a confluent drawing of $K_{2,2}$. We divide into cases based on the number of cube faces replaced by tracks.

Case 0: No faces are replaced by tracks. We get the usual planar drawing of a cube. It is unique because a cube is 3-connected.

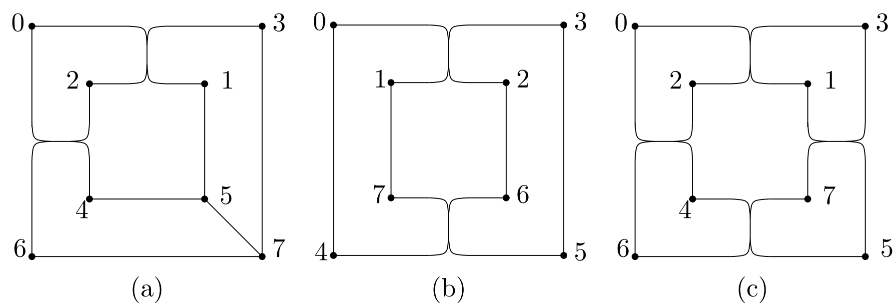


Figure 16: Three confluent drawings of a cube.

Case 1: One face is replaced by a track. This case is not possible, because the underlying graph of the drawing (formed by placing new vertices at track junctions) is non-planar.

Case 2: Only two adjacent faces are replaced by tracks. We have the drawing of Figure 16 (a). It is unique because the underlying planar graph is 3-connected.

Case 3: Two opposite faces are replaced by tracks. We have the drawing of Figure 16 (b). It is unique because the underlying planar graph is 3-connected.

Case 4: Three mutually adjacent faces are replaced by tracks. This case is not possible, even if we allow additional faces to be replaced by tracks as well. For example, suppose the faces $0-1-3-2$, $0-2-6-4$, and $0-1-5-4$ are replaced by tracks. The underlying graph of these replaced edges has a drawing with four faces, in which vertices 3, 5, and 6 are dangling and each may go in either of two faces (Figure 17). However, it is not possible for all three to be in the same face. So they can not all three be connected to vertex 7, as edges incident to 7 can not cross the existing tracks.

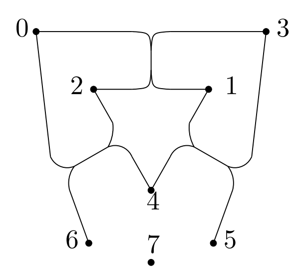


Figure 17: Attempt to use confluent tracks for three mutually-adjacent faces of a cube.

Case 5: Three non-mutually adjacent faces are replaced by tracks. This case is not possible because the underlying graph is non-planar (Figure 18).

Case 6: A ring of four faces is replaced by tracks. We have the drawing of

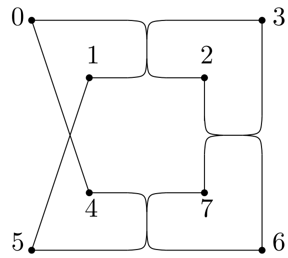


Figure 18: Attempt to use confluent tracks for three non-mutually-adjacent faces of a cube.

Figure 16 (c). It is unique too.

There are no other cases left. Thus a cube only has four confluent drawings. □

Theorem 6 *The* 4*-d hypercube is non-confluent.*

Proof: If we have a valid confluent drawing of the hypercube, and choose eight of its vertices in the form of a cube, the portion of the drawing connecting these vertices must be in one of the forms listed in the lemma above. We consider the four possible drawings of this cube, and attempt to add the other eight vertices (which also form a cube), showing that each case leads to a contradiction. Note that, among the edges of the first cube's drawing, only the ones drawn as single edges can take part in confluent tracks with the remaining eight vertices.

Case 0: In this drawing no faces are replaced. Since the hypercube is non-planar, at least one of its faces must be replaced, so we can always choose our first cube in such a way that this case does not occur.

Case 1: Two adjacent faces of the cube are replaced, as in Figure 16 (a). If only two adjacent faces f_1 and f_2 of the cube C_1 are replaced by tracks, find a different cube C_2 sharing f_1 but not f_2 with C_1. C_2 must have a second replaced face f_3 (it is not possible for a cube to have a confluent drawing with only one face replaced). Either $f_1 - f_3$ and $f_2 - f_3$ are non-adjacent faces of the same cube. So if this case exists, we can find a different cube that is in Case 2 or in Case 3.

Case 3: Two opposite faces of the cube are replaced, as in Figure 16 (b). In this drawing, each face of the cube has only two non-track edges, each of which can be crossed by at most one edge from the rest of the graph. Because the other eight vertices of the graph form a cube which is 3-connected, any subset of these eight vertices has more than two edges connecting to the complement of the subset. So putting any subset of these vertices, other than the whole set, in a single face of the cube drawing does not work. Putting the whole set of the remaining vertices in a single face of the cube drawing does not work either because there are four vertices of the first cube outside that single face to be reached, and only two of them can be reached across the two non-track edges.

Case 4: A ring of four faces of the cube is replaced, as in Figure 16 (c). Edges between the other eight vertices can not cross the tracks, so these vertices must all be placed within a single face of the first cube's drawing. However, these vertices would then be unable to connect to the four or more vertices of the first cube outside that face.

Since all cases fail, the 4-dimensional hypercube is non-confluent. □

5.3 Other non-confluent graphs

If we subdivide every edge of a non-planar graph, by adding a single vertex in the "middle" of each edge, the resulting graph is non-confluent, because the new vertices do not take part in any 4-cycles and so can not be included in any confluent tracks. For the same reason, if, for each edge of a non-planar graph, we add a new vertex and connect this new vertex to both the end points of that edge, the result is also non-confluent. In particular, adding new vertices in this way to the graph K_5 produces a non-confluent chordal graph, so despite our proofs that other graph families with tree-like structures are confluent, chordal graphs are not all confluent.

6 Conclusions

We introduce a new method of drawing non-planar graphs in a planar way. This can be very helpful for drawing graphs in the area of Software Visualization as well as in other areas. It generates aesthetically-pleasing drawings and makes it easier to understand the graph structures. Though we only show its applications on drawing function call graphs and object-interaction graphs, it is powerful for visualizing other kinds of graphs too.

Recently, Peter Hui, Marcus Schaefer and Daniel Štefankovič [25] introduce the concepts of *strongly confluent graphs* and *tree-confluent* graphs. They show that the strongly confluent graphs can be recognized in **NP**. They also show that tree-confluent graphs form a subclass of chordal bipartite graphs and can be recognized in polynomial time.

The complexity of deciding whether a general graph is confluent or not still remains an open problem.

If the planarity condition is relaxed to allow crossings in the drawing, the idea of confluency can be used to reduce crossings in dense graphs. Eppstein, Goodrich and Meng give the *confluent layered drawing* technique [18], which combines ideas of confluent drawing with Sugiyama-style layered drawing. A reduction to graph coloring is used to find and visualize sets of bicliques.

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