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On the approximation of Min Split-coloring and Min Cocoloring

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Abstract

We consider two problems, namely Min Split-coloring and Min Cocoloring, that generalize the classical Min Coloring problem by using not only stable sets but also cliques to cover all the vertices of a given graph. We prove the NP-hardness of some cases. We derive approximation results for Min Split-coloring and Min Cocoloring in line graphs, comparability graphs and general graphs. This provides to our knowledge the first approximation results for Min Split-coloring since it was defined only very recently [8, 9, 13]. Also, we provide some results on the approximability of Min Cocoloring and comparisons with Min Split-coloring and Min Coloring.

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1 Introduction

A generalization of the well known vertex coloring problem (Min Coloring) consists in partitioning the vertex set of a given graph into p cliques and k stable sets. Such a partition is called a *(p, k)-coloring*. In this paper we deal with two natural optimization problems in this context, namely Min Cocoloring and Min Split-coloring.

Given a graph G, the *Min Cocoloring* problem consists in finding the minimum number $(p + k)$ of cliques and stable sets covering the vertices of G. The corresponding optimal value is called *cochromatic number* of G and is denoted by $z(G)$. This problem was first introduced by Lesniak et al. in [23] and extensively studied since then [8, 15, 17].

The *Min Split-coloring*, problem defined first in [13], consists in minimizing the integer $\max(p, k)$ for which a (p, k)-coloring of G exists. This is equivalent to partitioning the vertices of G into a minimum number of *split graphs* (defined as graphs whose vertex set can be partitioned into a clique and a stable set). The optimal value is denoted by $\chi_S(G)$.

Min Coloring consists in minimizing the integer k for which G admits a $(0, k)$-coloring, i.e., is k-colorable. The minimum value is called the chromatic number and is denoted by $\chi(G)$; it satisfies $\chi_S(G) \leq z(G) \leq \chi(G)$. Max Stable and Max Clique consist in maximizing the size of a stable set and a clique, respectively, and $\alpha(G)$ is the maximum size of a stable set in G. A clique on p vertices is a p-clique; it is denoted by K_p.

It is clear that, in general, both Min Cocoloring and Min Split-coloring are NP-hard. There are numerous articles dealing with such coloring problems in general graphs [5, 14, 17] or in restricted classes of graphs [9, 8, 15, 20, 22] to detect polynomial cases and to approximate NP-hard cases. In this paper, we first consider the class of *line graphs*; given a graph G, in its line graph, denoted by $L(G)$, edges of G are replaced by vertices and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. So, coloring the vertices of a line graph $L(G)$ is equivalent to coloring the edges of G; an *edge coloring* is thus a partition of the edge set of G into matchings. We will observe that Min Cocoloring is NP-hard in line graphs. In [9], we show that Min Split-coloring is NP-hard in line graphs of bipartite graphs while Min Cocoloring is polynomial for this class. Here we approximate Min Split-coloring and Min Cocoloring in line graphs. Then, we give an improved approximation of Min Split-coloring in line graphs of bipartite graphs. In addition, noticing that Min Split-coloring is NP-hard in comparability graphs, we give a 2-approximation algorithm for this case; this result is the split counterpart of a result for cocoloring in comparability graphs [15].

A polynomial algorithm is said to guarantee a (standard) approximation ratio of ρ if, for every instance x, $\lambda(x)/\beta(x)$ is at most (for minimization case) ρ, where $\lambda(x)$ denotes the value of a solution of x given by the algorithm and $\beta(x)$ the value of an optimal solution of x. If some ambiguity arises, we write λ_S, β_S (respectively λ_C, β_C) in order to refer the Min Split-coloring (Min Cocoloring). In what follows, unless otherwise stated, approximation ratio stands for stan-

dard approximation ratio. Only in the last section, we will refer to another approximation ratio, called differential approximation ratio.
Differential approximation (which is also called z-approximation [18]) is an alternative way of looking at approximation algorithms. Min Coloring, for instance, is known to be approximable within a constant from this point of view [19, 12] while it is not the case from the usual point of view. This ratio is extensively discussed in [11, 10]; many studies in this area have pointed out that both ratios are complementary without trivial links between them, which emphasizes the interest to systematically study a problem by using both ratios. In the last section, we recall the definition of this ratio and we study the differential approximation behavior of Min Cocoloring and Min Split-coloring problems in general graphs. In particular, we show that Min Split-coloring and Min Cocoloring are better approximable than Min Coloring in terms of differential approximation ratio since they admit a differential polynomial time approximation scheme.
Let us state in Table 1 the results obtained in this paper; references are given whenever the results were known before. A "-" in an entry indicates that the corresponding problem has no meaning. Note that $\mathcal{G} = \{G \cup nK_{2n}\}$ is the class of graphs obtained by taking any arbitrary graph G of size n and adding n disjoint cliques of size $2n$ each.
A graph G is a *comparability graph* if there exists an orientation of its edges which is transitive (i.e., if $[xy]$, $[yz]$ are arcs of G, then there is also an arc $[xz]$). A graph is *perfect* if for any induced subgraph the chromatic number is equal to the maximum size of a clique.

Cl. of gr.	Pb.	Complexity	Approx.	Non-approx.
L(G)	χ_S	NP-hard	7/3	4/3-ϵ if P$\neq$NP
	z	NP-hard	2	
L(l.-perf.)	χ_S	NP-hard [9]	2	DFPTAS
	z	$O((m^2 + mn) \log n)$ [9]	-	-
L(Bipart.)	χ_S	NP-hard [9]	1.78	DFPTAS
	z	$O((m + n) \log n)$ [9]	-	-
Compar.	χ_S	NP-hard	2	DFPTAS
	z	NP-hard [26]	1.71 [15]	DFPTAS
$\mathcal{G} = \{G \cup nK_{2n}\}$	χ_S	$O(1)$ [9]	-	-
	z	NP-hard [9]	3/2	DFPTAS
General	χ_S	NP-hard [5]	DPTAS[a]	$n^{1/14-\epsilon}$ if P$\neq$NP, $n^{1/2-\epsilon}$ if coRP$\neq$NP, DFPTAS
	z	NP-hard [5]		

[a]Obviously, this result also holds for all subclasses of finite graphs.

Table 1: Summary of the results.

For a given graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, $\Delta(G)$ stands for the maximum degree of G, i.e., the largest degree $d(x)$ of a vertex x in G. Moreover, $\Gamma(x)$ denotes the set of neighbors of a vertex x and $\bar{G}$ stands for the complementary graph of G. For $V' \subset V$, $G[V']$ denotes the subgraph of G induced by V' while $G \setminus V' = G[V \setminus V']$. In general, graphs will be simple (no loops, no multiple edges). The complement $\bar{G}$ of a graph G is a graph constructed on the same vertex set as G where two vertices are linked if and only if they are not linked in G. See [4] for graph theoretical definitions not given here.

2 Preliminary remarks

Let us first mention the following preliminary result on approximation dealing with standard approximation ratio.

Proposition 1 *There is a reduction which preserves approximation between Min Split-coloring and Min Cocoloring: every r-approximation algorithm for one of these problems gives a $2r$-approximation algorithm for the other one.*

Proof: Suppose we have an r-approximation algorithm for Min Cocoloring giving a solution of value $\lambda_C(G)$ for any graph G. Consider the vertex partition of that solution as a split-coloring of value $\lambda_S(G)$. Clearly, we have $\lambda_S(G) \leq \lambda_C(G) \leq rz(G) \leq 2r\chi_S(G)$ since a minimum split-coloring of G provides a cocoloring of value $2\chi_S(G)$.
Similarly, if we have an r-approximation algorithm for Min Split-coloring giving a solution of value $\lambda_S(G)$ for any graph G, then the value of a cocoloring derived from that solution verifies $\lambda_C(G) \leq 2\lambda_S(G) \leq 2r\chi_S(G) \leq 2rz(G)$. □

Corollary 1 *For every class of graphs for which $z(G)$ (respectively $\chi_S(G)$) can be computed in polynomial time, Min Cocoloring (respectively Min Split-coloring) induces a 2-approximation for Min Split-coloring (respectively Min Cocoloring).*

It follows that $z(G)$ can be polynomially approximated within a factor of 2 in the class of graphs $\mathcal{G} = \{G \cup nK_{2n}\}$. In fact, a better approximation ratio can easily be obtained. It is shown in [9] that for any $G' \in \mathcal{G}$ we have $z(G') = n + z(G)$ where $G' = G \cup nK_{2n}$. Therefore, $z(G') \geq n + 1$ and a cocoloring of value $\lambda_C(G') \leq \frac{3n}{2} + 1$ can easily be obtained by taking a solution on G of value $\lceil \frac{n}{2} \rceil$ (since any pair of vertices forms either a clique or a stable set) and n cliques covering nK_{2n}. This provides an approximation ratio of 3/2.

3 Line graphs

Given a graph G, Min Split-coloring in $L(G)$ consists in covering the edges of G by either *bundles*, i.e., sets of edges adjacent to the same *central* vertex, or triangles (cliques in $L(G)$) and by matchings (stable sets in $L(G)$). We call *Min Edge Split-coloring* in G the Min Split-coloring problem in $L(G)$. The objective is to minimize the maximum between the number of triangles or bundles and the number of matchings covering all edges. The optimal value for G is $\chi'_S(G) = \chi_S(L(G))$. Analogously, we define *Min Edge Cocoloring* in G as being Min Cocoloring in $L(G)$. Here, we minimize the total number of triangles, bundles and matchings covering all edges. Then the optimal value of edge cocoloring for G is $z'(G) = z(L(G))$. Note that a graph is called *line-perfect* whenever its line graph is perfect. In what follows, we devise some approximation algorithms for both Min Edge Split-coloring and Min Edge Cocoloring.

3.1 Complexity results

First, let us mention the following theorem.

Theorem 1 ([9]) *In line-perfect graphs, Min Edge Cocoloring is polynomially solvable in time $O((m^2 + mn)\log n)$ while Min Edge Split-coloring is NP-hard.*

On the other hand, one can show the NP-hardness of both Min Edge Split-coloring and Min Edge Cocoloring.

Proposition 2 **(i)** *Edge 3-cocolorability is NP-complete.*
(ii) *Edge 3-split-colorability is NP-complete.*

Proof: **(i)** It is clearly in NP and we prove its NP-completeness by a reduction from edge 3-colorability (shown to be NP-complete in [21]). Let us consider an instance G of edge 3-colorability. We transform G into an instance $\tilde{G}$ of edge 3-cocolorability by adding 4 disjoint $K_{1,3}$, that is 4 bundles of size 3 each. Note that in any edge 3-cocoloring of $\tilde{G}$, edges of these 4 bundles have to be covered by 3 matchings. Consequently, $\tilde{G}$ is edge 3-cocolorable if and only if G is edge 3-colorable.
(ii) A similar argument shows that edge 3-colorability also reduces to edge 3-split-colorability. In order to show that, we obtain an instance $\tilde{G}_S$ of edge 3-split-colorability from an instance G of edge 3-colorability by adding 3 bundles of size 4 each. Then it suffices to observe that in any edge 3-split-coloring of $\tilde{G}_S$, edges of 3 disjoint $K_{1,4}$ have to be covered by 3 bundles. This implies that $\tilde{G}_S$ is edge 3-split-colorable if and only if G is 3-edge-colorable. □

Since both Min Edge Split-coloring and Min Edge Cocoloring have integral values, we can immediately deduce:

Corollary 2 *Both Min Edge Split-coloring and Min Edge Cocoloring are not approximable within a factor of $\frac{4}{3} - \epsilon$, unless P=NP.*

3.2 Approximation results

First of all, Corollary 1 combined with Theorem 1 allows us to state the following approximation result.

Proposition 3 *Min Edge Cocoloring provides a 2-approximation for Min Edge Split-coloring in line-perfect graphs in time $O((m^2 + mn)\log n)$.*

Indeed, an optimal edge cocoloring of an instance is a 2-approximation of the same instance now viewed as an instance of Min Edge Split-coloring.
It can be easily observed that this bound of 2 is tight for the graph $G = pK_{2p} \cup pK_p$ which is obviously the line graph of a line-perfect graph. More precisely, we have $z(G) = 2p$ by taking $2p$ cliques. This solution induces a split-coloring of value $2p$ as well. Nevertheless, we have $\chi_S(G) = p$ by choosing p cliques of size $2p$ and p stable sets covering the remaining p cliques of size p each.
Let $\mathcal{A}$ be a polynomial time algorithm computing a $(\Delta+1)$-edge-coloring for any graph of maximum degree Δ [24] and an optimal edge-coloring for line-perfect graphs [7]. We consider the following algorithm for Min Edge Split-coloring:

```
Greedy Edge Split-coloring
(1) R ← ∅;
(2) while |R| < Δ(G)
(3)       pick a vertex x of maximum degree in G;
(4)       R ← R ∪ {x};
(5)       remove x from G;
(6) Compute an edge coloring of the remaining edges by A
    (The solution is the set of edges incident to vertices in R
    completed by that edge coloring.)
```

The main idea is that, if $k = \min\{d : |\{x : d(x) > d\}| \leq d\}$, then by removing all vertices of degree greater than k (the maximum degree is at most k in the remaining graph) and by completing the solution by $k+1$ matchings [24], one finds an edge split-coloring of value $k+1$.

Proposition 4 (i) *For every graph G,* `Greedy Edge Split-coloring` *computes an edge split-coloring of cardinality at most $2\chi'_S(G)+1$.*
(ii) *It provides a 7/3-approximation for Min Edge Split-coloring.*
(iii) `Greedy Edge Split-coloring` *provides a 2-approximation for Min Edge Split-coloring in line-perfect graphs.*

Proof: Let us consider a graph $G = (V, E)$, it is straightforward to verify that `Greedy Edge Split-coloring` computes a split-coloring of G; we denote by λ_{Gr} its value. Let $k = \min\{d : |\{x : d(x) > d\}| \leq d\}$. In what follows, we show that $\lambda_{Gr} \leq k+1 \leq 2\chi'_S(G)+1$.
(i) Let us first note that if $\chi'_S(G) = 1$, then $\lambda_{Gr}(G)$ is either 1 or 2; on the other hand, if $\chi'_S(G) = 2$, then after 2 iterations of the while-loop the degree is less than 3 and no more than 3 matchings are used at line (6), computing also a solution of value 3 or less. In both cases, $\lambda_{Gr}(G)$ is at most $2\chi'_S(G)$. In what follows, we assume that $\chi'_S(G) \geq 3$.
Note that $\lambda_{Gr} \leq |R|+1$ since $|R| \geq \Delta(G \setminus R)$, where $G \setminus R = G[V \setminus R]$. Let r be the last vertex introduced in R and $R' = R \setminus \{r\}$; we have $|R'| < \Delta(G \setminus R')$ and consequently $d(r) \geq |R'|+1 = |R|$. Since vertices are introduced in R in decreasing order of their degree, every vertex in R has degree at least $|R|$. Consequently, $|\{x : d(x) \geq |R|\}| \geq |R|$. It means that $|R| < \min\{d : |\{x :$

$d(x) \geq d\}| < d\}$. It is straightforward to verify that $\min\{d : |\{x : d(x) \geq d\}| < d\} = k + 1$ and thus $\lambda_{Gr} \leq |R| + 1 \leq k + 1$.
In order to show $k \leq 2\chi'_S(G)$, we prove the following lemma:

Lemma 1 *Consider an optimal edge split-coloring of value $\chi'_S(G)$ minimizing the number of triangles among optimal edge split-colorings of G. Denote by T the set of triangles and by B the set of bundles in this solution ($|T|+|B| \leq \chi'_S(G)$). Let X be the set of vertices of degree at least $2\chi'_S(G) + 1$ that are not center of a bundle in B. Then $|X| \leq 3$.*

Proof: Let $x \in X$, we denote by T_x the set of triangles in T incident to x and by B_x the set of bundles centered on neighbors of x (by definition of X, x is not a center of a bundle in B). Since the solution minimizes the number of triangles, any two triangles in T are edge-disjoint and no center of a bundle in B belongs to a triangle in T. Consequently, $T_x \cup B_x$ contains exactly $|B_x| + 2|T_x|$ edges incident to x. Since only $\chi'_S(G)$ edges incident to x can be covered by matchings in the solution, $|B_x| + 2|T_x| \geq \chi'_S(G) + 1$. Let us then define a bipartite graph $I = (X, T \cup B, E_I)$ with $xr \in E_I \Leftrightarrow r \in T_x \cup B_x$, i.e., x is incident to an edge of bundle or triangle r. Vertices in T have a degree at most 3 in I and vertices in B have a degree at most $|X|$ in I. We then have:

$$\sum_{x \in X} (|B_x| + 2|T_x|) \geq (\chi'_S(G) + 1)|X| \tag{1}$$

$$\sum_{x \in X} (|B_x| + |T_x|) \leq 3|T| + |X||B| \tag{2}$$

We deduce by subtraction:

$$3|T| \geq \sum_{x \in X} |T_x| \geq (\chi'_S(G) - |B| + 1)|X| - 3|T| \geq (|T| + 1)|X| - 3|T|$$

Consequently $|X| \leq 5$. But, in this case, the number of triangles in T with degree 3 in I is at most 2 since a third triangle would have two vertices in common with one of the two other triangles. This contradicts the fact that the triangles are edge disjoint. Then, if $|T| \geq 2$, (2) can be replaced by $\sum_{x \in X}(|B_x| + |T_x|) \leq 2|T| + 2 + |X||B|$ implying $|X| \leq 4$. By the same argument as previously, since any graph generated by 2 triangles and at most 4 vertices can be covered by 2 bundles, at most 1 vertex in T has degree 3 in I implying $|X|(|T|+1) \leq 4|T|+2$ and thus $|X| \leq 3$. Finally if $|T| \leq 1$, (2) becomes $\sum_{x \in X}(|B_x| + |T_x|) \leq 3 + |X||B|$ implying $|X| \leq 3$, which concludes the proof. □

It implies that $|\{x : d(x) > 2\chi'_S(G)\}| \leq \chi'_S(G) + 3 \leq 2\chi'_S(G)$ since $\chi'_S(G) \geq 3$. Then, $k \leq 2\chi'_S(G)$ and $\lambda_{Gr} \leq 2\chi'_S(G) + 1$, which concludes the proof of **(i)**.
(ii) If $\chi'_S(G) \leq 2$, `Greedy Edge Split-coloring` uses clearly no more than 3 colors. If $\chi'_S(G) \geq 3$, then by **(i)** we have $\lambda_{Gr}(G) \leq 2\chi'_S(G) + 1 \leq 7\chi'_S(G)/3$.
(iii) Line-perfect graphs of maximum degree Δ can be edge colored in polynomial time (by $\mathcal{A}$) with Δ colors if $\Delta \geq 3$ and either with 2 or 3 colors if $\Delta = 2$.

If $\Delta = 2$, `Greedy Edge Split-coloring` uses at most 3 colors. If $\Delta \geq 3$, we just have to note that, in the proof of **(i)**, $\lambda_{Gr} \leq k \leq 2\chi'_S(G)$. □

Let us finally remark that the bound is tight in bipartite graphs. Consider namely an integer p, $V_1 = \{x_i, i = 1, \ldots, 2p\}$, $V_2 = \{y_{ij}, i = 1, \ldots, 2p, j = 1, \ldots, p+1\} \cup \{u_i, i = 1, \ldots, p\}$, $E = \{(x_i y_{ij}), i = 1, \ldots, 2p, j = 1, \ldots, p+1\} \cup \{(x_i u_j), i = 1, \ldots, 2p, j = 1, \ldots, p\}$. Every vertex in V_1 is of degree $2p+1 = \Delta(B)$, $d(u_i) = 2p, i = 1, \ldots, p$ and vertices $y_{ij}, i = 1, \ldots, 2p, j = 1, \ldots, p+1$ are of degree 1. The greedy algorithm removes vertices in V_1 (the related value being $2p$) while the optimal value $p+1$ is achieved by removing $u_1, \ldots, u_p$. The related ratio is $2 - 2/(p+1)$ and consequently the bound is asymptotically tight. The bound 2 is achieved for the same instance without vertices $y_{i(p+1)}$, but in this case, if the greedy algorithm makes the bad choices, it may compute a solution of value $2p$ only.

Proposition 5 *Min Edge Cocoloring is 2-approximable.*

Proof: Let us consider a minimum cocoloring minimizing the number of triangles. Then it is straightforward to verify that it contains either 2 disjoint triangles or 1 or none (since all other solutions can be replaced by solutions of the same value and containing less triangles).
By a similar method as in `Greedy Edge Split-coloring`, one can compute in polynomial time k minimizing $k + 1 + |\{x : d(x) > k\}|$; then there is an edge cocoloring consisting in bundles $\{x : d(x) > k\}$ (represented by their central vertices) completed by (at most) $k+1$ matchings. So we can construct such a solution with $k + 1 + |\{x : d(x) > k\}|$ color classes.
Let us first suppose that the fixed minimum edge cocoloring does not contain any triangle. Then, $|\{x : d(x) > z'(G)\}| \leq z'(G)$ since all bundles of size greater than $z'(G)$ have to be taken as bundles in an optimal solution. Moreover, if $|\{x : d(x) > z'(G)\}| = z'(G)$, an optimal solution (containing only bundles) has been detected at a stage of the computation of k. So we can assume $|\{x : d(x) > z'(G)\}| \leq z'(G) - 1$, but in this case, by definition of k we have:

$$k + 1 + |\{x : d(x) > k\}| \leq z'(G) + 1 + z'(G) - 1 = 2z'(G)$$

If the optimal solution contains some triangles (one or two), one can consider all possible triangles in a solution and then apply the previous argument to the remaining graph. This completes the proof showing that one can compute a 2-approximation of Min Edge Cocoloring in polynomial time. □

Let us now consider Min Edge Split-coloring in bipartite graphs. Given a bipartite graph $B = (V_1, V_2, E)$ and an integer k, let us denote by ${d'}^k(x) = |\Gamma(x) \cap \{y : d(y) \leq k\}|$ the degree of x in the graph obtained by removing all neighbors of x of degree greater than k. For $i = 1, 2$ we also denote by $V_i^k = \{x \in V_i : d(x) > k\}$ and by ${V'}_i^{k,k'} = \{x \in V_i : {d'}^k(x) > k'\}$. For instance, ${V'}_2^{k,k'}$ is the set of vertices in V_2 with a degree greater than k' in the graph obtained by deleting all vertices of V_1 of degree greater than k. Finally, for $i \in \{1, 2\}$, we set $\bar{i} = 3 - i$, i.e., $\{1, 2\} = \{i, \bar{i}\}$. Here $\mathcal{A}$ is an algorithm for computing an edge Δ-coloring of a bipartite graph with maximum degree δ; its complexity is $O(\Delta m)$ [25].

`Bipart. Edge Split-coloring`

$\epsilon \leftarrow (5 - \sqrt{17})/4$;

`For` $i = 1, 2$ `do`

(1) `for every` $x \in V_i$ `compute` $d(x)$;

(2) `for every` $y \in V_{\bar{i}}$ `and every` $x \in \Gamma(y)$, `compute` ${d'}^{d(x)}(y)$;

(3) `for every` $k \in \{1, \dots, \Delta(B)\}$ `compute` $|V_i^k|$ `and` $|{V'}_{\bar{i}}^{\frac{1+\epsilon}{2-\epsilon}k, k}|$;

(4) $d_i \leftarrow \min\{k : |V_i^{\frac{1+\epsilon}{2-\epsilon}k}| + |{V'}_{\bar{i}}^{\frac{1+\epsilon}{2-\epsilon}k, k}| \leq k\}; S_i \leftarrow V_i^{\frac{1+\epsilon}{2-\epsilon}d_i} \cup {V'}_{\bar{i}}^{\frac{1+\epsilon}{2-\epsilon}d_i, d_i}$;

(5) $d_0 \leftarrow \min\{k : |V_1^k| + |V_2^k| \leq k\}; S_0 \leftarrow V_1^{d_0} \cup V_2^{d_0}$;

(6) $i_0 \leftarrow \text{argmin}\{d_i, i = 0, 1, 2\}; S \leftarrow S_{i_0}$;

(7) `Compute an edge coloring of the remaining edges by` $\mathcal{A}$
`(The solution is the set of edges incident to vertices in` S
`completed by that edge coloring.)`

Theorem 2 *`Bipart. Edge Split-coloring` is a $O(mn)$-algorithm approximating Min Edge Split-coloring in bipartite graphs within ratio $2-(5-\sqrt{17})/4 \simeq 1.78$, where $m = |E|$ and $n = |V_1 \cup V_2|$.*

Proof: We take $\epsilon = (5 - \sqrt{17})/4 \simeq 0.22$ as defined in the algorithm. It is the root of $1 + \epsilon = 2(1 - \epsilon)^2$ which is smaller than 1. It follows that $2 - \epsilon \simeq 1.78$ and $\frac{1+\epsilon}{2-\epsilon} \simeq 0.68$.
Let us first note that $d = d_{i_0} = \min\{d_0, d_1, d_2\}$, where $d_0 = \min\{k : |V_1^k \cup V_2^k| \leq k\}$ and for $i = 1, 2$, $d_i = \min\{k : |V_i^{k\frac{1+\epsilon}{2-\epsilon}}| + |{V'}_{\bar{i}}^{k\frac{1+\epsilon}{2-\epsilon}, k}| \leq k\}$. Moreover, it is immediate to verify that `Bipart. Edge Split-coloring` computes a feasible edge split-coloring of value d. More precisely, d is such that the graph obtained by removing at most d vertices is of degree at most d: the maximum degree of the graph obtained by removing $V_1^{d_0} \cup V_2^{d_0}$ is at most d_0 and the graph obtained by removing $V_i^{d_i\frac{(1+\epsilon)}{(2-\epsilon)}} \cup {V'}_{\bar{i}}^{d_i\frac{(1+\epsilon)}{(2-\epsilon)}, d_i}$ has degrees at most d_i, $i = 1, 2$ (note that $\frac{(1+\epsilon)}{(2-\epsilon)} \leq 1$). Concerning the complexity, lines (1), (3), (4) and (5) need $O(m)$ time, line (7) needs $O(\Delta m)$ time and finally line (2) needs $O(mn)$.
Let us now analyze the approximation behavior of the algorithm. Denote by $\ell = \chi'_S(B)$: $\exists L_1 \subset V_1, L_2 \subset V_2, |L_1| = \ell_1, |L_2| = \ell_2, \ell_1 + \ell_2 = \ell$ and $\Delta(B \setminus (L_1 \cup L_2)) \leq \ell$, where $B \setminus (L_1 \cup L_2) = B[(V_1 \cup V_2) \setminus (L_1 \cup L_2)]$. In the sequel, we consider the following cases:

(1) $\ell_1 \geq \ell\epsilon$ and $\ell_2 \geq \ell\epsilon$

(2) $\ell_2 < \ell\epsilon$ with 2 sub-cases **(2.1)**$|V_1^{\ell(1+\epsilon)}| \geq \ell\epsilon$ and **(2.2)** $|V_1^{\ell(1+\epsilon)}| < \ell\epsilon$

(3) $\ell_1 < \ell\epsilon$

Let us point out the following property **(P)** which will be useful:
(P) If $x \in V_i \setminus L_i, i \in \{1,2\}$ and $d(x) \geq \ell + r$, then $|\Gamma(x) \cap L_{\bar{i}}| \geq r$ where $\bar{i} = 3 - i$.
This holds because after removal of $L_{\bar{i}}$, vertex x has degree at most ℓ.
Case (1) $\ell_1 \geq \ell\epsilon$ and $\ell_2 \geq \ell\epsilon$.
By property **(P)**, $V_i^{\ell+l_{\bar{i}}} \subset L_i; i = 1,2$ and then:

$$|V_1^{\ell+\max(\ell_1,\ell_2)} \cup V_2^{\ell+\max(\ell_1,\ell_2)}| \leq |V_1^{\ell+\ell_2}| + |V_2^{\ell+\ell_1}| \leq \ell_1 + \ell_2 = \ell \leq \ell + \max(\ell_1, \ell_2)$$

We deduce $d_0 \leq \ell + \max(\ell_1, \ell_2) \leq \ell(2 - \epsilon)$, where the last inequality holds since we are considering case **(1)**.
Case (2) $\ell_2 < \ell\epsilon$.
By property **(P)**, we have $V_1^{\ell(1+\epsilon)} \subset L_1$.
Sub-case (2.1) $|V_1^{\ell(1+\epsilon)}| \geq \ell\epsilon \Rightarrow |(L_1 \setminus V_1^{\ell(1+\epsilon)})| \leq \ell(1 - \epsilon)$.
Then, property **(P)** implies that ${V'}_2^{\ell(1+\epsilon),\ell(2-\epsilon)} \subseteq L_2$. It follows from the above relations that $|V_1^{\ell(1+\epsilon)} \cup {V'}_2^{\ell(1+\epsilon),\ell(2-\epsilon)}| \leq \ell \leq (2-\epsilon)\ell$, which implies by definition of d_1 (consider $k = \ell(2 - \epsilon)$ in the definition), $d_1 \leq \ell(2 - \epsilon)$.
Sub-case (2.2) $|V_1^{\ell(1+\epsilon)}| < \ell\epsilon$.
For every $x \in V_2 \setminus L_2$ such that ${d'}^{\ell(1+\epsilon)}(x) > \ell(2 - \epsilon)$, we have by property **(P)** $|\Gamma(x) \cap (L_1 \setminus V_1^{\ell(1+\epsilon)})| \geq \ell(1 - \epsilon)$. Then, by considering the number $\mathcal{E}$ of edges between $(L_1 \setminus V_1^{\ell(1+\epsilon)})$ and $({V'}_2^{\ell(1+\epsilon),\ell(2-\epsilon)} \setminus L_2)$ we deduce:

$$(|{V'}_2^{\ell(1+\epsilon),\ell(2-\epsilon)}| - \ell_2)\ell(1 - \epsilon) \leq \mathcal{E} \leq (\ell_1 - |V_1^{\ell(1+\epsilon)}|)\ell(1 + \epsilon) \leq \ell_1\ell(1 + \epsilon)$$

since the maximum degree of V_1 after removing $V_1^{\ell(1+\epsilon)}$ is at most $\ell(1 + \epsilon)$.
We deduce:

$$|{V'}_2^{\ell(1+\epsilon),\ell(2-\epsilon)}| \leq \frac{\ell(1+\epsilon)}{1-\epsilon} = \ell(2 - 2\epsilon)$$

Consequently $|V_1^{\ell(1+\epsilon)}| + |{V'}_2^{\ell(1+\epsilon),\ell(2-\epsilon)}| \leq \ell(2 - \epsilon)$, which implies $d_1 \leq \ell(2 - \epsilon)$.
Case (3) $\ell_1 < \ell\epsilon$.
It corresponds to the second case by interchanging V_1 and V_2. So $d_2 \leq \ell(2 - \epsilon)$ and in all cases, $d = \min\{d_0, d_1, d_2\}$ satisfies the expected ratio. □

4 Comparability graphs

Let us first note the following result allowing us to deduce the hardness of Min Split-coloring in comparability graphs.

Proposition 6 *Let $\mathcal{G}$ be a class of graphs closed under addition of disjoint cliques without link to the rest of the graph and under addition of a complete k-partite graph completely linked to the rest of the graph. If Min Split-coloring is polynomial in class $\mathcal{G}$, then so is Min Cocoloring.*

Proof: Let us consider a graph G of order n such that $z(G) = p + k$ (where p is the number of cliques and k is the number of stable sets in an optimum solution) and let us first assume that $p \leq k$. Consider the graph G' consisting of G and $l = k - p \leq n$ disjoint cliques, each of size $n + 1$, without any link to the rest of the graph. Note that $k - p$ new cliques completed by p cliques and k stable sets of the optimal cocoloring of G form a split-coloring of value k, implying that $\chi_S(G') \leq k \leq n$. Consequently a minimum split-coloring of G' necessarily contains the $k - p$ new cliques completed by p' cliques and k' stable sets of G. Since $\chi_S(G') = \max((k - p + p'), k') \leq k$, we have $p' \leq p$ and $k' \leq k$. On the other hand, $p' + k' \geq k + p$ since the restriction to G of the split-coloring of G' provides a cocoloring of value $p' + k'$. So $p' + k' = p + k$ and this cocoloring of G is optimal.
If $z(G) = p + k$ with $p \geq k$, we show by the same arguments that a minimum cocoloring of G can immediately be deduced from a minimum split coloring of G'', the graph obtained from G by adding $p - k \leq n$ stable sets, each of size $n + 1$ and completely linked to the rest of the graph.
Finally, in both cases, $|k - p| \leq k + p \leq 2\chi_S(G)$, consequently, the reduction runs as follows:

```
>From Split to Coco
(1) P ← ∅; (*P will contain cocolorings of G*)
(2) compute an optimal split-coloring of G;
(3) store in P the related partition; L ← 2χ_S(G);
(4) for every l ∈ {1,...,L} do
(5)      construct G' obtained from G by adding l cliques, each of size
              n+1 without link with the rest of the graph;
(6)      compute an optimal split-coloring of G'
              and store its restriction to G in P;
(7)      construct G'' obtained from G by adding l stable sets, each of
              size n+1 and completely linked to the rest of the graph;
(8)      compute an optimal split-coloring of G''
              and store its restriction to G in P;
(9) Return the best cocoloring stored in P.
```

□

Corollary 3 *Min Split-coloring is NP-hard in comparability graphs.*

Proof: Min Cocoloring is NP-hard even in permutation graphs [26] (a graph G is a *permutation graph* if G and $\bar{G}$ are comparability graphs). This class of graphs is clearly closed under addition of disjoint cliques and under complementation and consequently it satisfies the conditions of Proposition 6. Therefore Min Cocoloring polynomially reduces to Min Split-coloring that is consequently NP-hard in permutation graphs, and then also in comparability graphs. □

In this section, we show that the method proposed in [15] for approximating Min Cocoloring in comparability graphs within a factor of 1.71, can be adapted to

Min Split-coloring with another ratio. Note that a graph G is a *cocomparability graph* if $\bar{G}$ is a comparability graph.

Theorem 3 *Min Split-coloring is 2-approximable for comparability and cocomparability graphs in time $O(n^{7/2})$.*

Proof: Let us first establish the split counterpart of Lemma 2 in [15]:

Lemma 2 *Let $G = (V, E)$ be a perfect graph of order n and let k satisfy $k \geq \sqrt{n}$, then $\chi_S(G) \leq k$ and a split-coloring of size k can be computed in polynomial time.*

Proof: Let $G = (V, E)$ be a perfect graph, we consider a slight modification of procedure SQRTPartition of [15]. It takes k as input and runs as follows:

`SQRT-split-partition`

(1) `while` $k \neq 0$ `and the graph is not empty do`
(2) `If` $\min\{\alpha(G); \alpha(\bar{G})\} \leq k$
(3) `then compute a` k`-coloring of` G `or` $\bar{G}$`, include each clique or stable set in the solution and set` $k \leftarrow 0$
(4) `else find a stable set and a clique of size` $k+1$ `and color the related split graph of size at least` $2k+1$ `with a new color;`
(5) `Set` $k \leftarrow k-1$ `and remove from` G `all already colored vertices.`

It is straightforward to verify that this procedure runs in polynomial time. Moreover if line (3) is executed or if the graph becomes empty it computes a split-coloring of size k . If line (3) is not computed and if k loops are performed, then at least $\sum_{i=0}^{k-1} 2(k-i)+1 = k(k+2) \geq k^2$ vertices are covered and consequently the graph is also covered by k split graphs. □

Let us adapt the algorithm `APPROX COCOLOURING` of [15] for Min Split-coloring:

`Compar.-Split-coloring`

(1) `compute a maximum` r`-colorable subgraph` (C_r, E_r) `of` $\bar{G}$ `and a maximum` r`-colorable subgraph` (S_r, E'_r) `of` G `such that` r `is minimum subject to` $|C_r| + |S_r| \geq n$`;`
(2) `introduce in the solution an` r`-split-coloring of` $C_r \cup S_r$`;`
(3) `remove` $C_r \cup S_r$ `from` G`;`
(4) `complete the solution by the split graphs computed by SQRT-split-partition in the remaining graph.`

The complexity of lines `(1)`,`(2)` and `(3)` is $O(\chi_S(G)n^3) \leq O(n^{7/2})$; it follows from the fact that a maximum r-colorable subgraph of G and $\bar{G}$ can be computed in time $O(n^3)$ in comparability graphs [16] and that $\chi_S(G) \leq \sqrt{n}$. Let us now analyze the complexity of `SQRT-split-partition` for comparability and cocomparability graphs. Line `(4)` of `SQRT-split-partition` is computed at most t times where t is the smallest integer such that $(2k+1)+(2(k-1)+1)+\ldots+(2(k+1-t)+1)\geq n$ or equivalently $t^2-(2k+2)t+n\leq 0$; hence, recalling that we have $k\geq\sqrt{n}$,

$$\begin{aligned} t &= \left\lceil \frac{(2k+2)-\sqrt{(2k+2)^2-4n}}{2} \right\rceil \\ &= \left\lceil \frac{4n}{2\big((2k+2)+\sqrt{(2k+2)^2-4n}\big)} \right\rceil \\ &\leq \lceil \sqrt{n} \rceil \\ &\leq \sqrt{n}+1. \end{aligned}$$

Finding a maximum clique and a maximum stable set in a comparability graph can be done respectively in time $O(n+m)$ and $O(nm)$; therefore, the complexity of this step is dominated by $O(n^{3/2}m)$. This completes the proof of the overall complexity.

Since G can be decomposed into $\chi_S(G)$ cliques and $\chi_S(G)$ stable sets, $r\leq\chi_S(G)$ where r is as defined in `Compar.-Split-coloring`. On the other hand, since $|C_r\cap S_r|\leq r^2$, $n-|C_r\cup S_r|\leq r^2$ and consequently, by Lemma 2, at most $r\leq\chi_S(G)$ split graphs are computed at line (4), the computed split-coloring is of size at most $2\chi_S(G)$ and the proof is complete. Note that this result remains valid for every class of perfect graphs for which subgraphs such as described in line `(1)` of `Compar.-Split-coloring` can be polynomially computed. □

5 General graphs

5.1 Standard approximation ratio

Min Coloring is known to be particularly difficult to approximate since it is not approximable within $n^{1-\epsilon}$ if coRP≠NP and not approximable within $n^{(1/7)-\epsilon}$ if P≠NP [2]. Similar hardness results can immediately be deduced for Min Split-coloring and Min Cocoloring:

Proposition 7 **(i)** *If Min Cocoloring is $n^{(1/2)-\epsilon}$-approximable for $0<\epsilon<1/2$, then Min Coloring is $n^{1-\epsilon}$-approximable.*
(ii) *If coRP≠NP, then for every $\epsilon>0$, Min Cocoloring is not approximable within $n^{(1/2)-\epsilon}$; if P≠NP, then for every $\epsilon>0$, Min Cocoloring is not approximable within $n^{(1/14)-\epsilon}$.*
(iii) *The same holds up to a constant factor for Min Split-coloring.*

Proof: Let $\mathcal{O}$ be an oracle for Min Cocoloring guaranteeing the ratio $n^{(1/2)-\epsilon}$, with $\epsilon<1/2$; the reduction constructs $\tilde{G}$ consisting in $(\lfloor n^{1-\epsilon}\rfloor+1)$ copies of G without link and computes a cocoloring of $\tilde{G}$ by using $\mathcal{O}$. If a copy of G in $\tilde{G}$ is covered only by stable sets, then it outputs this coloring; else it outputs any greedy coloring.

If $\chi(G)\leq n^{\epsilon}$, then $z(\tilde{G})\leq\chi(\tilde{G})=\chi(G)\leq n^{\epsilon}$. As the cocoloring computed by the oracle on $\tilde{G}$ guarantees the ratio $n(\tilde{G})^{(1/2)-\epsilon}$ and $n(\tilde{G})\leq n^2$, it uses at most $(n^2)^{(1/2)-\epsilon} n^\epsilon = n^{1-\epsilon}$ colors. Consequently at least one copy of G in $\tilde{G}$ is covered only by stable sets in the cocoloring computed by $\mathcal{O}$, which leads to a coloring of G using at most $n^{1-\epsilon}$ colors and the ratio $n^{1-\epsilon}$ is guaranteed. If now $\chi(G) > n^\epsilon$, then any coloring of G guarantees the expected ratio, which concludes the proof of **(i)**. **(ii)** follows from hardness results for Min Coloring. Finally **(iii)** is immediately deduced by using Proposition 1. □

This hardness result considerably limits the possibilities for approximating Min Split-coloring or Min Cocoloring in general graphs. A master-slave strategy [1] enables us to reduce these problems to Max Stable and Max Clique with an increase of the ratios by a factor $O(\log n)$ (the approximation counterpart of the algorithm `GREEDY COCOLOURING` of [15]), leading trivially to a $O(n/\log n)$-approximation for both problems; but it seems not so easy to reduce these problems to Min Coloring in order to refine the comparison of their approximation behavior.

5.2 Differential approximation ratio

The framework of the *differential approximation ratio*, also called z-approximation (see for instance [10, 11, 18] for more details about this area) allows such a comparison. Let x be an instance where the value of an optimum solution is $\beta(x)$; given an approximation algorithm, we denote by $\lambda(x)$ the value of an approximate solution for the instance x. Let $\omega(x)$ be the value of a worst solution; it is in general obtained by interchanging minimization and maximization. In some cases $\omega(x)$ is trivial to compute. For instance, for a Min Coloring instance x with n vertices, we have $\omega(x) = n$. Then, the differential approximation ratio is defined by $\delta(x) = [\omega(x) - \lambda(x)]/[\omega(x) - \beta(x)]$ and an algorithm guarantees a differential ratio of r if, for every instance x, $\delta(x) \geq r$. Note that $\delta(x) \in [0, 1]$ and the larger the ratio is, the better, without distinction between maximization and minimization problems. Roughly speaking, this ratio gives the position of the approximated value between the worst and the best one. This ratio has been used since a long time (see for instance [27]) and is extensively discussed in [11]. In particular, it has the advantage of respecting some affine equivalence such as the equivalence between maximum stable set and minimum vertex covering problems while both problems are known to have radically different approximation behaviors for the usual ratio. Works in this context have pointed out that it is often interesting to simultaneously consider both points of view since these ratios provide different pieces of information about combinatorial problems.
For instance, Min Coloring admits constant differential approximation algorithms, the best ratio currently known being 59/72 [12], while it is hard to approximate from the usual ratio framework. On the other side, it does not admit any differential PTAS (differential ratio $1 - \epsilon$, for every $\epsilon > 0$), unless P=NP ([3]). On the contrary, some other problems are constant approximated from the usual ratio and hard to approximate from the differential point of view and, finally, some problems have similar behavior from both points of view. Moreover, every approximation ratio is more or less appropriate to compare

the approximation behavior of different given problems. In what follows, we show that, as for Min Coloring, Min Cocoloring and Min Split-coloring are well approximated from the differential point of view; moreover they appear to be better approximated than the Min Coloring problem from this point of view.
More precisely, we devise a differential PTAS for Min Split-coloring and Min Cocoloring, i.e., a $(1-\epsilon)$-differential approximation algorithm with complexity $O(n^{1+3/\epsilon})$, for every $0 < \epsilon < 1$. On the other hand, a differential FPTAS (the same ratio with complexity polynomial in $1/\epsilon$) cannot be guaranteed, unless P=NP.
For Min Split-coloring, we consider $\lceil n(G)/3 \rceil$ as worst value since one can always assume that each color (except at most one) contains at least 3 vertices (every set of 3 vertices induces in G a split graph). The ratio associated to G is $\delta(G) = [\lceil n(G)/3 \rceil - \lambda(G)]/[\lceil n(G)/3 \rceil - \chi_S(G)]$. Similarly, the differential ratio for Min Cocoloring is $[\lceil n(G)/2 \rceil - \lambda(G)]/[\lceil n(G)/2 \rceil - z(G)]$. Note that a larger worst value such as n could be also used, leading to better approximation ratios. But it is reasonable to consider the little more restrictive values $\lceil n(G)/3 \rceil$ and $\lceil n(G)/2 \rceil$, respectively, in order to avoid this artificial increasing of the final ratio (see [11] where the notion of worst value is discussed). It simply corresponds to restrict the analysis to "reasonable" solutions.

Theorem 4 *DPTAS-split-coco is a $O(n^{3p+1})$-algorithm guaranteeing a differential approximation ratio of $(1-1/p)$ for both Min Split-coloring and Min Cocoloring.*

```
DPTAS-split-coco
(1) while the current graph contains a 3p-stable or a 3p-clique,
                          color such a stable set or clique with a new color;
(2) complete the solution by an exhaustive search on the remaining graph.
```

Proof: For the whole complexity, note that step (2) is computed for a graph without a stable set or a clique of order $3p$ and consequently the order of which is less than the related Ramsey number $R_2(3p, 3p) \le K^p$ for a constant K [6].
It is straightforward to verify `DPTAS-split-coco` computes either a split-coloring or a cocoloring of the instance. The only difference between the two cases arises in line (2) that computes either an optimal split-coloring or an optimal cocoloring in the remaining graph.
We propose an analysis valid for both problems. The problem being fixed, we denote respectively by $\omega(H)$ and $\beta(H)$ the worst value and the optimal value of H, with respect to this problem (consequently $\beta(H)$ stands either for $\chi_S(H)$ or for $z(H)$).
The approximation ratio is proved by induction on $n(G)$ (see also [19]).
If $n(G) < 3p$, then only step (2) is computed and the algorithm finds an optimal solution corresponding to a ratio of 1. Let us now assume that the expected ratio is guaranteed for every graph of order n or less, where $n \ge 3p$ and consider a graph G_{n+1} of order $n+1$. If no clique or stable set of order $3p$ is detected

at step (1), then G_{n+1} is optimally colored at step (2). Else, the algorithm attributes a new color either to a stable set or to a clique of size $3p$ and is then executed on the graph G' obtained from G_{n+1} by deleting these $3p$ vertices. Since G' is of order less than n, the ratio is guaranteed for G'. Note also that:

$$\begin{array}{rcl} \lambda(G_{n+1}) & = & 1 + \lambda(G') \\ \beta(G_{n+1}) & \geq & \beta(G') \\ \omega(G_{n+1}) & \geq & \omega(G') + p \geq \lambda(G_{n+1}) \end{array}$$

which implies:

$$\begin{array}{rl} \omega(G') + p - \lambda(G_{n+1}) & \geq (1 - 1/p)(\omega(G') - \beta(G')) + p - 1 \\ & \geq (1 - 1/p)(\omega(G') + p - \beta(G_{n+1})) \end{array}$$

and then, since $\omega(G_{n+1}) \geq \omega(G') + p$ and δ is increasing with respect to ω, we have:

$$\frac{\omega(G_{n+1}) - \lambda(G_{n+1})}{\omega(G_{n+1}) - \beta(G_{n+1})} \geq \frac{\omega(G') + p - \lambda(G_{n+1})}{\omega(G') + p - \beta(G_{n+1})} \geq (1 - 1/p)$$

which concludes the proof. □

It is straightforward to verify that, since Min Split-coloring (respectively Min Cocoloring) has integral values and $\omega(G) - \chi_S(G)$ is polynomially bounded, an DFPTAS (differential fully polynomial time approximation scheme) would allow to solve it polynomially for any finite graph. Moreover, a result of [3] implies that both problems are PTAS-complete under a Turing reduction preserving FPTAS.

6 Conclusion

We have essentially considered two extensions of the classical coloring problems, namely Min Cocoloring and Min Split-coloring. The complexity status of these problems has been settled for some classes of graphs and approximability has been studied as well. Further research should examine how the approximation algorithms sketched here could be improved; in particular the case of edge-cocoloring could be handled.
Also, subclasses of graphs could be characterized where these problems become polynomially solvable or admit better approximations, like the permutation graphs which will be studied in a forthcoming paper.

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