NP-completeness of the Planar Separator Problems

Junichiro Fukuyama

Department of Mathematics and Computer Science
Indiana State University
http://math.indstate.edu
jxf140@yahoo.com

Abstract

For a given graph G, the *Separator Problem* asks whether a vertex or edge set of small cardinality (or weight) exists whose removal partitions G into two disjoint graphs of approximately equal sizes. Called the *Vertex Separator Problem* when the removed set is a vertex set, and the *Edge Separator Problem* when it is an edge set, both problems are **NP**-complete for general unweighted graphs [6].

Despite the significance of planar graphs, it has not been known whether the *Planar Separator Problem*, which considers a planar graph and a threshold as an input, is **NP**-complete or not.

In this paper, we prove that the Vertex Separator Problem is in fact **NP**-complete when G is a vertex weighted planar graph. The Edge Separator Problem will be shown **NP**-complete when G is a vertex and edge weighted planar graph.

In addition, we consider how to treat the constant $\alpha \in \mathbb{R}^+$ of the α-*Separator Problem* that partitions G into two disjoint graphs of size at most $(1 - \alpha)|V(G)|$. The α-Separator Problem is not **NP**-complete for all real numbers $\alpha \in (0, 1/2]$, because it would imply uncountably many Non Deterministic Turing Machines. We will present a general scheme for treating a constant in computer arithmetic, by introducing the notion of *real numbers comparable with rationals in polynomial time*. This approach allows us to prove **NP**-completeness for each such real number α.

1 Introduction

The *Separator Problem* questions whether a vertex or edge set of small cardinality (or weight) exists in a given graph G whose removal partitions G into two disjoint graphs of approximately equal sizes. It is called the *Vertex Separator Problem* when the removed set is a vertex set, and the *Edge Separator Problem* when it is an edge set. Both are **NP**-complete for general unweighted graphs [6].

This problem originated in the area of VLSI design, and has both theoretical and practical importance, especially for planar graphs [2, 10]. When G is planar, it is called the *Planar Separator Problem*. Recursively using an algorithm/heuristics for the problem is a standard method of designing the layout of a small VLSI chip [14]. In terms of theory, it is closely related to the Planar Separator Theorem proven by Lipton and Tarjan in 1979 [9]. The theorem guarantees the existence of a vertex separator of size $O\left(\sqrt{|V(G)|}\right)$ in planar graph G. Because of the amount and difficulty of the research carried out to date, the class of planar graphs is recognized as one of the most significant classes of graphs not only for the Separator Problem, but also for other important questions in theoretical computer science.

Some polynomial time approximation algorithms for the Edge Separator Problem are known [11, 12, 13]. Most notably, an algorithm of poly-logarithmic approximation ratio has been discovered for the *Graph Bisection Problem* [5], which is a variant of the Edge Separator Problem that partitions G into two disjoint subgraphs of the exactly equal size.

Despite its significance and the efforts made by researchers, it has been unknown whether the Planar Separator Problem is **NP**-complete or not. The only specific hardness result of the Separator Problem except for the general one is the hardness of approximating the Graph Bisection Problem within an additive term [3].

In this paper, we prove that the Vertex Separator Problem is in fact **NP**-complete when G is a vertex weighted planar graph. The Edge Separator Problem will be shown **NP**-complete when G is a vertex and edge weighted planar graph.

In addition, we consider how to treat the constant $\alpha \in \mathbb{R}^+$ of the *α-Separator Problem* that partitions G into two disjoint graphs of size at most $(1-\alpha)\,|V(G)|$. The α-Separator Problem is not **NP**-complete for all real numbers $\alpha \in (0, 1/2]$, because it would imply uncountably many Non Deterministic Turing Machines.

We will present a general scheme for treating a constant in computer arithmetic by introducing the notion of *real numbers comparable with rationals in polynomial time*. Denote by $\mathbb{R}_{\mathbf{Q}}$ the class of such real numbers. We will show **NP**-completeness of the α-Separator Problems for each $\alpha \in (0, 1/2] \cap \mathbb{R}_{\mathbf{Q}}$. By Liouville's Approximation Theorem known in number theory, we can see that an algebraic number is comparable with rationals in polynomial time. As a result, we will prove the **NP**-completeness for a much larger class of numbers than $(0, 1/2] \cap \mathbb{Q}$.

Section 2 of this paper provides general definitions. In Section 3, we will define the notion of comparability with rationals in polynomial time and discuss related facts. Section 4 proves the **NP**-completeness results. The last section includes the conclusions and open problems.

2 General Definitions

Let S be a weighted finite set, *i.e.*, S is associated with a weight function $w : S \to \mathbb{N}$. The weight w is extended to a subset P of S in the natural manner, *i.e.*, $w(P) = \sum_{x \in P} w(x)$. A pair (P, Q) is called a *partition* of S if $P \cup Q = S$ and $P \cap Q = \emptyset$. It is an *even* partition if $w(P) = w(Q)$.

Denote by $V(G)$ and $E(G)$ the vertex and edge sets of a graph G, respectively. Let G have the vertex weight $w_V : V(G) \to \mathbb{N}$. It may be edge weighted. If it is not, let every edge have weight one.

A pair (P, Q) is an *edge separator* of G if it is a partition of $V(G)$. It is called a *vertex separator of G* if $P \cap Q = \emptyset$, $P \cup Q \subseteq V(G)$ and $E(G)$ contains no edge between P and Q. A *separator* is a vertex or edge separator. If (P, Q) is an edge separator, it also denotes the set of the edges between P and Q. If it is a vertex separator, we identify (P, Q) with $V(G) - P - Q$. Its *weight* is the sum of the weights of the elements in (P, Q).

Let $\alpha \in (0, 1/2]$ and $B \in \mathbb{Z}^+$ be given. A separator (P, Q) of G is an *α-separator* if

$$\max \left(w_V(P), w_V(Q) \right) \leq (1 - \alpha) w_V(V(G)). \tag{1}$$

An *α-Separator Problem* asks whether there exists an α-separator of weight at most B in a (restricted) graph. The *Planar α-Vertex and Edge Separator Problems*, which ask similar questions for planar graphs, are the α-Separator Problems that we are most concerned with.

To prove **NP**-completeness of α-Separator Problems in Section 4, we will show reductions from the Partition Problem. It is one of the classical **NP**-complete problems [6], which asks whether a given weighted finite set has an even partition.

A number α is called *algebraic*, if it is a root of a rational polynomial P. Its *degree* is the minimum degree of P with root α. If α is irrational, d is at least 2.

3 Treatment of a Constant in Computer Arithmetic

It will be shown **NP**-complete to ask whether there exists an α-vertex separator of weight at most B in a vertex weighted planar graph G, for a given constant $\alpha \in (0, 1/2]$. One thing that must be considered carefully here is the general question of how to treat constants in computer arithmetic.

Observe that such α cannot be an arbitrary positive real number at most $1/2$. If it were the case, there would be uncountably many **NP**-complete problems, each of which is computed by a Non Deterministic Turing Machine (NTDM).

The standard way to handle a constant in terms of computability is presented in various articles [4, 8]. We say that $\alpha \in \mathbb{R}^+$ is *polynomially computable* if there exists a Deterministic Turing Machine (DTM), which computes $\hat{\alpha} \in \mathbb{Q}$ such that

$$\left| \alpha - \hat{\alpha} \right| \leq 2^{-n}$$

in time bounded by a polynomial in n, for each given non-negative integer n. The following theorem is well-known [8]:

Theorem 1 *The class of polynomially computable real numbers forms a real closed field.*

Using classical numerical algorithms, we see that real algebraic numbers, *i.e.*, real roots of rational polynomials, are polynomially computable, as are all the values of expressions formed with integers and "elementary" functions such as *ln*, *sin*, *arctan* etc.

However, it is possible that we need more than the notion of computability of a real constant when we investigate a discrete structure. To prove that an α-Separator Problem is in **NP**, we need to show Ineq. (1). If α is just a polynomially computable real number, we have only an efficient algorithm to approximate α to any desired accuracy. Suppose we obtain a real number r' that approximates $r = \lfloor (1 - \alpha) w_V (V(G)) \rfloor$ by such an algorithm. Since we do not know the sign of $r' - r$, we can only compute $r + \epsilon$ for a truncation error $\epsilon \in \{-1, 0, 1\}$. There is no precise way to see if the inequality is true, without further information about r. This is our motivation to consider the following subclass of polynomially computable real numbers.

We say that α is *comparable with rationals in polynomial time*, if there exists a DTM M that decides whether or not $\alpha \leq \frac{p}{q}$ in poly-logarithmic time of $\max(|p|, q)$, for any two integers $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$.

Such M can be used to compute both $\lfloor q\alpha \rfloor$ and $\lceil q\alpha \rceil$ in time bounded by a polynomial in $\lceil \log_2 q \rceil$, the number of bits of q. Assume that α is irrational; otherwise, the two integers can be computed easily. Since α is fixed and $q \in \mathbb{Z}^+$ is given, we know the value of $q \lceil \alpha \rceil$ that is no less than both $\lfloor q\alpha \rfloor$ and $\lceil q\alpha \rceil$. In other words, we know the maximum possible number k of bits of the two integers. Using M for $O(k)$ times, we can decide each bit of the two integers in the topdown manner. The maximum integer p such that $q\alpha \geq p$ is $\lfloor q\alpha \rfloor$, and the minimum p such that $q\alpha \leq p$ is $\lceil q\alpha \rceil$. Notice that $q\alpha \notin \mathbb{Z}$, since α is irrational.

Denote by $\mathbb{R}_{\mathbf{Q}}$ the class of real numbers comparable with rationals in polynomial time. We have the following basic properties on $\mathbb{R}_{\mathbf{Q}}$:

Lemma 1 *If α is comparable with rationals in polynomial time,*
i) α is polynomially computable, and
ii) $\alpha^{-1}, r\alpha, \alpha + r \in \mathbb{R}_{\mathbf{Q}}$ for each rational number r.

Proof: i) Given a non-negative integer n, set $q = 2^n$. We can compute $\lfloor q\alpha \rfloor /q$ in polynomial time in n, since $\alpha \in \mathbb{R}_{\mathbf{Q}}$. The real number α is polynomially computable.

ii) The comparability with rationals of $r\alpha$ and $\alpha + r$ is obvious. We show the claim for α^{-1}.

For each given $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$, we want to decide if $\alpha^{-1} \leq \frac{p}{q}$ in time $O\left(\log^k \max(|p|, q)\right)$ for some $k \in \mathbb{Z}^+$.

Since $\alpha \in \mathbb{R}_{\mathbf{Q}}$, there exists a DTM M that decides whether or not $\alpha^{-1} \geq \frac{p'}{q'}$ in poly-logarithmic time of $\max(|p'|, q')$ for any two integers $p' \in \mathbb{Z}$ and $q' \in \mathbb{Z}^+$. Using this M, compute $\lceil q\alpha^{-1} \rceil$ in time $O\left(\log^k \max(|p|, q)\right)$. Return true if $\lceil q\alpha^{-1} \rceil \leq p$, false otherwise.

This algorithm decides if $\alpha^{-1} \leq \frac{p}{q}$ in the desired time bound. $\qquad\square$

Lemma 2 *A real algebraic number α is comparable with rationals in polynomial time.*

Proof: Assume $\alpha \notin \mathbb{Q}$. We use Liouville's Approximation Theorem that is known [1, 7] equivalent to

$$\exists c \in \mathbb{R}^+, \; \forall p \in \mathbb{Z}, \; \forall q \in \mathbb{Z}^+, \; \left|\alpha - \frac{p}{q}\right| > c \cdot q^{-d}.$$

Here d is the degree of α. Since α is irrational, d is at least 2. The difference between $z = q\alpha$ and the integer closest to z is larger than $\epsilon = c \cdot q^{-d+1}$. The integer d and real number c are constants determined by α, and z is algebraic. Compute an approximate value z' of z with error smaller than ϵ. We have $\lfloor z \rfloor = \lfloor z' \rfloor$. It takes $O\left(\log^k \max(|p|, q)\right)$ time for a constant $k \in \mathbb{N}$ to determine whether or not $z = \alpha q \leq p$. The real number α is comparable with rationals in polynomial time. $\qquad\square$

We will show **NP**-completeness of α-Separator Problems for every $\alpha \in (0, 1/2] \cap \mathbb{R}_{\mathbf{Q}}$. Due to Lemma 2, the claims will cover a much larger class of real numbers than $(0, 1/2] \cap \mathbb{Q}$ for α.

4 NP-completeness of the Planar Separator Problems

In this section, we show **NP**-completeness of two α-Separator Problems. Formally, the following two are proven **NP**-complete. Let $\alpha \in (0, 1/2]$ be a fixed real number that is comparable with rationals in polynomial time.

1. Planar $\alpha-$Vertex Separator

  Instance: Planar graph G, weight function $w_V : V(G) \to \mathbb{N}$,
      and integer $B \in \mathbb{N}$.
  Question: Does G have an $\alpha-$vertex separator of weight at most B?

2. Planar $\alpha-$Edge Separator with Vertex and Edge Weights

  Instance: Planar graph G, weight functions $w_V : V(G) \to \mathbb{N}, w_E : E(G) \to \mathbb{N}$,
      and integer $B \in \mathbb{N}$.
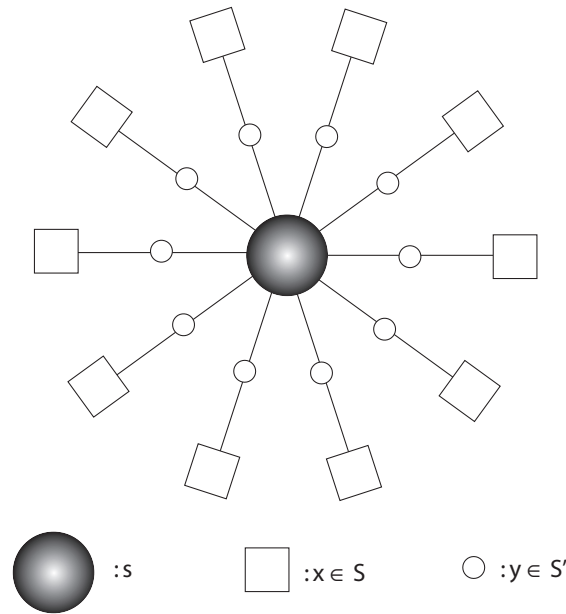 Question: Does G have an $\alpha-$edge separator of weight at most B?

Figure 1: Construction for Problem 1

   The basic proof scheme for **NP**-completeness of the two problems is the following: We reduce the Partition Problem to Problem 2. Given a weighted finite set S, construct a vertex and edge weighted planar graph G, whose vertex set $V(G)$ consists of S and a special vertex s. The weight of each $x \in S$ is $w(x)$, the original weight, and that of s is $r = \left(\frac{1}{2\alpha} - 1\right) w(S)$. Let us assume that r is an integer at this moment. Connect s to every $x \in S$, so that the weight of (s,x) is $w(x)$. The finite set S has an even partition if and only if T has an α-edge separator of weight at most $\frac{1}{2}w(S)$ when $r \in \mathbb{N}$.

   The case of $\alpha = \frac{1}{2}$ is trivially NP-complete for vertex as well as for edge separators. Just consider the graph $G = (S, \emptyset)$ with the given weights on S and with $B = 0$. In both cases, a $\frac{1}{2}$-separator of weight 0 exists if and only if S has an even partition. From now on, we assume $\alpha < \frac{1}{2}$.

To prove **NP**-completeness of Problem 1 for general r, replace the edge (s, x) for each $x \in S$ in the above construction by a path of length two, say $\{(s, y), (y, x)\}$. Denote the set of all y by S', and let $f : S \to S'$ be the bijection with $f(x) = y$. Let $w_V : V(G) \to \mathbb{N}$ be the vertex weight of G. We define

$$
w_V(s) = \left\lfloor (L+1) \left(\frac{1}{2\alpha} - 1 \right) w(S) \right\rfloor
$$

$$
= (L+1) \left(\frac{1}{2\alpha} - 1 \right) w(S) - \epsilon \ \text{ with } 0 \leq \epsilon < 1 \tag{2}
$$

$$
w_V(x) = L \cdot w(x) \text{ for each } x \in S, \text{ and}
$$

$$
w_V(y) = w \left(f^{-1}(y) \right) \text{ for each } y \in S'
$$

where $L > 0$ is a sufficiently large positive integer, e.g.,

$$
L = \max \left(\left\lfloor \frac{2\alpha}{1 - 2\alpha} \right\rfloor, \frac{1}{2} w(S) + 1 \right).
$$

Figure 1 illustrates the construction.

Assuming $\alpha \in \mathbb{R}_{\mathbf{Q}}$, the above weight is computable in polynomial time in $|S|$ due to Lemma 1 ii). Notice that it has been the convention to regard the weight w to be bounded by $2^{p(|S|)}$ for some rational polynomial p [6]. Otherwise, it would take super-polynomial time to fetch it. We will also use the comparability of rationals of α to show that the problems belong to **NP** for each $\alpha \in (0, 1/2] \cap \mathbb{R}_{\mathbf{Q}}$.

We will prove that the constructed graph G has an α-vertex separator of weight at most $\frac{1}{2} w(S)$ if and only if S has an even partition. If we choose L much larger than $w(S)$, it "absorbs" the sum of $w(y) = w_V(y)$ for all $y \in S'$. To see this more clearly, consider a vertex separator of G such that only elements in S' are removed. Let (P, Q) be the corresponding partition of S such that Q and s belong to the same component of the separator.

If $w_V(Q) > w_V(P)$, it means $w_V(Q) \geq w_V(P) + L$. Since L is large, it cannot be an α-vertex separator, no matter how $w_V(y)$ for $y \in S'$ are distributed in both components. If $w_V(P) > w_V(Q)$, the removed vertex set contains the elements in S' that correspond to all $x \in P$. The size of the vertex separator is larger than $\frac{1}{2} w(S)$. As a result, if suffices to consider only when S is partitioned into exact halves.

In the actual proof, we will find that the above value of L is large enough. We are now ready to prove our main claim.

Theorem 2 *Planar $\alpha-$ Vertex Separator is **NP**-complete for each fixed number $\alpha \in (0, 1/2] \cap \mathbb{R}_{\mathbf{Q}}$.*

Proof: It is in **NP**. Suppose we are given a planar graph G with vertex weight $w_V : V(G) \to \mathbb{N}$, and its vertex separator (P, Q). Since α is in $\mathbb{R}_{\mathbf{Q}}$, there exists a DTM that decides whether $q\alpha \leq p$ in poly-logarithmic time of $\max(|p|, q)$, for every $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^+$. Since $\log w_V$ is bounded by a polynomial in $|V(G)|$,

one can decide whether or not (1) holds in polynomial time. The problem belongs to **NP**.

To prove its **NP**-hardness, we show a reduction from the Partition Problem to this problem. Given a finite set S with weight $w : S \to \mathbb{N}$, construct a vertex weighted planar graph G (defined in Figure 1) with $w_V : V(G) \to \mathbb{N}$ as follows: Duplicate S into S' (with the bijection $f : S \to S'$). The vertex set of G is $S \cup S' \cup \{s\}$. Set w_V by (2). The weighted planar graph is constructible in time bounded by a polynomial in $|S|$.

It suffices to show the following lemma:

Lemma 3 *This α-Vertex Separator instance has a solution if and only if the weighted set S has an even partition.*

Proof: Let S have an even partition (P, Q), i.e.,

$$\sum_{x \in P} w(x) = \sum_{x \in Q} w(x).$$

Let $P' = f(P)$ and $Q' = f(Q)$. We will show that the set P' forms an α-vertex separator. On one side, we have P with weight

$$w_V(P) = L \cdot w(P) = \frac{L}{2} \cdot w(S),$$

on the other side, we have $\tilde{Q} = Q \cup Q' \cup \{s\}$ with weight

$$w_V(\tilde{Q}) = w_V(Q) + w_V(Q') + w_V(s)$$

$$= L \cdot w(Q) + w(Q) + (L+1)\left(\frac{1}{2\alpha} - 1\right)w(S) - \epsilon$$

$$= \frac{1}{2}(L+1)\left(\frac{1}{\alpha} - 1\right)w(S) - \epsilon.$$

The total weight of G is

$$w_V(V(G)) = w_V(S) + w_V(S') + w_V(s)$$

$$= (L+1)\frac{1}{2\alpha}w(S) - \epsilon.$$

The weight of $\tilde{Q}$ does not exceed $(1 - \alpha)w_V(V(G))$, because

$$w_V(\tilde{Q}) \leq \frac{1}{2}(L+1)(\frac{1}{\alpha} - 1)w(S) - (1 - \alpha)\epsilon$$

$$= (1 - \alpha)w_V(V(G)).$$

Trivially,

$$v_V(P) \leq \frac{1}{2}w_V(V(G))$$

$$\leq (1 - \alpha)w_V(V(G)).$$

Hence, P' is an α-vertex separator.

Now assume we have an α-vertex separator P' of weight at most $B = \frac{1}{2}w(S)$. Then $w(P) = w_V(P') \leq \frac{1}{2}w(S)$. We will show that $w(P)$ has an exact weight of $\frac{1}{2}w(S)$. From the definition of L, we know that

$$L \geq \frac{2\alpha}{1 - 2\alpha} - 1$$

implying

$$(L + 1)\left(\frac{1}{2\alpha} - 1\right) \geq 1$$

and therefore

$$w_V(s) = \left\lfloor (L+1)\left(\frac{1}{2\alpha} - 1\right)w(S)\right\rfloor \geq w(S) > \frac{1}{2}w(S) = B.$$

Hence, s cannot be in an α-vertex separator of weight at most B. Furthermore, no vertex $x \in S$ can be in an α-Vertex Separator of weight B, because

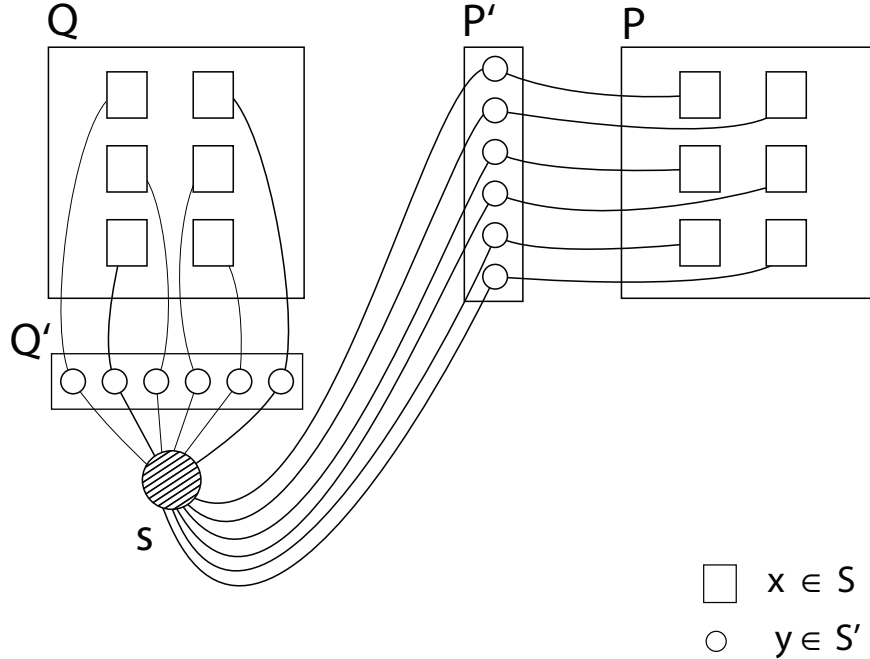$$w_V(x) \geq L > \frac{1}{2}w(S) = B.$$

Figure 2: When $|P'| \leq \frac{1}{2}w(S)$.

Thus our separator P' contains only vertices of S'. Let the separator P' produce the partition $(P, Q \cup Q' \cup \{s\})$ with $P, Q \subseteq S$, $P' = f(P)$ and $Q' = f(Q)$.

As the separator P' has a weight at most B, we obtain

$$w(P) = w_V(P) \leq B = \frac{1}{2}w(S),$$

implying

$$w_V(P) + w_V(P') = L \cdot w(P) + w(P) \leq \frac{1}{2}(L+1)w(S).$$

P' being an α-vertex separator with P on one side implies

$$w_V(P) + w_V(P') \geq \alpha w_V(V(G))$$
$$= \frac{L+1}{2}w(S) - \alpha\epsilon.$$

Therefore, we obtain

$$w(P) = \frac{1}{L+1}(w_V(P) + w_V(P'))$$
$$\geq \frac{1}{2}w(S) - \frac{\alpha}{L+1}\epsilon$$
$$> \frac{1}{2}(w(S) - 1).$$

Hence, $w(P) = \frac{1}{2}w(S)$ and (P, Q) is an even partition of S. □

This completes the proof of Theorem 2. □

Now, it is clearly true that

Corollary 1 *Planar α−Edge Separator with Vertex and Edge Weights is* **NP**-*complete for each $\alpha \in (0, 1/2] \cap \mathbb{R}_{\mathbf{Q}}$.*

Notice that the two problems remain **NP**-complete even if we restrict G to be a weighted tree.

5 Conclusions

We have introduced the notion of a real number comparable with rationals in polynomial time, and have shown that the Planar α-Vertex Separator Problem for a vertex weighted planar graph G is **NP**-complete for every fixed real number $\alpha \in (0, 1/2]$ that is comparable with rationals in polynomial time. The Planar α-Edge Separator Problem for vertex and edge weighted G has also been proven **NP**-complete.

Future research may ask whether the α-Separator Problems remain **NP**-complete, when G is an unweighted planar graph. We do not believe that

NP-completeness can be proven for a planar graph with no vertex weight. It may be possible to show that the Edge Separator Problem is **NP**-complete when G is only vertex weighted.

Moreover, it may be worthwhile to investigate whether some special number such as π is comparable with rationals in polynomial time.

Acknowledgement

J. Fukuyama, *The Planar Separator Problems*, JGAA, 10(2) 317–328 (2006)328

References

[1] T. M. Apostol. *Modular Functions and Dirichlet Series in Number Theory.* Springer-Verlag, 1990.

[2] S. N. Bhatt and F. T. Leighton. A framework for solving VLSI graph layout problems. *J. of Computer and System Sciences*, 28(2):300–343, 1984.

[3] T. N. Bui and C. Jones. Finding good approximate vertex and edge partitions is **NP**-hard. *Information Processing Letters*, 42(3):153–159, 1992.

[4] C. Calude and P. Hertling. Computable approximations of reals: An information-theoretic analysis. *Fundamenta Informaticae*, 33:105–120, 1998.

[5] U. Feige and R. Krauthgamer. A polylogarithmic approximation of the minimum bisection. *Proceeding of FOCS*, 33:105–115, 2001.

[6] M. R. Garey and D. S. Johnson. *Computers and Intractability.* W. H. Freeman and Company, 1979.

[7] G. Hardy and E. Wright. *An Introduction to the Theory of Numbers.* Oxford University Press, 1998.

[8] Ker-I Ko. *Complexity Theory of Real Functions.* Birkhauser, 1991.

[9] R. Lipton and R. E. Tarjan. A separator theorem for planar graphs. *SIAM J. Appl. Math.*, 36(2):177–189, 1979.

[10] R. Lipton and R. E. Tarjan. Applications of a planar separator theorem. *SIAM J. Comp.*, 9(3):615–627, 1980.

[11] G. L. Miller. Finding small simple cycle separators for 2-connected planar graphs. *JCSS*, 32:265–279, 1986.

[12] S. Rao. Finding near optimal separators in planar grpahs. *Proceeding of FOCS*, 20:225–237, 1987.

[13] S. Rao. Faster algorithms for finding small separators in planar graphs. *Proceeding of STOC*, 24:229–240, 1992.

[14] J. D. Ullman. *Computational Aspect of VLSI.* W.H. Freeman and Company, 1984.