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k-colored Point-set Embeddability of Outerplanar Graphs

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Abstract

This paper addresses the problem of designing drawing algorithms that receive as input a planar graph G, a partitioning of the vertices of G into k different semantic categories $V_0, \cdots, V_{k-1}$, and k disjoint sets $S_0, \cdots, S_{k-1}$ of points in the plane with $|V_i| = |S_i|$ ($i \in \{0, \cdots, k-1\}$). The desired output is a planar drawing such that the vertices of V_i are mapped onto the points of S_i and such that the curve complexity of the edges (i.e. the number of bends along each edge) is kept small. Particular attention is devoted to outerplanar graphs, for which lower and upper bounds on the number of bends in the drawings are established.

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1 Introduction

Semantic constraints for the vertices of a graph G define the placement that these vertices must have in a readable visualization of G [5, 10, 13]. For example, in the context of database design some particularly relevant entities of an ER schema may be required to be represented in the center and/or along the boundary of the diagram (see, e.g., [14]). Similarly in social network visualization the positions of the vertices can be defined to reflect their centrality (see, e.g., [4]). A possible way of modeling semantic constraints for a (sub)set $\{v_1, v_2, \cdots, v_h\}$ of the vertices of a graph G is to specify a set $\{p_1, p_2, \cdots, p_h\}$ of locations for their placement. Often, it is sufficient for the application that every vertex v_i ($i = 1, \cdots, h$) is placed at any of the given locations p_j ($1 \leq j \leq h$), that is the mapping of each vertex to a specific location is not part of the input. A key reference in this scenario is the work by Kaufmann and Wiese [11]. Given a planar graph G with n vertices and a set S of n distinct points in the plane, they show how to compute a planar drawing of G such that each vertex is mapped to any point of S and every edge bends at most twice, which is proved to be worst case optimal. It is also known that for specific classes of graphs, such as outerplanar graphs and trees, the number of bends per edge can be reduced to zero (see, e.g., [2, 3, 8]). The work by Kaufmann and Wiese, however, does not seem to be immediately scalable to applications where the vertices of G are grouped based on their relevance or meaning, and for each of these groups a different semantic constraint should be applied (for example a subset of the vertices on the boundary, some others in a central position, and so on). A solution in this case could be to fix in advance the location of each vertex in each semantic category and then route the edges. Halton [9] and, independently, Pach and Wenger [12], showed that a planar graph G always admits a planar drawing such that the location of each vertex is part of the input; however, fixing the vertex positions in advance may give rise to drawings with high visual complexity. Pach and Wenger [12] show that a linear number of bends per edge is asymptotically optimal in the worst case even for graphs as simple as paths. The bounds of Pach and Wenger have been refined in [1].

This paper studies the above mentioned problem without imposing that the position of the vertices is part of the input. The input is a planar graph G, a partitioning of the vertices of G into k different semantic categories $V_0, \cdots, V_{k-1}$, and k disjoint sets $S_0, \cdots, S_{k-1}$ of points in the plane with $|V_i| = |S_i|$ ($i \in \{0, \cdots, k-1\}$). The required output is a planar drawing such that the vertices of V_i are mapped to the points of S_i; for each vertex $v \in V_i$ the drawing algorithm can choose which point of S_i represents v. We study the visual complexity of these types of drawings, expressed in terms of the number of bends per edge. The intuition is that if the number of categories is constant, then a constant number of bends per edge may be sufficient, at least for simple classes of planar graphs. This type of investigation was started in [6, 7], where the apparently simple case of $k = 2$ is studied. In [7] and in [6] a constant number of bends per edge is proved to be sufficient for constructing planar poly-line drawings of subclasses of outerplanar graphs, including paths, cycles, caterpillars, and wreaths. In [6] it is also shown that there exists a 2-outerplanar graph G with a vertex partition V_0, V_1 and two disjoint sets S_0, S_1 of points such that any planar drawing of G that maps a vertex $v \in V_i$ to a distinct point of S_i ($i = 0, 1$) has at least one edge with a linear number of bends. The 2-outerplanarity of the counterexample on one hand, and the outerplanarity of the families of graphs for which a constant number of bends per edge is possible, motivated us to further investigate how many bends are required for general outerplanar graphs and then extend the research to cases where $k > 2$.

In this paper, each integer $i \in \{0, \cdots, k-1\}$ identifying a partition set of the vertices of G is called a *color*, G is called a *k-colored graph*, and the set of points $S = S_0 \cup \cdots \cup S_{k-1}$ such that $|V_i| = |S_i|$ (for each color $i \in \{0, \cdots, k-1\}$) is called a *k-colored set compatible with* G. A planar drawing of G such that each $v \in V_i$ is mapped to a distinct point $p \in S_i$ is a *point-set embedding* of G on S. Graph G is *k-colored point-set embeddable* if it admits a point-set embedding on *every* k-colored set compatible with G. It may be worth remarking that [11] and [9, 12] can be regarded as studies about 1-colored point-set embeddable graphs and n-colored point-set embeddable graphs, respectively. The main focus of this paper is the study of k-colored point-set embeddable graphs for values of k such that $1 \leq k \leq n$. Our main results are as follows.

- Every outerplanar 2-colored graph is 2-colored point-set embeddable with at most 5 bends per edge. Also, a 2-colored embedding of this type can be computed in $O(n \log n)$ time. (See Section 3).
- For every positive integer $h > 0$, there exists an outerplanar 3-colored graph G, whose number of vertices depends on h, and a set of points S compatible with G such that every point-set embedding of G on S has an edge with more than h bends. (See Section 4).
- For k colors in which one restricts all the points of the point set with the same color to vertical regions separated by vertical lines, at most $4k + 1$ bends per edge are required for outerplanar graphs. The drawings can be computed in $O(n \log n + kn)$ time. (See Section 5).

2 Preliminaries

Let $G = (V, E)$ be a graph. A *k-coloring* of G is a partition $\{V_0, V_1, \dots, V_{k-1}\}$ of V where the integers $0, 1, \dots, k-1$ are called *colors*. In the rest of this section the index i is $0 \leq i \leq k-1$ if not differently specified. For each vertex $v \in V_i$ we denote by $col(v)$ the color i of v. For any subset of vertices $U \subseteq V_i$ we denote by $col(U)$ the color i of all the elements of U. A graph G with a k-coloring is called a *k-colored graph*.

Let S be a set of distinct points in the plane. For any point $p \in S$ we denote by $x(p)$ and $y(p)$ the x- and y-coordinates of p, respectively. A *k-coloring* of S is a partition $\{S_0, S_1, \dots, S_{k-1}\}$ of S. A set of points S with a k-coloring is called a *k-colored set*. For each point $p \in S_i$ $col(p)$ denotes the color i of p, and for any subset $R \subseteq S_i$ $col(R)$ denotes the color i of all the elements of R. A k-colored set S is *compatible with* a k-colored graph G if $|V_i| = |S_i|$ for every i; if G is planar we say that G has a *point-set embedding* on S if there exists a planar drawing of G such that: (i) every vertex v is mapped to a distinct point p of S with $col(p) = col(v)$, (ii) each edge e of G is drawn as a polyline λ; a point shared by any two consecutive segments of λ is called a *bend* of e.

A *k-colored sequence* σ is a sequence of (possibly repeated) colors c_0, c_1, ..., c_{n-1} such that $0 \leq c_j \leq k-1$ $(0 \leq j \leq n-1)$. We say that σ is *compatible with* a k-colored graph G if color i occurs $|V_i|$ times in σ. Let S be a k-colored set. Throughout the paper we always assume that the points of S have different x-coordinates (if not we can rotate the plane so to achieve this condition). Let $p_0, p_1, \ldots, p_{n-1}$ be the points of S with $x(p_0) < x(p_1) < \ldots < x(p_{n-1})$. The k-colored sequence $col(p_0), col(p_1), \ldots col(p_{n-1})$ is called the *k-colored sequence induced by S*, and is denoted as $seq(S)$.

A graph G is *Hamiltonian* if it has a simple cycle that contains all its vertices; such a cycle is called a *Hamiltonian cycle* of G. If G is a k-colored graph and $\sigma = c_0, \ldots, c_{n-1}$ is a k-colored sequence compatible with G, a *k-colored Hamiltonian cycle of G consistent with σ* is a Hamiltonian cycle $v_0, v_1, \ldots, v_{n-1}$ such that $col(v_j) = c_j$ $(0 \leq j \leq n-1)$. If such a cycle exists, G is said to be *k-colored Hamiltonian consistent with σ*. Let G be a planar k-colored graph and let σ be a k-colored sequence compatible with G. It is always possible to augment G with dummy edges so that the resulting (not necessarily planar) graph has a k-colored Hamiltonian cycle $\mathcal{H}$ consistent with σ and including all dummy edges. Let Ψ be a planar embedding of G and suppose that Γ is a drawing of $G \cup \mathcal{H}$ such that: (i) The drawing of G in Γ preserves Ψ; (ii) no two dummy edges of $\mathcal{H}$ cross; (iii) each edge e of G crosses the dummy edges of $\mathcal{H}$ at most d times, for some positive integer d. Each crossing between an edge e of G and a dummy edge of $\mathcal{H}$ is replaced in Γ with a dummy vertex that we call a *division vertex for e* and we say that $\mathcal{H}$ is an *augmenting k-colored Hamiltonian cycle of G consistent with σ with at most d division vertices per edge.* We also say that $\mathcal{H}$ (with at most d division vertices per edge) *is constructed on* Ψ.

Lemma 1 *Let G be a planar k-colored graph, let σ be a k-colored sequence compatible with G, and let b be a positive integer. If G admits a point-set embedding with at most b bends per edge on any k-colored set S such that $seq(S) = \sigma$, then G admits an augmenting k-colored Hamiltonian cycle consistent with σ and with at most $b-1$ division vertices per edge constructed on some planar embedding of G.*

Proof: Let S be a set of points $p_0, p_1, \ldots, p_{n-1}$ that lie on a line l parallel to the x-axis such that $x(p_j) < x(p_{j+1})$ $(j = 0, \ldots, n-2)$ and $seq(S) = \sigma$. By hypothesis G admits a point-set embedding Γ on S with at most b bends per edge.

We now describe how to obtain an augmenting k-colored Hamiltonian cycle of G consistent with σ and with at most $b-1$ division vertices per edge. For an illustration refer to Figure 1. Assume vertex v_i of G is placed at point p_i for all i. For each pair of consecutive vertices v_i and v_{i+1} $(0 \leq i \leq n-2)$, if v_i and v_{i+1} are not adjacent in G we add edge (v_i, v_{i+1}) in Γ as a straight-line segment connecting v_i and v_{i+1}. If v_{n-1} and v_0 are not adjacent in G we add edge (v_{n-1}, v_0). This edge is drawn as a polyline as follows. Let p be the leftmost point shared by l and Γ (point p may or may not coincide with v_0) and let q be the rightmost point shared by l and Γ (point q may or may not coincide with v_{n-1}). Edge (v_{n-1}, v_0) consists of three pieces: segment $\overline{pv_0}$ (if p and v_0 coincide this segment is empty), a polyline connecting p and q on the external face of Γ and segment $\overline{qv_{n-1}}$ (if q and v_{n-1} coincide this segment is empty). Cycle $\mathcal{C} = v_0, v_1, \ldots, v_{n-1}$ is an augmenting k-colored Hamiltonian cycle consistent with σ. The edges of $\mathcal{C}$ cross an edge e of Γ a number of times that is at most the number of times that e crosses the line l. Any of the segments of the polyline representing e crosses l only if its endpoints are on different half-planes defined by l. Since the endvertices of e are points of l, a crossing can happen only when two consecutive bends of e are on different half-planes defined by l. Since e has at most b bends, it crosses l at most $b-1$ times, so the result follows. □

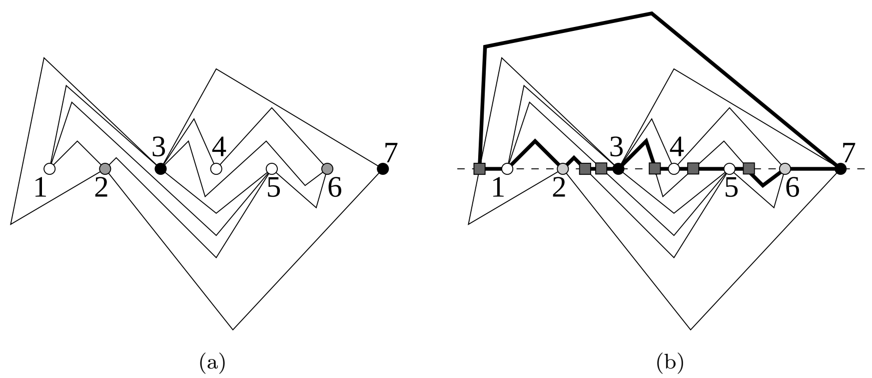


Figure 1: (a) A point-set embedding of a planar graph G with at most 4 bends per edge. (b) An augmenting 3-colored Hamiltonian cycle of G (bold edges). Every edge of G has at most 3 division vertices.

Lemma 2 *Let G be a planar k-colored graph, let σ be a k-colored sequence compatible with G, and let b be a positive integer. If G has a planar embedding on which an augmenting k-colored Hamiltonian cycle consistent with σ and with at most b division vertices per edge can be constructed, then G admits a point-set embedding with at most $2b+1$ bends per edge on any k-colored set S such that $seq(S) = \sigma$.*

Proof: Let S be a k-colored set such that $seq(S) = \sigma$. By hypothesis it is possible to find an augmenting k-colored Hamiltonian cycle $\mathcal{H}$ of G consistent with σ such that each edge has at most b division vertices. We will now use

$\mathcal{H}$ in order to construct the point-set embedding of G on S. The technique is similar to the one described in [11]. For an illustration see Figure 2.

Let $\mathcal{H} = w_0, w_1, \ldots, w_{n'-1}$ be the augmenting k-colored Hamiltonian cycle of G, where the starting vertex w_0 of $\mathcal{H}$ is chosen so that the vertices of G appear in $\mathcal{H}$ ordered coherently with σ. Cycle $\mathcal{H}$ contains also the division vertices, which are not vertices of G. We give these vertices a new color k. In order to host them we define a new set of points S' by adding a suitable number of points to S, all colored k and placed so that if $q_0, q_1, \ldots, q_{n'-1}$ are the points of S' ordered according to their x-coordinates, then $c(q_j) = c(w_j)$ $(j = 0, \ldots, n'-1)$. In the following we denote as G' the augmented graph obtained by adding edges and dummy vertices to G in order to find $\mathcal{H}$.

Determine a planar embedding Ψ of G' such that edge $(w_0, w_{n'-1})$ lies on the external face (notice that Ψ always exists). Map each vertex w_j to point q_j $(j = 0, \ldots, n'-1)$ in S' and draw the edges of path $\mathcal{P} = \mathcal{H} \setminus \{(w_0, w_{n'-1})\}$ as straight-line segments between their end-vertices. Draw each remaining edge e using two segments, one with slope $s > 0$ and the other with slope $-s$. We prevent e from crossing the previously drawn edges in $\mathcal{P}$ by choosing our slope s to be greater than the absolute value of the slope of each edge in $\mathcal{P}$. With segments of slope $\pm s$, it is possible to draw e above or below $\mathcal{P}$. In order for the drawing to preserve the planar embedding Ψ, draw e above $\mathcal{P}$ if e is on the left-hand side when walking from w_0 to $w_{n'-1}$ in G, and below $\mathcal{P}$, otherwise.

The resulting drawing is planar except that edges outside $\mathcal{P}$ that are incident on the same vertex may contain overlapping segments. To eliminate overlapping, perturb overlapping edges by decreasing the absolute value of their segment slopes by slightly different amounts. The slope changes are chosen to be small enough to avoid creating edge crossings while preserving the same planar embedding. For details about this rotation see [11].

The drawing obtained by the technique described above is a point-set embedding of G' on S' with at most one bend per edge. Removing the vertices and edges added to obtain G' from G we have a point-set embedding of G on S. Consider an edge e of G and suppose that e is split by means of b division vertices in G'. Then there are $b+1$ edges in G' corresponding to e. This implies that e has at least $b+1$ bends in the point-set embedding of G on S. Every division vertex w of e has two straight-line segments incident on it in the drawing of G'. If these two segments have different slopes, when w is removed there is an additional bend at the point where w was drawn. This implies that, in the worst case, e has b additional bends for a total of $2b+1$ bends. □

3 Outerplanar 2-colored Graphs

A graph is *outerplanar* if it admits a planar embedding such that all vertices are on the external face. A graph is 2-*outerplanar* if it admits a planar embedding such that removing all vertices on the external face, results in all the remaining vertices being on the external face. The following result is proved in [6].

Theorem 1 [6] *For every $n \geq 4$ there exists a 2-outerplanar 2-colored graph G*

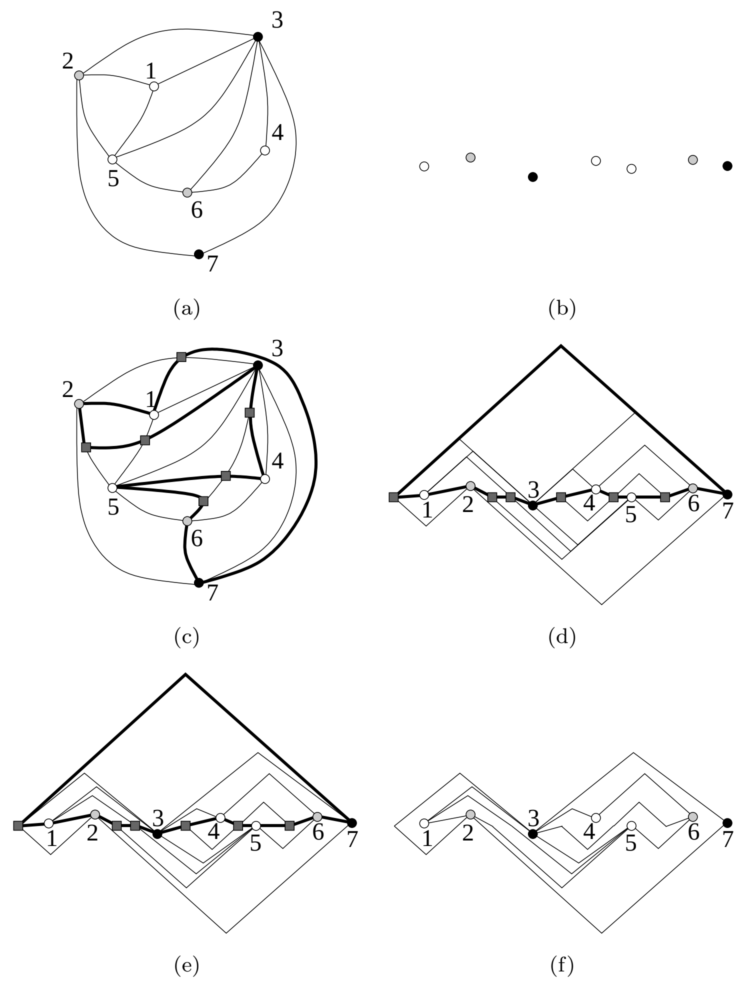
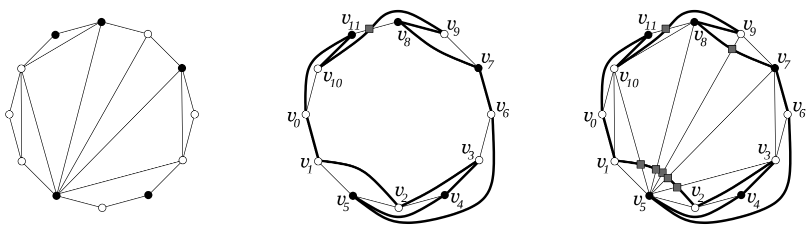


Figure 2: (a) A planar 3-colored graph. (b) A 3-colored set of points S. (c) An augmenting k-colored Hamiltonian cycle of G. Graph G has been augmented to a planar 4-colored graph G' by means of division vertices and dummy edges. Every edge has at most $b = 3$ division vertices. (d) Construction of the point-set embedding of G' on a set of points S' obtained by adding more points to the set S in order to host the division vertices. In this drawing some edges can overlap. (e) The final point-set embedding of G' on S'. Overlaps are removed. Every edge e has at most one bend. (f) The point-set embedding of G on S obtained by removing the division vertices and the dummy edges. Every edge has at most $2b + 1 = 7$ bends per edge.

with $2n$ *vertices and a* 2*-colored set compatible with* G *such that the maximum number of bends per edge of every point-set embedding of* G *on* S *is* $\Omega(n)$*.*

Motivated by Theorem 1, we prove that every outerplanar 2-colored graph G admits a point-set embedding with at most 5 bends per edge on any 2-colored set S compatible with G.

Let Ψ be a planar embedding of G with all vertices on the external face. We prove that G admits an augmenting 2-colored Hamiltonian cycle constructed on Ψ, consistent with $\sigma = seq(S)$, and with at most 2 division vertices per edge. The result then follows from Lemma 2. Since every outerplanar graph can be made biconnected by adding edges while maintaining the outerplanarity, we can assume, without loss of generality, that G is biconnected. In this case the boundary of the external face of G is a simple cycle $\mathcal{C}$ containing all vertices of G. The edges of G that are not in $\mathcal{C}$ are called *chords*. We start by proving that every simple cycle $\mathcal{C}$ admits an augmenting 2-colored Hamiltonian cycle $\mathcal{H}$ consistent with σ and with at most 1 division vertex per edge. To this aim we describe an algorithm that computes $\mathcal{H}$. We then shall describe how it is possible to obtain from $\mathcal{H}$ an augmenting 2-colored Hamiltonian cycle of G consistent with σ and with at most two division vertices per edge. The idea is illustrated in Figure 3.



(a) (b) (c)

Figure 3: (a) An outerplanar 2-colored graph G. (b) An augmenting 2-colored Hamiltonian cycle of the external boundary of G, with at most 1 division vertex per edge. (c) An augmenting 2-colored Hamiltonian cycle of G with at most 2 division vertices per edge.

Let $\sigma = c_0, \ldots, c_{n-1}$ be the color sequence. We order the vertices of $\mathcal{C}$ counterclockwise starting from an arbitrary vertex whose color is c_0. More formally, let v_0 be a vertex of $\mathcal{C}$ such that $col(v_0) = c_0$; we walk counterclockwise along $\mathcal{C}$ starting from v_0 and write $u < v$ if u is encountered before v. Also, given a subset of vertices $U \subseteq V(\mathcal{C})$ we write $\text{first}(U)$ to denote the first vertex of U according to $<$ and with $\text{last}(U)$ the last vertex of U according to $<$. Given a vertex $v \in U$, $\text{next}(v)$ denotes the vertex of U that is immediately after v according to $<$.

The construction of $\mathcal{H}$ begins with v_0 and adds one vertex of $\mathcal{C}$ per step (plus possibly one division vertex). The vertex of $\mathcal{C}$ added at Step i will be denoted as v_i. Also, we denote as H_i the set of vertices added to $\mathcal{H}$ up to Step i and as G_i the augmented graph constructed up to Step i, i.e. the graph consisting of $\mathcal{C}$ plus all dummy edges and division vertices possibly added during Steps

$0, 1, \ldots, i$. We also define the following sets:

- $NB_i = \{v \mid v \in V(\mathcal{C}), v \notin H_i, \text{first}(H_i) < v < \text{last}(H_i)\}$
- $F_i^c = \{v \mid v \in V(\mathcal{C}), \text{last}(H_i) < v, col(v) = c\}$, where $c = 0, 1$ (notice that $F_i^c \cap H_i = \emptyset$)

Sets NB_i and F_i^c partition the set of vertices of $\mathcal{C}$ that must be still added to $\mathcal{H}$ at the end of Step i. Intuitively, the vertices in NB_i have been already encountered moving counterclockwise on $\mathcal{C}$, while the vertices in F_i^c have yet to be encountered. At the end of Step i, the following invariants are maintained:

Invariant 1 *All vertices of NB_i have the same color.*

Invariant 2 *All vertices of NB_i are on the external face of G_i.*

Invariant 3 *Vertex v_i is on the external face of G_i.*

Invariant 4 *If $v_i \neq \text{last}(H_i)$, then for each vertex u such that $v_i < u < \text{last}(H_i)$, we have $u \in H_i$.*

At Step $i+1$ the algorithm chooses the vertex v_{i+1} of $\mathcal{C}$ to be added to $\mathcal{H}$; the addition of v_{i+1} may imply the addition to $\mathcal{H}$ of some dummy edges and one division vertex. We say that a dummy edge is added *inside* $\mathcal{C}$ if it is inserted on the left-hand side when walking counterclockwise around $\mathcal{C}$. In order to choose and add v_{i+1}, the algorithm distinguishes between the following cases:

Case 1: $NB_i \neq \emptyset$ and $col(NB_i) = c_{i+1}$. The algorithm chooses $v_{i+1} = \text{last}(NB_i)$. If v_i and v_{i+1} are not adjacent in $\mathcal{C}$, a dummy edge (v_i, v_{i+1}) is added on the external face of G_i (see, e.g., the addition of vertices v_4 and v_5 in Figure 3(b)).

Case 2: $NB_i = \emptyset$ or $col(NB_i) \neq c_{i+1}$. The algorithm chooses $v_{i+1} = \text{first}(F_i^{c_{i+1}})$. Vertices v_i and v_{i+1} are connected in $\mathcal{H}$ according to the following subcases:

Case 2.a: $v_i = \text{last}(H_i)$. If $v_{i+1} = \text{next}(\text{last}(H_i))$, then v_i and v_{i+1} are adjacent in $\mathcal{C}$ and therefore no dummy edge needs to be added (see, e.g., the addition of vertex v_7 in Figure 3(b)). If $v_{i+1} \neq \text{next}(\text{last}(H_i))$, a dummy edge (v_i, v_{i+1}) is added inside $\mathcal{C}$ (see, e.g., the addition of vertex v_8 in Figure 3(b)).

Case 2.b: $v_i \neq \text{last}(H_i)$ and $v_{i+1} = \text{next}(\text{last}(H_i))$. A dummy edge (v_i, v_{i+1}) is added on the external face of G_i (see, e.g., the addition of vertex v_6 in Figure 3(b)).

Case 2.c: $v_i \neq \text{last}(H_i)$ and $v_{i+1} \neq \text{next}(\text{last}(H_i))$. The algorithm splits edge $(\text{last}(H_i), \text{next}(\text{last}(H_i)))$ by means of a division vertex d, which is added to $\mathcal{H}$ between v_i and v_{i+1}. A dummy edge (v_i, d) is added on the external face of G_i and a dummy edge (d, v_{i+1}) is added inside $\mathcal{C}$. For example, in Figure 3(b), the addition of vertex v_{10} is done by inserting a division vertex d that splits (v_8, v_{11}), and the two dummy edges (v_9, d), (d, v_{10}).

Lemma 3 *Let $\mathcal{C}$ be a 2-colored simple cycle and let σ be a 2-colored sequence compatible with $\mathcal{C}$. Then $\mathcal{C}$ admits an augmenting 2-colored Hamiltonian cycle consistent with σ and with at most 1 division vertex per edge.*

Proof: In order to prove the statement we show that the algorithm described above correctly computes an augmenting 2-colored Hamiltonian cycle of $\mathcal{C}$. To this aim we prove that at the end of Step i the graph G_i is planar (i.e. the dummy edges added by the algorithm do not violate planarity) and that Invariants 1, 2, 3, and 4 hold. We prove this by induction. At step 0 G_0 is planar since $G_0 = \mathcal{C}$ and the four invariants trivially hold because NB_0 is empty and v_0 is on the external face of G_0. Assume now that at the end of Step $i > 0$, G_i is planar and Invariants 1-4 hold. We prove that this is true also at the end of Step $i + 1$.

Planarity In Cases 1 and 2.b the dummy edge (v_i, v_{i+1}) is (possibly) added on the external face of G_i. This edge can be added without violating planarity since both v_i and v_{i+1} are on the external face of G_i. Vertex v_i is on the external face of G_i by Invariant 3, and v_{i+1} is on the external face of G_i either by Invariant 2 because $v_{i+1} \in NB_i$ (Case 1), or because $\text{last}(H_i) < v_{i+1}$ and therefore no dummy edge of G_i has an end-vertex after v_{i+1}. In Case 2.a either no dummy edge is added, or a dummy edge is added inside $\mathcal{C}$. Since $v_i = \text{last}(H_i)$ no dummy edge of G_i has an end-vertex after v_i. Also $v_i < v_{i+1}$ and therefore edge (v_i, v_{i+1}) does not cross any other dummy edge. In Case 2.c two dummy edges are added: (v_i, d) and (d, v_{i+1}). Edge (v_i, d) does not violate planarity because v_i and d are both on the external face of G_i. Vertex v_i is on the external face of G_i by Invariant 3 and d is on the external face of G_i because no dummy edge of G_i is incident on a vertex after $\text{last}(H_i)$. Edge (d, v_{i+1}) can be added without violating the planarity constraint since no dummy edge of G_i is incident on a vertex that is after $\text{last}(H_i)$.

Invariant 1 In Cases 1 and 2.b $NB_{i+1} \subseteq NB_i$ and therefore Invariant 1 holds by induction. In Cases 2.a and 2.c $NB_{i+1} = NB_i \cup U$ where $U = \{u \mid u \in \mathcal{C}, \text{last}(H_i) < u < v_{i+1}\}$. Since $v_{i+1} = \text{first}(F_i^{c_{i+1}})$, then $col(U) \neq c_{i+1}$. We have that either $NB_i = \emptyset$ and hence Invariant 1 holds, or $col(NB_i) \neq c_{i+1}$, i.e. $col(U) = col(NB_i)$ and Invariant 1 holds also in this case.

Invariant 2 In Case 2.a Invariant 2 holds by induction, because in this case either no dummy edge is added, or it is added inside $\mathcal{C}$.

In Cases 1 and 2.b $NB_{i+1} \subseteq NB_i$ and therefore any vertex $u \in NB_{i+1}$ is on the external face of G_i by induction. If u is not on the external face of G_{i+1}, then it must be $v < u < w$, where (v, w) is the dummy edge (possibly) added on the external face of G_i at Step $i+1$. In Case 1 we have $v_{i+1} < v_i$. Namely, if $v_i = \text{last}(H_i)$, then trivially $v_{i+1} < v_i$; if, otherwise, $v_i \neq \text{last}(H_i)$ then, by Invariant 4, no vertex of NB_i is between v_i and $\text{last}(H_i)$ and therefore $v_{i+1} < v_i$ also in this case. Since $v_{i+1} = \text{last}(NB_i)$ then $u < v_{i+1}$ for any $u \in NB_{i+1}$. Thus Invariant 2 holds in Case 1. In

Case 2.b $v_i < v_{i+1}$ because $v_{i+1} = \text{next}(\text{last}(H_i))$. Also, by Invariant 4, no vertex of NB_i is between v_i and $\text{last}(H_i)$. This implies that $u < v_i$ for any $u \in NB_{i+1}$. Also in Case 2.b Invariant 2 holds.

In Case 2.c two dummy edges are added, edge (v_i, d) and edge (d, v_{i+1}). Edge (d, v_{i+1}) is added inside $\mathcal{C}$ and therefore it does not affect the validity of Invariant 2. If $u \in NB_{i+1}$, then either $u \in NB_i$ or $\text{last}(H_i) < u$. In both cases u is on the external face of G_i. Namely, if $u \in NB_i$, then u is on the external face of G_i by induction, otherwise it is on the external face of G_i because no dummy edge is incident of G_i on a vertex that is after $\text{last}(H_i)$. If u is not on the external face of G_{i+1}, it must be $v_i < u < d$. However, if $u \in NB_i$, then $u < v_i$ because no vertex of NB_i is between v_i and $\text{last}(H_i)$; if, instead, $\text{last}(H_i) < u$, then $d < u$ because d splits edge $(\text{last}(H_i), \text{next}(\text{last}(H_i)))$. In both cases u is on the external face of G_{i+1} and Invariant 2 holds.

Invariant 3 In Cases 2.a, 2.b and 2.c, we have $v_{i+1} = \text{last}(H_{i+1})$. Since none of the dummy edges of G_i is incident on a vertex that is after $\text{last}(H_{i+1})$, v_{i+1} is on the external face of G_{i+1} and Invariant 3 holds. In Case 1 v_{i+1} is on the external face of G_i by induction (since $v_{i+1} \in NB_i$). The addition of edge (v_i, v_{i+1}) leaves v_{i+1} on the external face of G_{i+1} and therefore Invariant 3 holds also in this case.

Invariant 4 In Cases 2.a, 2.b and 2.c, we have $v_{i+1} = \text{last}(H_{i+1})$ and therefore Invariant 4 does not apply. In Case 1 $NB_{i+1} \subset NB_i$ and therefore for any vertex $u \in NB_{i+1}$ we have $u < v_{i+1}$ because $v_{i+1} = \text{last}(NB_i)$. This implies that for any vertex v such that $v_{i+1} < v < \text{last}(H_{i+1})$, we have $v \in H_{i+1}$, i.e. Invariant 4 holds.

This concludes the proof that the algorithm described above correctly computes an augmenting 2-colored Hamiltonian cycle of $\mathcal{C}$. Also, since the algorithm adds at most one division vertex per edge, the augmenting 2-colored Hamiltonian cycle of $\mathcal{C}$ has at most one division vertex per edge. □

Lemma 4 *Let G be an outerplanar 2-colored graph and let Ψ be a planar embedding of G having all vertices on the external face. Let σ be a 2-colored sequence compatible with G. Then G admits an augmenting 2-colored Hamiltonian cycle constructed on Ψ, consistent with σ, and with at most 2 division vertices per edge.*

Proof: Since every outerplanar graph can be made biconnected by adding edges while maintaining outerplanarity, we assume that G is biconnected. Let $\mathcal{C}$ be the boundary of the external face of G in Ψ. We remove all the chords from G and compute an augmenting 2-colored Hamiltonian cycle of $\mathcal{C}$ consistent with σ and with at most one division vertex per edge, by using the algorithm described above. If we add back the chords of G to the graph G_{n-1}, i.e. the augmented graph constructed at the end of the algorithm, these edges will cross the edges of $\mathcal{H}$ that are inside $\mathcal{C}$. For each crossing between an edge $e_{\mathcal{H}}$ of $\mathcal{H}$ and a chord

e_{ch} of G we add a division vertex d that splits both $e_{\mathcal{H}}$ and e_{ch}. Thus we obtain an augmenting 2-colored Hamiltonian cycle of G. To complete the proof we must show that every chord e_{ch} is split by at most two division vertices. To this aim observe that an edge $e_{\mathcal{H}} = (u, w)$ of $\mathcal{H}$ must cross e_{ch} only if an endvertex v of e_{ch} is such that $u < v < w$. We show that for each vertex v there can be at most one edge $e_{\mathcal{H}} = (u, w)$ of $\mathcal{H}$ such that $u < v < w$. Since edge $e_{\mathcal{H}}$ is inside the cycle $\mathcal{C}$, it is a dummy edge added at some Step i ($0 \leq i \leq n-1$) when Cases 2.a or 2.c apply. After the addition of this edge, vertex v and all the other vertices between v and w become vertices of NB_i. According to the algorithm the vertices of NB_i are added to $\mathcal{H}$ by means of edges that are either edges of $\mathcal{C}$ or dummy edges on the external face of G_i (Case 1). This implies that any other dummy edge incident on a vertex between v and w is not inside $\mathcal{C}$ and therefore edge $e_{\mathcal{H}} = (u, w)$ is the only edge of $\mathcal{H}$ such that $u < v < w$. Since each chord e_{ch} has two endvertices, there is at most one division vertex on e_{ch} for each of them and therefore at most two division vertices for each chord. □

Theorem 2 *An outerplanar 2-colored graph G admits a point-set embedding with at most 5 bends per edge on any 2-colored set compatible with G. Such a point-set embedding can be computed in $O(n \log n)$ time, where n is the number of vertices of G.*

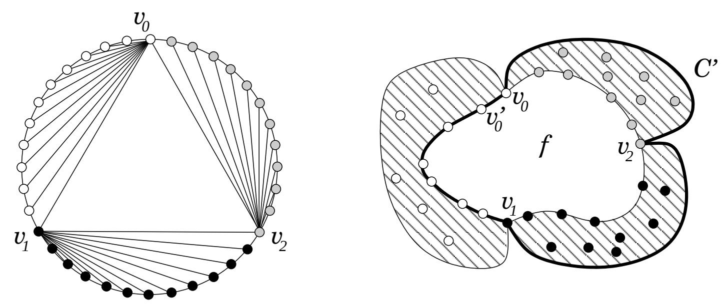
Proof: Let G be any outerplanar 2-colored graph, and let S be any 2-colored set compatible with G. Let $\sigma = seq(S)$. By Lemma 4, G admits an augmenting 2-colored Hamiltonian cycle consistent with σ and with at most 2 division vertices per edge. Therefore by Lemma 2 G admits a point-set embedding with at most 5 bends per edge on S. This concludes the proof concerning the curve complexity.

About the time complexity of the algorithm that computes a point-set embedding, we can consider that: (i) Ordering the points of S according to increasing values of the x-coordinates takes $O(n \log n)$ time; (ii) An augmenting 2-colored Hamiltonian cycle with at most 2 division vertices per edge can be computed in $O(n)$ time using the algorithm described at the beginning of this section; (iii) From the 2-colored Hamiltonian cycle a point-set embedding of G on S can be constructed in $O(n)$ time with the technique illustrated in the proof of Lemma 2. Therefore the overall time complexity is $O(n \log n)$. □

4 Outerplanar 3-colored Graphs

Since, by Theorem 2, outerplanar 2-colored graphs are 2-colored point-set embeddable with $O(1)$ bends per edge, one may wonder whether the number of bends per edge remains constant for values of k larger than 2. Unfortunately, this may not be the case even for 3 colors.

In this section we describe an infinite family of outerplanar 3-colored graphs such that for any integer $h \geq 0$ there exists a graph in the family and a set of points S such that any point-set embedding of the graph on S contains an edge having more than h bends. Our family of outerplanar 3-colored graphs is parametric with n; every member of this family, denoted as G_n, has $3n$ vertices and is defined as follows. G_n consists of a simple cycle formed by n vertices of color 0, followed (in the counterclockwise order) by n vertices of color 1, followed by n vertices of color 2 (notice that G_n actually has $3n$ vertices). The vertex of color 1 adjacent in the cycle to a vertex of color 0 is denoted as v_1; the vertex of color 2 adjacent in the cycle to a vertex of color 1 is denoted as v_2; the vertex of color 0 adjacent in the cycle to a vertex of color 2 is denoted as v_0. Also, in G_n every vertex colored i is adjacent to v_i ($i = 0, 1, 2$) and vertices v_0, v_1, v_2 form a 3-cycle. Figure 4(a) is an example of G_n for $n = 12$.



(a) (b)

Figure 4: (a) Graph G_{12}. (b) An illustration for the proof of Lemma 5. Cycle C' is highlighted in bold.

To prove our lower bound, we consider all possible planar embeddings of G_n; for each planar embedding Ψ of G_n we will prove that every augmenting k-colored Hamiltonian cycle constructed on Ψ and consistent with $seq(S)$ has more than h division vertices for some edge of G_n.

For a planar embedding Ψ of G_n and a cycle $C \in G_n$ we say that C *separates* a subset V' from a subset V'' of the vertices of G_n if all vertices of V' lie in the interior of the region bounded by C and all vertices of V'' are in the exterior of this region.

Lemma 5 *Let $h > 0$ be a positive integer and let G_n be such that $n = (h+1)(25h^2+2)+25h^2$. Let Ψ be a planar embedding of G_n and let C be the cycle v_0, v_1, v_2. If C does not separate any two vertices, then at least one of the following conditions holds for Ψ:*

C.1 *There exists a face f in Ψ having at least $25h^2$ vertices of each color and containing vertices v_0, v_1, and v_2.*

C.2 *There exists a cycle C' containing vertices v_0, v_1, and v_2 and such that: (i) C' has at most $25h^2+2$ edges, (ii) at least $(h+1)(25h^2+2)$ vertices of a color i ($i = 0, 1, 2$) are inside C', (iii) every vertex of $V - \{v_0, v_1, v_2\}$ with color different from i is outside C'.*

Proof: It is simple to observe that in every planar embedding of G_n there are exactly two faces whose boundary contain vertices of the three colors. If C does not separate any two vertices, one of these two faces has C as its boundary. Let f be the face of Ψ containing vertices of the three colors whose boundary is not C. For each color i in G_n, exactly two vertices of color i are adjacent to vertices of different color, and each vertex of color i except v_i has degree two or three and is adjacent to two vertices of the same color; this implies that all vertices of f having color i are consecutive along the boundary of f $(i = 0, 1, 2)$.

If Condition **C.1** does not hold, then the number of vertices of at least one color, say 0, along the boundary of f is $k < 25h^2$. Let v_0' be the vertex of f having color 0 and adjacent to v_0; if v_0 is the only vertex of f having color 0, let $v_0' = v_1$. Refer to Figure 4(b). Consider the cycle $C' = v_0, v_0', \pi(v_0', v_1), v_1, v_2$, where $\pi(v_0', v_1)$ is the path along the boundary of f that goes from v_0' to v_1. Cycle C' has $k+2 < 25h^2+2$ vertices, therefore it has at most $25h^2+1$ vertices, and hence edges.

As cycle C does not separate any two vertices, and C and C' share exactly two edges $((v_1, v_2), (v_2, v_0))$, it follows that C' separates at least $n - 25h^2 = (h + 1)(25h^2 + 2)$ vertices of color 0 from all vertices of color 1 and 2 distinct from v_1 and v_2. □

We are now in the position to prove the following theorem.

Theorem 3 *For every $h > 0$, there exists an outerplanar 3-colored graph G with $3 \cdot ((h+1)(25h^2 + 2) + 25h^2)$ vertices and a set of points S compatible with G, such that every point-set embedding of G on S has an edge with more than h bends.*

Proof:

We prove the theorem by reductio ad absurdum. Therefore consider that there exists $h > 0$ such that for any outerplanar 3-colored graph G and for any set of points S compatible with G, G admits at least a point-set embedding with at most h bends per edge on S. This also holds for G_n and therefore, by Lemma 1, for any set of points S and for any planar embedding Ψ of G there exists an augmenting 3-colored Hamiltonian with at most $h-1$ division vertices per edge built on Ψ and consistent with $seq(S)$. We shall prove that this is not true. As a matter of fact we shall show that, for each planar embedding Ψ of G_n, every augmenting 3-colored Hamiltonian cycle constructed on Ψ and consistent with the alternating sequence σ of the 3 colors (i.e., $\sigma = 0, 1, 2, 0, 1, 2, \ldots, 0, 1, 2$) has more than $h - 1$ division vertices for some edge of G_n. Let $\mathcal{H}$ be one such augmenting 3-colored Hamiltonian cycle and let C be the cycle v_0, v_1, v_2. Note that in any planar embedding of G_n all vertices having a same color are either inside or outside the region bounded by C; hence if C separates any two vertices u and v, it separates all vertices with the same color as u from all vertices with the same color as v (except, of course, v_0, v_1, and v_2). We distinguish between two cases.

Case 1. Ψ is such that C separates all vertices of a color i, except v_i, from all vertices of a different color j, except v_j.

Case 2. Ψ is such that C does not separate any two vertices of G_n. Without loss of generality, assume that all vertices of G_n are on or outside C.

In Case 1, since $\mathcal{H}$ is consistent with σ, each vertex of color i is adjacent in $\mathcal{H}$ to a vertex of color j. Therefore there are $n-1$ edges of $\mathcal{H}$ that cross C in Ψ; it follows that one edge of C has at least $\frac{n-1}{3} > h$ division vertices.

In Case 2, either Condition **C.1** or Condition **C.2** of Lemma 5 holds for Ψ.

If Condition **C.1** holds, there exists a face f in Ψ having at least $25h^2$ vertices of each color, containing vertices v_0, v_1, and v_2, and such that all vertices of f having color i are consecutive along the boundary of f ($i = 0, 1, 2$). The three edges (v_0, v_1), (v_1, v_2), and (v_2, v_3) are not on the boundary of f.

Consider the $25h^2$ vertices of color 0 (denote them as U_0) that are on the boundary of f. Each of these vertices must be connected in $\mathcal{H}$ to a vertex of color 1 and to a vertex of color 2. Hence, there are $50h^2$ edges of $\mathcal{H}$ incident on the vertices of U_0. If more than h of these edges are completely outside face f, then edge (v_0, v_1) is crossed more than h times, i.e. it has more than h division vertices and the statement is true.

Suppose otherwise that at most h edges incident on the vertices of U_0 are completely outside face f. This implies that at least $50h^2 - h$ edges intersect face f. An edge e of $\mathcal{H}$ intersecting f could not be completely inside f (i.e. it could cross the boundary of f); the portion of e inside f is called a *sub-edge* and will be regarded as an edge possibly having division vertices as its endvertices. Denote the set of sub-edges as SE. We write $u < v$ if u is encountered before v when walking clockwise along the boundary of f, starting from v_0.

We partition SE into six sets: each set $SE_{i,j} \subseteq SE$ ($i, j \in \{0, 1, 2\}$) contains those sub-edges such that one end-vertex is a (division) vertex u such that $v_i < u < v_{(i+1) \bmod 3}$ and the other one is (division) vertex v such that $v_j < v < v_{(j+1) \bmod 3}$. One of these sets $SE_{i,j}$ ($0 \leq i, j \leq 2$) contains at least $\frac{50h^2-h}{6}$ sub-edges; assume this set is $SE_{0,1}$. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be two edges of $SE_{0,1}$. Since e_1 and e_2 cannot cross because they belong to $\mathcal{H}$ which is a simple cycle, it follows that if $u_1 < u_2$, then $v_2 < v_1$. Let $e_m = (u_m, v_m) \in SE_{0,1}$ and $e_M = (u_M, v_M) \in SE_{0,1}$ be the two sub-edges of $SE_{0,1}$ such that for any other sub-edge $(u, v) \in SE_{0,1}$, $u_m < u < u_M$. See also Figure 5.

In order to avoid crossings between sub-edges of $SE_{0,1}$, we have $v_M < v < v_m$. If the number of edges in the path from v_M to v_m along the boundary of f is less than $\frac{50h^2-h}{6h} = \frac{50h-1}{6}$, then at least one such edge has more than h division vertices and the statement is true. Otherwise, the number of edges (and hence the vertices) in the path from v_M to v_m along the boundary of f is at least $\frac{50h-1}{6}$. Denote by v'_M (u'_m respectively) either v_M (u_m respectively), if v_M (u_m respectively) is not a division vertex, or the vertex that is immediately before v_M (u_m respectively), if v_M (u_m respectively) is a division vertex. Analogously, denote by v'_m (u'_M respectively) either v_m (u_M respectively), if v_m (u_M respectively) is not a division vertex, or the vertex that is immediately after v_m (u_M respectively), if v_m (u_M respectively) is a dummy vertex. The cycle $C' = v_0, u'_m, u_m, v_m, v'_m, v_1, v'_M, v_M, u_M, u'_M, v_0$ separates at least $\frac{50h-1}{6}$ vertices of color 1 (denote them as U_1) from all vertices of color 2. Every vertex of U_1

must be connected to a vertex of color 2 in $\mathcal{H}$. Therefore $\frac{50h-1}{6}$ edges must cross cycle C' in order to connect a vertex inside C' to a vertex outside C'. Since in C' there are only 8 edges that can be crossed by edges of $\mathcal{H}$ (the two sub-edges e_m and e_M cannot be crossed because they are in $\mathcal{H}$), then at least one of these 8 edges is crossed $\frac{50h-1}{48}$ times, i.e. has $\frac{50h-1}{48}$ division vertices. Since $\frac{50h-1}{48} > h$ for every $h \geq 1$, the statement holds.

If Condition **C.2** holds, denote by U_i the set of vertices of color i inside the cycle C' in the statement of Lemma 5. Since $\mathcal{H}$ is consistent with σ, each vertex of U_i is adjacent to vertices with color different from i. Thus at least $(h+1)(25h^2+2)$ edges cross cycle C'. Since C' has at most $25h^2+2$ edges, at least one edge of C' has more than h division vertices. □

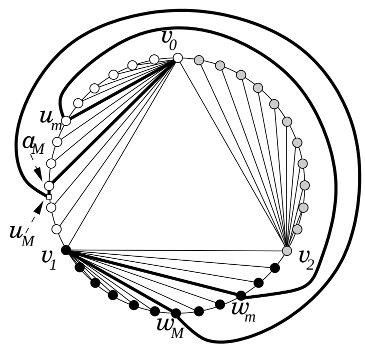


Figure 5: Illustration for the proof Theorem 3.

5 Outerplanar k-colored Graphs

In contrast with the result of Theorem 3, we prove that given any outerplanar k-colored graph G (for any constant $k > 2$), there exist infinitely many k-colored sets compatible with G for which a point-set embedding of G with a constant number of bends per edge is possible.

Let S be a k-colored set such that no two points have the same x-coordinate; assume that the points are ordered by increasing x-values. S is an *ordered k-colored set* if, for every color i ($0 \leq i \leq k-1$), all points of color i appear consecutively in the ordering (that is, the colors never alternate in the ordering). The sequence of colors induced by an ordered k-colored set is called an *ordered k-colored sequence*.

We first describe an algorithm that, given a k-colored simple cycle $\mathcal{C}$ and a k-colored sequence σ compatible with $\mathcal{C}$, computes an augmenting k-colored Hamiltonian cycle $\mathcal{H}$ of $\mathcal{C}$ consistent with σ and with at most k division vertices. We then use this algorithm to obtain an augmenting k-colored Hamiltonian cycle of an outerplanar k-colored graph G consistent with σ and with at most

$2k$ division vertices. Also in this case, the augmenting k-colored Hamiltonian cycle of G is constructed on a planar embedding of G having all vertices on the external face.

Let $\mathcal{C}$ be a k-colored simple cycle and let $\sigma = c_0, \ldots, c_{n-1}$ be an ordered k-colored sequence consistent with $\mathcal{C}$. The algorithm computes $\mathcal{H}$ by performing at most k rounds. At each round, it walks along the cycle $\mathcal{C}$ either in the counterclockwise direction, or in the clockwise direction. At each round we refer to the direction followed in that round as the *walking direction.* We order the vertices of $\mathcal{C}$ from a starting vertex and according to the walking direction. For each walking direction, we use the relation $<$ and the notation $\text{first}(U)$, $\text{last}(U)$, and $\text{next}(v)$ defined in Section 3.

At Step 0, the algorithm adds to $\mathcal{H}$ any vertex v_0 of color c_0. At each Step i ($i = 1, \ldots, n-1$), a new vertex v_i of color c_i is added to $\mathcal{H}$; adding v_i may imply adding some division vertices. Denote by H_i the set of vertices added to $\mathcal{H}$ up to Step i, and as G_i the augmented graph constructed up to Step i, i.e. the graph consisting of $\mathcal{C}$ plus all edges and division vertices added during Steps $0, 1, \ldots, i$. For each Step i we define the following two sets:

- $F_i^c = \{v \mid v \in V(\mathcal{C}), v \notin H_i, \text{last}(H_i) < v, col(v) = c\}$, where $c = 0, \ldots, k-1$
- $N_i(u, w) = \{v \mid v \in V(\mathcal{C}), v \notin H_i, u < v < w\}$, where $u < w$.

At the end of Step i, the following invariants are maintained:

Invariant 5 *Any vertex $v \notin H_i$ is on the external face of G_i.*

Invariant 6 *Vertex v_i is on the external face of G_i.*

At Step $i+1$ the algorithm chooses a vertex v_{i+1} of $\mathcal{C}$ to be added to $\mathcal{H}$. The addition of v_{i+1} to $\mathcal{H}$ may require the addition to $\mathcal{H}$ of some dummy edges and some division vertices. We say that a dummy edge is added *inside* $\mathcal{C}$ if it is added on the left-hand side when walking counterclockwise around $\mathcal{C}$. If $F_i^{c_{i+1}} \neq \emptyset$ we choose $v_{i+1} = \text{first}(F_i^{c_{i+1}})$, else invert the walking direction (which starts a new round) and repeat the test. Vertices v_i and v_{i+1} can be connected in $\mathcal{H}$ according to different cases:

Case 1: $N_i(v_i, v_{i+1}) = \emptyset$**.** Refer to Figure 6(a). If v_i and v_{i+1} are not adjacent in $\mathcal{C}$, a dummy edge (v_i, v_{i+1}) is added on the external face of G_i.

Case 2: $N_i(v_i, v_{i+1}) \neq \emptyset$**.** Refer to Figure 6(b). The vertices of $N_i(v_i, v_{i+1})$ may not form a consecutive sequence along $\mathcal{C}$, because there can be vertices of H_i intermixed with them. Let $\sigma_0, \sigma_1, \ldots, \sigma_{l-1}$ be the maximal sub-sequences of consecutive vertices of $N_i(v_i, v_{i+1})$. For each sub-sequence σ_j, let $s_j = \text{first}(\sigma_j)$ and $t_j = \text{last}(\sigma_j)$. Also, let s'_j be the vertex (either a "real" vertex of $\mathcal{C}$ or a division vertex) that is immediately before s_j and let t'_j be the vertex (either a "real" vertex of $\mathcal{C}$ or a division vertex) that is immediately after t_j. Edges (s'_j, s_j) and (t_j, t'_j) are split by means

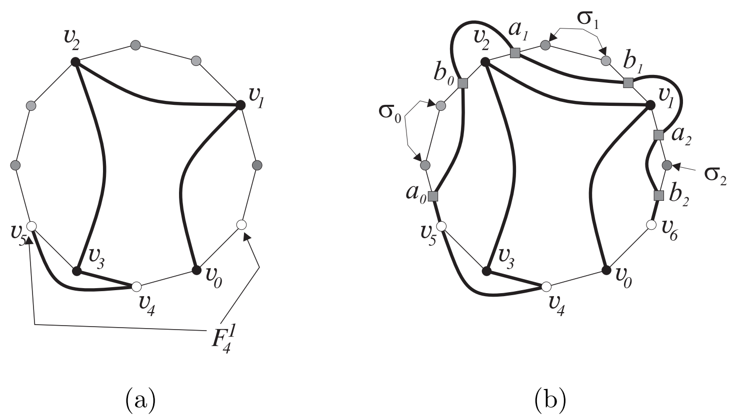


Figure 6: An illustration of different cases of the algorithm to compute $\mathcal{H}$ for a k-colored cycle. (a) Case 1 with $v_i = v_4$ and $v_{i+1} = v_5$. Edge (v_4, v_5) is added on the external face because $N_4(v_4, v_5) = \emptyset$. (b) Case 2 with $v_i = v_5$ and $v_{i+1} = v_6$. Sub-sequences σ_0, σ_1, and σ_2 are highlighted; dummy vertices are drawn as small squares.

of the division vertices a_j and b_j, respectively. The algorithm connects v_i to v_{i+1} in $\mathcal{H}$ by means of the path $v_i, a_0, b_0, a_1, b_1, \ldots, a_{l-1}, b_{l-1}, v_{i+1}$. Edges (v_i, a_0), (b_j, a_{j+1}) $(0 \leq j \leq l-1)$ and (b_{l-1}, v_{i+1}) are added on the external face of G_i, while edges (a_j, b_j) $(0 \leq j \leq l-1)$ are added inside $\mathcal{C}$.

To prove Theorem 4 we first prove the following lemmas.

Lemma 6 *Let $\mathcal{C}$ be a k-colored simple cycle and let σ be an ordered k-colored sequence compatible with $\mathcal{C}$. Then $\mathcal{C}$ admits an augmenting k-colored Hamiltonian cycle consistent with σ and with at most k division vertices per edge.*

Proof: In order to prove the statement we show that the algorithm described in Section 5 correctly computes an augmenting k-colored Hamiltonian cycle of $\mathcal{C}$ consistent with σ. To this aim we prove that at the end of Step i the graph G_i is planar (i.e. the dummy edges added by the algorithm do not violate planarity) and that Invariants 5 and 6 hold. The proof is by induction.

At Step 0, G_0 is planar since $G_0 = \mathcal{C}$ and the two invariants trivially hold. Assume now that at the end of Step $i > 0$, G_i is planar and Invariants 5 and 6 hold. We prove that this is true also at the end of Step $i + 1$.

Planarity In Case 1 the dummy edge (v_i, v_{i+1}) is (possibly) added on the external face of G_i. This edge can be added without violating planarity since both v_i and v_{i+1} are on the external face of G_i by Invariants 5 and 6. In Case 2 the algorithm adds $2l + 1$ dummy edges (where l is the number of maximal sequences of vertices of $N_i(v_i, v_{i+1})$). The $l + 1$ edges (v_i, a_0), (b_j, a_{j+1}) $(0 \leq j \leq l-1)$ and (b_{l-1}, v_{i+1}) are added on the external face of G_i; these edges do not violate planarity because the endvertices of each of them are on the external face of G_i. Namely, v_i is on the external face by Invariant 6 and v_{i+1} is on the external face by Invariant 5 (because $v_{i+1} \in F_i^{c_{i+1}}$). Again by Invariant 5, each vertex s_j and t_j $(0 \leq j \leq l-1)$ is on the external face of G_i and therefore each vertex s'_j and t'_j is also on the external face of G_i; this implies that each vertex a_j and b_j $(0 \leq j \leq l-1)$ is on the external face of G_i.

The l edges (a_j, b_j) $(0 \leq j \leq l-1)$ are added inside $\mathcal{C}$. Let (a_j, b_j) be one such edge and let (u, w) be another dummy edge of G_i. Since σ_j is a maximal sequence of vertices of $N_i(v_i, v_{i+1})$, no dummy edge is incident on a vertex v such that $a_j < v < b_j$. This implies that $u < a_j < b_j < w$ and therefore the two edges do not cross.

Invariant 5 Trivially $v \notin H_{i+1} \Rightarrow v \notin H_i$. For this reason a vertex $v \notin H_{i+1}$ is on the external face of G_i by Inv. 5, and it cannot be anymore on the external face of G_{i+1} only if $u < v < w$, where (u, w) is a dummy edge added during Step $i+1$. If $v < v_i$ or $v_{i+1} < v$, then v is on the external face of G_{i+1} because none of the dummy edges added at Step $i+1$ has an endvertex before v_i and after v_{i+1}. If $v_i < v < v_{i+1}$ then $v \in \sigma_j$ for some j $(0 \leq j \leq l-1)$; since edge (a_j, b_j) is added inside $\mathcal{C}$ and any other edge added at Step $i+1$ has endvertices either before a_j or after b_j, vertex v is on the external face of G_{i+1}. This implies that Inv. 5 holds for any vertex that is not in H_{i+1}.

Invariant 6 Vertex v_{i+1} is on the external face of G_i by Inv. 5 (because $v_{i+1} = \text{first}(F_i^{c_{i+1}})$). Since none of the dummy edges added at Step $i+1$ has an endvertex after v_{i+1}, v_{i+1} is on the external face of G_{i+1}.

This concludes the proof that the algorithm described above correctly computes an augmenting k-colored Hamiltonian cycle of $\mathcal{C}$ consistent with an ordered sequence σ. It remains to prove that the number of division vertices per edge is at most k. Observe that when we connect v_i to v_{i+1} in Case 2, each edge of $\mathcal{C}$ is split at most once. Also, a single edge split during a round is no longer split until next round (i.e. until we change the walking direction). Namely, since we always choose v_{i+1} as $F_i^{c_{i+1}}$, while the same walking direction is maintained, later steps split edges that are after those split in previous steps. This implies that the number of division vertices is at most the number of rounds. Since at each round all the vertices of at least one color are added to H_i, then the number of rounds is at most k and the statement follows. □

Lemma 7 *Let G be an outerplanar k-colored graph and let Ψ be a planar embedding of G having all vertices on the external face. Let σ be an ordered k-colored sequence compatible with G. Then G admits an augmenting k-colored Hamiltonian cycle constructed on Ψ, consistent with σ and with at most $2k$ division vertices per edge.*

Proof: Since every outerplanar graph can be made biconnected by adding edges while maintaining outerplanarity, we assume that G is biconnected. Let $\mathcal{C}$ be the boundary of the external face of G. We remove all the chords from G and compute an augmenting k-colored Hamiltonian cycle of $\mathcal{C}$ consistent with σ and with at most one division vertex per edge, by using the algorithm described above. If we add back the chords of G to the graph G_{n-1}, i.e. the augmented graph constructed at the end of the algorithm, these edges will cross the edges of $\mathcal{H}$ that are inside $\mathcal{C}$. For each crossing between an edge $e_{\mathcal{H}}$ of $\mathcal{H}$ and a chord e_{ch} of G we add a division vertex d that splits both $e_{\mathcal{H}}$ and e_{ch}. Thus we obtain an augmenting k-colored Hamiltonian cycle of G. To complete the proof we must show that every chord e_{ch} is split by at most $2k$ division vertices. Observe that an edge $e_{\mathcal{H}} = (u, w)$ of $\mathcal{H}$ must cross e_{ch} only if an endvertex v of e_{ch} is such that $u < v < w$. We show that for each vertex v there can be at most k edges $e_{\mathcal{H}} = (u, w)$ of $\mathcal{H}$ such that $u < v < w$. Since edge $e_{\mathcal{H}}$ is inside the cycle $\mathcal{C}$, it is a dummy edge added at some Step i $(0 \leq i \leq n-1)$ when Case 2 applies. As observed in the proof of Lemma 6, while the same walking direction is maintained, the dummy edges added at steps following Step i are incident on vertices that are after w. That means that the number of edges $e_{\mathcal{H}} = (u, w)$ of $\mathcal{H}$ such that $u < v < w$ is at most equal to the number of rounds, i.e. it is at most k. Since each chord e_{ch} has two endvertices, we have at most k division vertices on e_{ch} for each of them and therefore at most $2k$ division vertices for each chord. □

Theorem 4 *Let G be an outerplanar k-colored graph and let S be an ordered k-colored set compatible with G. Then G admits a point-set embedding with at most $4k + 1$ bends per edge on S. Such a point-set embedding can be computed in $O(n \log n + kn)$ time, where n is the number of vertices of G.*

Proof: By Lemma 2 and Lemma 7, G admits a point-set embedding with at most $4k + 1$ bends per edge on S.

Regarding the time complexity of the algorithm that computes a point-set embedding we can consider that: (i) Ordering the points of S according to increasing values of the x-coordinates takes $O(n \log n)$ time; (ii) an augmenting 2-colored Hamiltonian cycle with at most $2k$ division vertices per edge can be computed in $O(kn)$ time using the algorithm described at the beginning of this section; (iii) from the 2-colored Hamiltonian cycle a point-set embedding of G on S can be constructed in $O(n)$ time with the technique illustrated in the proof of Lemma 2. Therefore the overall time complexity is $O(n \log n + kn)$. □

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