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The Multi-Commodity Source Location Problems and the Price of Greed

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Abstract

Given a graph $G = (V, E)$, we say that a vertex subset $S \subseteq V$ covers a vertex $v \in V$ if the edge-connectivity between S and v is at least a given integer k, and also say that S covers an edge $vw \in E$ if v and w are both covered. We propose the *multi-commodity source location problem*, which is such that given a vertex- and edge-weighted graph G, p players each select q vertices, and obtain a profit that is the total over all players of the weight of each player's covered vertices and edges. However, vertices selected by one player cannot be selected by the other players. The goal is to maximize the total profits of all players. We show that the *price of greed*, which indicates the ratio of the total profit of cooperating players to that of selfish players based on an ordered strategy, is tightly bounded by $\min\{p, q\}$. Also when $k = 2$, we obtain tight bounds for vertex-unweighted trees when sources are located on the leaves.

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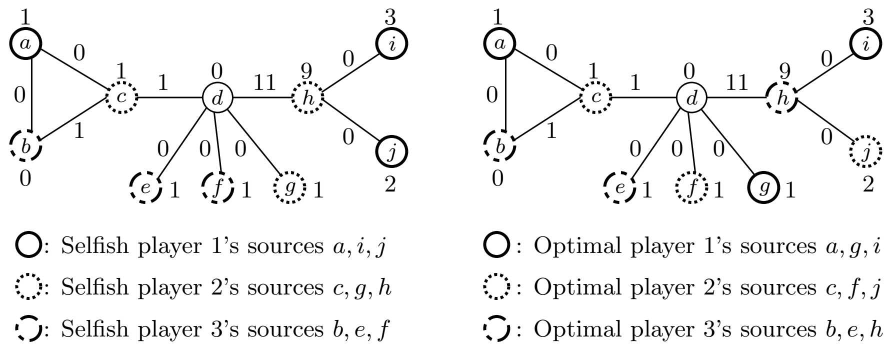
1 Introduction

Given an undirected graph $G = (V, E)$ with vertex set V, edge set E and an edge capacity function, the *edge-connectivity* between $S \subseteq V$ and $v \in V$ is the minimum total capacity of a set of edges such that v is disconnected from S by removal of these edges. For a given integer k, where $k \geq 1$, we say that a set $S \subseteq V$ of vertices (called sources) *covers* $v \in V$ if the edge-connectivity between S and v is at least k. The source location problem is to find a minimum-size source set $S \subseteq V$ covering all vertices in V. This problem has been studied widely [2, 3, 6, 7, 8, 10, 13, 15], and such problems are important in the design of networks resistant to the failure of edges.

In real networks, there are multiple service providers, and they locate servers in networks in order to supply services. Thus we propose the *multi-commodity source location problem*. In this problem, a network $N = (G = (V, E), w, c)$ and positive integers k, p and q are given, where G is an undirected connected graph with $|V| \geq pq$, $c : E \to \mathbf{Z}_+$ is an edge capacity function, $w : V \cup E \to \mathbf{R}_+$ is a vertex- and edge-weight function, and p is the number of players. Here $\mathbf{Z}_+$ (resp., $\mathbf{R}_+$) denotes the set of non-negative integers (resp., real numbers). Players $1, 2, \ldots, p$ each locate q sources on vertices of G. However, if a player locates a source on a vertex, then it is unavailable for the other players to locate a source on. Let player i's profit be the total weight of vertices and edges covered by the sources located by player i, where a source set S covers an edge $e = vw$ if both v and w are covered by S. The goal of this problem is to maximize the sum of the profits of all players. When $p = 1$, the problem is the same as the maximum-cover source-location problem [13]. This problem is NP-hard [14], but it can be solved in polynomial time when $k \leq 3$ [13, 12, 14].

In recent years, game theory has attracted attention in certain fields of computer science. In various real problems, e.g., routing, network design and scheduling, the selfish actions of agents are obstacles to the optimization of social welfare. Such phenomena are modeled as games, and the influence of selfish actions of players has been extensively analysed [1, 5, 11]. In this paper, we consider the influence of selfish actions of providers on network reliability. Generally the quality of service of newcomers is influenced by the actions of preceding providers. For example, the location of servers becomes restricted and consequently their profits may be smaller. We analyze the phenomenon by means of a selfish model in the multi-commodity source location problem. In our model, each player plays just once, in a fixed order, i.e., the "game" consists of a single round.

The selfish model is such that players $1, 2, \ldots, p$ in this order locate q sources on vertices of G so as to maximize their own profits. Note that as described previously, player i cannot locate sources on the vertices on which one of players $1, \ldots, i-1$ has already located sources. We compare the social welfare of this model to that of the case where all players cooperate, that is, optimal solutions of the problem. Figure 1 shows an example when $k = 2$, $p = 3$ and $q = 3$. The numbers beside vertices and edges denote their weights, and the edge capacities are uniformly one. Selfish player 1 locates sources to maximize his/her profit



(a) (b)

Figure 1: An example of behaviours of selfish and optimal players when $k = 2$, $p = 3$ and $q = 3$.

(see Fig. 1(a)). The source set $\{a, i, j\}$ of player 1 covers vertices a, b, c, d, h, i, j and he/she gets profit 29. Selfish player 2 then locates sources on vertices not already occupied such that his/her own profit is maximum in this situation. The source set $\{c, g, h\}$ of player 2 covers vertices a, b, c, d, g, h and his/her profit is 25. Then, selfish player 3 does similarly and gets profit 6 since his/her source set $\{b, e, f\}$ covers vertices a, b, c, d, e, f. The total profit is $29 + 25 + 6 = 60$. In contrast, optimal players 1, 2 and 3 locate their sources as in Fig. 1(b), so that the players share as many covered vertices and edges as possible. Optimal players 1, 2 and 3 obtain profits 28, 27 and 25, respectively, and the total is 80.

As a measure of the influence of the selfish behaviour based on the ordering of players, we propose the *price of greed*, which represents the ratio of the maximum total profit of the cooperating players to the worst, i.e., minimum, total profit of the selfish players. Formally, let the price of greed for the multi-commodity source location problem be

$$POG_k(N, p, q) = \frac{\text{the optimal total profit}}{\text{the worst selfish total profit}} .$$

The price of greed of the example in Fig. 1 is $POG_2(N, 3, 3) = (28 + 27 + 25)/(29 + 25 + 6) = 4/3$.

A well-known similar measure, the price of anarchy is the ratio of the worst cost of Nash equilibria of selfish players to the optimal cost [11]. The locations of sources derived by our selfish model are Nash equilibria, since no player can gain profit by changing their locations. However, the locations are only a component of all Nash equilibria, and we analyze the influence of the greedy behaviour based on the ordered strategy in our model. Thus we introduce the new measure. Note also that for POG we deal with *profits* whereas for the price of anarchy we consider *costs*. Thus, for POG we have the optimal profit in the numerator (the selfish profit in the denominator) whereas in the price of anarchy the optimal cost is in the denominator (the selfish cost is in the numerator). Our choice is such that the "price" of "selfish behaviour" is greater than one, and in most of our results will be an integer.

Our Results Our goal is to analyze the maximum value of the price of greed $POG_k(p,q) = \max_N POG_k(N,p,q)$. When $k = 1$, it is clear that $POG_1(p,q) = 1$ for any $p, q \geq 1$, since all vertices and all edges are covered, wherever the sources are located. Hence we assume $k \geq 2$.

Our main results are stated in the following two theorems which give tight bounds on the price of greed. The first of these, for general k and arbitrary networks N, is proved in Sect. 2.

Theorem 1 *$POG_k(p,q) = \min\{p,q\}$ for any $k \geq 2$, $p \geq 1$ and $q \geq 1$.*

Furthermore, we consider the case where $k = 2$ and the input graph G is restricted to a vertex-unweighted tree where sources are located only on the leaves; as in the general case, each edge has capacity at least 1 and weight at least 0. This vertex-unweighted tree case is equivalent to the problem in which p players find the p subtrees induced by q leaves of the input tree such that the total edge-weight of these p subtrees is a maximum. Maximum edge-weight trees have many applications, e.g., communication networks [4, 9]. This is a special case of the original problem. However, $POG_2(p,q)$ is less than that for the vertex- and edge-weighted case by at most one. Note that if $q = 1$, then any optimal and selfish player obtains no profit and hence we assume $q \geq 2$.

Theorem 2 *For vertex-unweighted trees and any $p \geq 1$, $POG_2(p,2) = \min\{p,2\}$ and $POG_2(p,q) = \min\{p,q-1\}$ for $q \geq 3$ when sources are located only on the leaves.*

Moreover, we also consider the case where sources are located on any vertex. Tight bounds $POG_2(p,q) = \min\{p,q-1\}$ for $q \geq 3$ and $POG_2(2,2) = \frac{4}{3}$ are obtained in this case. However, when $q = 2$ and $p \geq 3$, $POG_2(p,2)$ may be smaller than that of the case of locating sources only on leaves.

2 Analysis of $POG_k(p,q)$ for the General Case

In this section we consider the case of an arbitrary network N and any number k for the edge-connectivity.

Theorem 1 *$POG_k(p,q) = \min\{p,q\}$ for any $k \geq 2$, $p \geq 1$ and $q \geq 1$.*

Clearly, when $p = 1$ or $q = 1$, $POG_k(N,p,q) = 1$ for any network N. We will now assume that $p \geq 2$ and $q \geq 2$.

We begin by showing the upper bound in Lemma 2. Given a network N, let W_i ($1 \leq i \leq p$) be the profit of optimal player i. We assume without loss of generality that $W_1 \geq W_2 \geq \cdots \geq W_p$. Let W'_i ($1 \leq i \leq p$) be the

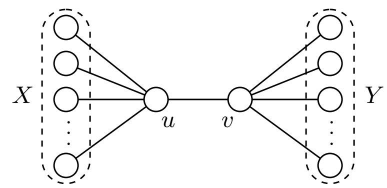


Figure 2: An instance of $POG_k(N, p, q) = \min\{p, q\}$.

profit of selfish player i in the worst case, i.e., where the total profit is the least. From the definition, $W'_1 \geq W'_2 \geq \cdots \geq W'_p$ holds, and $POG_k(N, p, q) = (\sum_{i=1}^{p} W_i)/(\sum_{i=1}^{p} W'_i)$. Let S_i $(1 \leq i \leq p)$ be the set of sources located by optimal player i and S'_i $(1 \leq i \leq p)$ be the set of sources located by selfish player i. For a source set $S \subseteq V$, let $w_k(S)$ denote the total weight of vertices and edges covered by S. Note that $w_k(S_i) = W_i$ and $w_k(S'_i) = W'_i$ for $i \in \{1, \ldots, p\}$.

Lemma 2 relies on the following lemma.

Lemma 1 *For any i $(1 \leq i \leq \lfloor \frac{p-1}{q} \rfloor + 1)$, we have $W'_i \geq W_{(i-1)q+1}$.*

Proof: When $i = 1$, the inequality $W'_1 \geq W_1$ evidently holds. Then we consider $i \geq 2$. Since $|\bigcup_{j=1}^{i-1} S'_j| = (i-1)q$, at least one, S_r say, of $S_1, \ldots, S_{(i-1)q+1}$ has no common source with any of $S'_1, \ldots, S'_{i-1}$. The profit W'_i of selfish player i is the largest profit when he/she locates sources on vertices in $V \setminus \bigcup_{j=1}^{i-1} S'_j$. From the above discussion, we obtain $W'_i \geq W_r \geq W_{(i-1)q+1}$. □

Lemma 2 *For any $k \geq 2$, $p \geq 1$ and $q \geq 1$, we have $POG_k(p, q) \leq \min\{p, q\}$.*

Proof: From $W'_1 \geq W_i$ for any i, it is clear that $POG_k(N, p, q) \leq (\sum_{i=1}^{p} W_i)/W'_1 \leq p$ for any N, $p \geq 1$ and $q \geq 1$. Then we show $POG_k(p, q) \leq q$ for $q < p$ as follows. From Lemma 1, for $i \in \{1, \ldots, \lfloor \frac{p-1}{q} \rfloor + 1\}$, we have $qW'_i \geq W_{(i-1)q+1} + \cdots + W_{iq}$ (where we take $W_v = 0$ when $v > p$), and hence $q \sum_{i=1}^{p} W'_i \geq \sum_{i=1}^{p} W_i$. Therefore, we have $POG_k(p, q) \leq q$ for $q < p$. □

Proof of Theorem 1: We prove that the upper bound is tight by showing an instance (N, k, p, q) in Fig. 2 that has $POG_k(N, p, q) = \min\{p, q\}$ where $p \geq 2$ and $q \geq 2$. Let $|X| = \min\{p, q\} - 1$ and $|Y| = pq - \min\{p, q\} - 1$. The weight of u is 1, the other vertices and all edges have weight 0, and the capacities of all edges are $\lceil k/2 \rceil$. In this case, a vertex is covered when it is on a path between sources. If selfish player 1 obtains profit 1 by locating sources on $X \cup \{u\}$ (i.e., $\min\{p, q\}$ sources) and $q - \min\{p, q\}$ ones on $Y \cup \{v\}$, then the other selfish players obtain no profits. Hence the worst total selfish profit is 1. On the other hand, each of the optimal players $1, \ldots, \min\{p, q\}$ obtains profit 1 by locating one source on $X \cup \{u\}$ and $q - 1$ sources on $Y \cup \{v\}$. Then the other optimal players cannot obtain any profit. Hence the optimal total profit is $\min\{p, q\}$. Therefore, this instance has $POG_k(N, p, q) = \min\{p, q\}$. This completes the proof of Theorem 1. □

3 Analysis of $POG_2(p,q)$ for Vertex-Unweighted Trees

3.1 The Case of Locating Sources Only on Leaves

In this section, we deal with the case of $k = 2$, vertex-unweighted trees with every edge of capacity 1 and weight at least 0, and where the players locate sources only on the leaves of the tree. Locating sources on leaves does not make the problem weak, since the case where sources can be located on any vertex can be reduced to this case by adding a leaf l to every non-leaf vertex v and letting the weight of the edge lv be 0. This problem is equivalent to the problem of finding p subtrees induced by q leaves of the input tree such that the p subtrees have maximum total edge-weight, and it is a basic and important problem in network optimization problems [13].

Theorem 2 *For vertex-unweighted trees and any $p \geq 1$, $POG_2(p,2) = \min\{p,2\}$ and $POG_2(p,q) = \min\{p,q-1\}$ for $q \geq 3$ when sources are located only on the leaves.*

Proof: The upper bounds for the cases $q = 2$ and $q \geq 3$ are established in Lemma 4; Lemma 5 shows that these bounds are tight. □

Before proving Lemma 4, we prove the following useful lemma. Recall that $w_2(s,v)$ is the total weight of edges in the the path from s to v.

Lemma 3 *Let $S^0 = \{s_1, s_2, \ldots, s_q\} \subseteq V$ where $q \geq 3$, $v_0 \in V \setminus S^0$, and S a set of sources. For $j = 1, 2, \ldots, q$, let $S^j = (S^0 \cup \{v_0\}) \setminus \{s_j\}$. If*
(a) S satisfies $w_2(S^j) \leq w_2(S)$ for any $j \in \{1, \ldots, q\}$, and
(b) there exists $t \in \{1, \ldots, q\}$ such that $w_2(S^t) \leq w_2(S^0)$,
then $\sum_{j=1}^{q} w_2(s_j, v_0) \leq (q-1)w_2(S)$.

Proof: Let P_j be the set of edges on the path between s_j and v_0 for $j \in \{1, \ldots, q\}$. Let $X = \bigcap_{j=1}^{q} P_j$ be the set of edges contained in all of $P_1, \ldots, P_q$, and let $Y = \bigcup_{1 \leq j < \ell \leq q} (P_j \cap P_\ell) \setminus X$ be the set of edges contained in at least two of $P_1, \ldots, P_q$ and not contained in X (see Fig. 3). Let x (resp., y) be the total weight of the edges in X (resp., Y). Moreover, let a_j be the total weight of the edges in $P_j \setminus (X \cup Y)$ (see Fig. 3). We abbreviate $\sum_{j=1}^{q} a_j$ by A.

From the definition, the following inequality holds.

$$\sum_{j=1}^{q} w_2(s_j, v_0) \leq qx + (q-1)y + \sum_{j=1}^{q} a_j = qx + (q-1)y + A \ . \tag{1}$$

From condition (a) in Lemma 3, for any $j \in \{1, \ldots, q\}$,

$$\begin{aligned} w_2(S^j) &= x + y + \sum_{1 \leq i \leq q : i \neq j} a_i \\ &= x + y + A - a_j \leq w_2(S) \ . \end{aligned} \tag{2}$$

Summing (2) for all $j \neq t$ with $1 \leq j \leq q$, where t satisfies the condition (b), yields

$$(q-1)(x+y+A)-(A-a_t) \leq (q-1)w_2(S) \ .$$

From (b), $x \leq a_t$, since $w_2(S^t) = x+y+A-a_t \leq y+A = w_2(S^0)$. Thus,

$$qx+(q-1)y+(q-2)A \leq (q-1)w_2(S) \ . \tag{3}$$

The proof can then be completed, as

$$\begin{aligned}
\sum_{j=1}^{q} w_2(s_j, v_0) &\leq qx+(q-1)y+A && \text{(from (1))} \\
&\leq (q-1)w_2(S)-(q-3)A && \text{(from (3))} \\
&\leq (q-1)w_2(S) \ . && \text{(since } q \geq 3\text{)}
\end{aligned}$$

□

Lemma 4 *For vertex-unweighted trees and any $p \geq 1$, $POG_2(p,2) \leq \min\{p,2\}$ and $POG_2(p,q) \leq \min\{p,q-1\}$ for $q \geq 3$.*

Proof: From Lemma 2, $POG_2(p,q) \leq \min\{p,q\}$ for any $p \geq 1$ and $q \geq 2$. It therefore remains to show $POG_2(p,q) \leq q-1$ for $p \geq q \geq 3$ in the following part. Let v_0 be a source in the set $\bigcup_{i=1}^{p} S_i$ of sources located by all optimal players; suppose without loss of generality that $v_0 \in S_t$ for some t: $1 \leq t \leq p$. Since for any subset $X \subseteq V$ of at least 3 elements, it holds that $w_2(X) \leq w_2(X \setminus \{x_j\}) + w_2(x_i, x_j)$ for any two distinct elements x_i, x_j of X, we have

$$W_t = w_2(S_t) \leq \sum_{s \in S_t \setminus \{v_0\}} w_2(s, v_0) \ ,$$

and for $1 \leq i \leq p$ and $i \neq t$

$$W_i = w_2(S_i) \leq w_2(S_i \cup \{v_0\}) \leq \sum_{s \in S_i} w_2(s, v_0) \ .$$

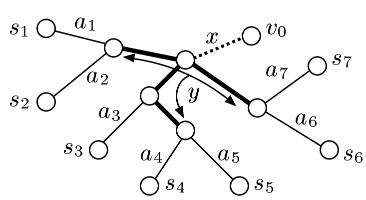


Figure 3: The edges on the dotted line are in X and those on the bold lines are in Y.

If we sum each side of the inequalities, then for any $v_0 \in \bigcup_{i=1}^{p} S_i$,

$$\sum_{i=1}^{p} W_i \leq \sum_{s \in \left(\bigcup_{i=1}^{p} S_i\right) \setminus \{v_0\}} w_2(s, v_0) \ . \tag{4}$$

In the rest of the proof, we consider two cases (i) $\bigcup_{i=1}^{p} S'_i = \bigcup_{i=1}^{p} S_i$ and (ii) $\bigcup_{i=1}^{p} S'_i \neq \bigcup_{i=1}^{p} S_i$.

Case (i): $\bigcup_{i=1}^{p} S'_i = \bigcup_{i=1}^{p} S_i$. Pick $v_0 \in S'_p$. Since $v_0 \in \bigcup_{i=1}^{p} S'_i$, then $v_0 \in \bigcup_{i=1}^{p} S_i$.

For $i \in \{1, \ldots, p-1\}$, let $S^0 := S'_i$ and $S := S'_i$ $(= S^0)$, and let $S^j := (S^0 \cup \{v_0\}) \setminus \{s_j\}$ for each $s_j \in S^0$ with $j \in \{1, \ldots, q\}$. Then $|S^0| \geq 3$ from the assumption, and $v_0 \notin S^0$, since $S^0 = S'_i$ for $i \in \{1, \ldots, p-1\}$ and $v_0 \in S'_p$. For S^0, v_0, and S, conditions (a) and (b) of Lemma 3 hold. Because $w_2(S^j) \leq w_2(S) = W'_i$ $(= w_2(S'_i))$ for any $j \in \{1, \ldots, q\}$ (i.e., (a) holds) from $v_0 \in S'_p$ and the behaviour of selfish player i for $1 \leq i \leq p-1$; moreover, condition (b) clearly holds, since (a) implies (b) in the case of $S = S^0$. Then, from Lemma 3, for each $i \in \{1, \ldots, p-1\}$,

$$\sum_{s' \in S'_i} w_2(s', v_0) \leq (q-1) w_2(S'_i) = (q-1) W'_i \ . \tag{5}$$

Inequality (5) holds for each i with $1 \leq i \leq p-1$. On the other hand, for any $s' \in S'_p$, $w_2(s', v_0) \leq W'_p$, since $v_0 \in S'_p$. Hence

$$\sum_{s' \in S'_p \setminus \{v_0\}} w_2(s', v_0) \leq (q-1) W'_p \ . \tag{6}$$

Consequently, if we sum up (5) for $1 \leq i \leq p-1$ and (6), we get

$$\sum_{s' \in \left(\bigcup_{i=1}^{p} S'_i\right) \setminus \{v_0\}} w_2(s', v_0) \leq (q-1) \sum_{i=1}^{p} W'_i \ . \tag{7}$$

Since $\bigcup_{i=1}^{p} S'_i = \bigcup_{i=1}^{p} S_i$, the left side of (7) is equal to the right side of (4), so

$$\begin{aligned}
\sum_{i=1}^{p} W_i &\leq \sum_{s \in \left(\bigcup_{i=1}^{p} S_i\right) \setminus \{v_0\}} w_2(s, v_0) \\
&= \sum_{s' \in \left(\bigcup_{i=1}^{p} S'_i\right) \setminus \{v_0\}} w_2(s', v_0) \\
&\leq (q-1) \sum_{i=1}^{p} W'_i \ .
\end{aligned}$$

Therefore, $POG_2(p, q) \leq q-1$, for any p, q with $p \geq q \geq 3$.

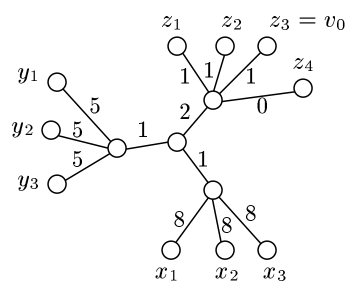


Figure 4: An example of Case (i).

Figure 4 shows an example. When each of 3 players locates 3 sources, source sets $S_1 = \{x_1, y_1, z_1\}, S_2 = \{x_2, y_2, z_2\}$ and $S_3 = \{x_3, y_3, z_3\}$ are optimal whereas the sets $S'_1 = \{x_1, x_2, x_3\}, S'_2 = \{y_1, y_2, y_3\}$ and $S'_3 = \{z_1, z_2, z_3\}$ are the selfish source sets. Hence $\bigcup_{i=1}^{3} S'_i = \bigcup_{i=1}^{3} S_i$. We select an arbitrary source v_0 in S'_3, and let $v_0 = z_3$ (Fig. 4).

For the set S'_1 and vertex v_0, let $S^0 := S'_1$; then, $|S^0| \geq 3$ and $v_0 \notin S^0$, since $v_0 \in S'_3$. Let $S := S'_1$ (i.e., $S = S^0 = S'_1$) and $S^j := (S^0 \cup \{v_0\}) \setminus \{x_j\}$ for $x_j \in S^0$ with $j \in \{1, 2, 3\}$. Then $w_2(S^j) \leq w_2(S) = w_2(S^0)$ $(= w_2(S'_1))$ for each $j \in \{1, 2, 3\}$ clearly. Hence conditions (a) and (b) of Lemma 3 hold. Similarly, for the set S'_2 and vertex v_0, conditions (a) and (b) hold. In fact,

$$\begin{aligned} 32 = \sum_{i=1}^{3} w_2(x_i, v_0) &\leq 2w_2(S'_1) = 48 \text{ , and} \\ 27 = \sum_{i=1}^{3} w_2(y_i, v_0) &\leq 2w_2(S'_2) = 30 \text{ .} \end{aligned}$$

On the other hand, since $z_1, z_2, v_0 \in S'_3$, $w_2(z_1, v_0) \leq w_2(S'_3)$ and $w_2(z_2, v_0) \leq w_2(S'_3)$. Hence

$$4 = w_2(z_1, v_0) + w_2(z_2, v_0) \leq 2w_2(S'_3) = 6 \text{ .}$$

Thus from (4),

$$\begin{aligned} W_1 + W_2 + W_3 &\leq \sum_{i=1}^{3} (w_2(x_i, v_0) + w_2(y_i, v_0)) + \sum_{i=1}^{2} w_2(z_i, v_0) \\ &\leq 2(w_2(S'_1) + w_2(S'_2) + w_2(S'_3)) = 2(W'_1 + W'_2 + W'_3) \text{ .} \end{aligned}$$

Case (ii): $\bigcup_{i=1}^{p} S'_i \neq \bigcup_{i=1}^{p} S_i$. In this case, there exist selfish source sets S'_i with $S'_i \not\subseteq \bigcup_{j=1}^{p} S_j$. Let $S'_{\ell(1)}, S'_{\ell(2)}, \ldots, S'_{\ell(h)}$ with $\ell(1) < \ell(2) < \cdots < \ell(h)$ be such source sets.

For each $S'_{\ell(i)}$ with $1 \leq i \leq h$, we make a new source set $S''_{\ell(i)} \subseteq \bigcup_{j=1}^{p} S_j$ with $|S''_{\ell(i)}| = q$ in the following way. Let $T_{\ell(i)} = S'_{\ell(i)} \cap (\bigcup_{j=1}^{p} S_j)$ be a source

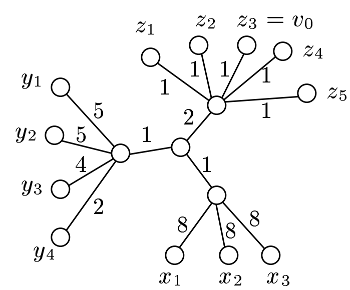


Figure 5: An example of Case (ii).

set that consists of the sources contained not only in $S'_{\ell(i)}$ but also contained in some of the optimal sources $S_1, \ldots, S_p$. For $T_{\ell(1)}$, we select r $(= q - |T_{\ell(1)}|)$ sources $s_1, \ldots, s_r$ from $\bigcup_{i=1}^{p} S_i \setminus \bigcup_{i=1}^{p} S'_i$ and let $S''_{\ell(1)} = T_{\ell(1)} \cup \{s_1, \ldots, s_r\}$; the vertices $s_1, \ldots, s_r$ are selected so as to maximize the total weight $W''_{\ell(1)} := w_2(S''_{\ell(1)})$ of edges covered by $S''_{\ell(1)}$. Similarly, $S''_{\ell(2)}$ is constructed by augmenting $T_{\ell(2)}$ with $q - |T_{\ell(2)}|$ more sources from not-yet-selected vertices in $\bigcup_{i=1}^{p} S_i \setminus \bigcup_{i=1}^{p} S'_i$ so as to maximize the total weight $W''_{\ell(2)} := w_2(S''_{\ell(2)})$. By repeating the above operations, $S''_{\ell(1)}, \ldots, S''_{\ell(h)}$ are obtained. Note that $W''_{\ell(i)} \leq W'_{\ell(i)}$ for $i \in \{1, \ldots, h\}$, from the behaviour of selfish players.

We show an example of selecting $S''_{\ell(i)}$ in Fig. 5, where $p = 3$ and $q = 3$. Optimal source sets are $S_1 = \{x_1, y_1, z_1\}$, $S_2 = \{x_2, y_2, z_2\}$ and $S_3 = \{x_3, y_3, z_3\}$, while $S'_1 = \{x_1, x_2, x_3\}$, $S'_2 = \{y_1, y_2, z_4\}$ and $S'_3 = \{y_3, y_4, z_5\}$ are selfish source sets. Since $S'_2 \not\subseteq \bigcup_{i=1}^{3} S_i$ and $S'_3 \not\subseteq \bigcup_{i=1}^{3} S_i$, $\ell(1) = 2$ and $\ell(2) = 3$. Moreover, $T_{\ell(1)} = S'_2 \cap (\bigcup_{i=1}^{3} S_i) = \{y_1, y_2\}$ and $T_{\ell(2)} = S'_3 \cap (\bigcup_{i=1}^{3} S_i) = \{y_3\}$. Next, we construct S''_2 and S''_3. For each source v in $\bigcup_{i=1}^{3} S_i \setminus \bigcup_{i=1}^{3} S'_i = \{z_1, z_2, z_3\}$, we add it to $T_{\ell(1)} = \{y_1, y_2\}$, trying to maximize $w_2(y_1, y_2, v)$. The maximum is obtained for $v = z_1$, and so $S''_2 = \{y_1, y_2, z_1\}$. Similarly, we select not-yet-selected sources z_2 and z_3, and let $S''_3 = \{y_3, z_2, z_3\}$. Clearly, $w_2(S''_2) \leq w_2(S'_2)$ and $w_2(S''_3) \leq w_2(S'_3)$. Moreover, we select an arbitrary source $v_0 \in \{z_2, z_3\}$ $(= S''_3 \setminus T_{\ell(2)})$, e.g., let $v_0 = z_3$ in Fig. 5.

Let $S''_{\ell(i)} = \{s''_{\ell(i),1}, \ldots, s''_{\ell(i),q}\}$ for $i \in \{1, \ldots, h\}$ and let $v_0 \in S''_{\ell(h)} \setminus T_{\ell(h)}$ be a source of selfish player $\ell(h)$ which is selected in the above operation. Note that $v_0 \in \bigcup_{i=1}^{p} S_i$.

Next, we apply Lemma 3 to $S''_{\ell(i)}$, v_0 and S'_i for $i \in \{1, \ldots, h-1\}$ as follows. Let $S^0 := S''_{\ell(i)}$ for $i \in \{1, \ldots, h-1\}$, then $|S^0| = q \geq 3$ from the assumption, and $v_0 \notin S^0$, since $v_0 \in S''_{\ell(h)}$. Let $S := S'_i$ (i.e., $w_2(S) = W'_{\ell(i)}$). Moreover, for each $j \in \{1, \ldots, q\}$, let $S^j := (S^0 \cup \{v_0\}) \setminus \{s''_{\ell(i),j}\}$ for $s''_{\ell(i),j} \in S^0$ $(= S''_{\ell(i)})$ with $i \in \{1, \ldots, h-1\}$. Now we verify that condition (a) of Lemma 3 holds. Because the source set S'_i $(= S)$ has the maximum profit W'_i $(= w_2(S))$ among sets of q sources located on vertices in $V \setminus (\bigcup_{1 \leq r \leq i-1} S'_r)$. Thus, $w_2(S^j) \leq w_2(S)$

for any $j \in \{1, \ldots, q\}$. Moreover, condition (b) also holds. If there exists no $t \in \{1, \ldots, q\}$ such that $w_2(S^t) \leq w_2(S^0)$ $(= w_2(S''_{\ell(i)}))$, then $w_2(S^j) > w_2(S''_{\ell(i)})$ for each $j \in \{1, \ldots, q\}$, and hence it contradicts the definition of $S''_{\ell(i)}$, since we should have selected v_0 instead of $s''_{\ell(i),j}$ in the above operation. Therefore, from Lemma 3, for $q \geq 3$,

$$\sum_{j=1}^{q} w_2(s''_{\ell(i),j}, v_0) \leq (q-1)W'_{\ell(i)} \ . \tag{8}$$

On the other hand, for any $s'' \in S''_{\ell(h)}$, $w_2(s'', v_0) \leq W''_{\ell(h)}$ since $v_0 \in S''_{\ell(h)}$. Hence

$$\sum_{s'' \in S''_{\ell(h)} \setminus \{v_0\}} w_2(s'', v_0) \leq (q-1)W''_{\ell(h)} \leq (q-1)W'_{\ell(h)} \ . \tag{9}$$

In addition, we consider the selfish source sets S'_i other than $S'_{\ell(1)}, S'_{\ell(2)}, \ldots, S'_{\ell(h)}$. For each such S'_i, inequality (5) holds, i.e.,

$$\sum_{s' \in S'_i} w_2(s', v_0) \leq (q-1)W'_i \ . \tag{10}$$

Let $S' = \bigcup_{i=1}^{p} S'_i \setminus \bigcup_{i=1}^{h} S'_{\ell(i)}$, and $S'' = \bigcup_{i=1}^{h} S''_{\ell(i)}$. Now $S' \cup S''$ is equal to the union $\bigcup_{i=1}^{p} S_i$ of the optimal source sets. Thus, the summation of the left side of inequalities (8), (9) and (10) is equal to the right side of (4), i.e.,

$$\begin{aligned}
\sum_{i=1}^{p} W_i &\leq \sum_{s \in \left(\bigcup_{i=1}^{p} S_i\right) \setminus \{v_0\}} w_2(s, v_0) \\
&= \sum_{s'' \in S'' \setminus \{v_0\}} w_2(s'', v_0) + \sum_{s' \in S'} w_2(s', v_0) \\
&= \sum_{i=1}^{h-1} \sum_{j=1}^{q} w_2(s''_{\ell(i),j}, v_0) + \sum_{s'' \in S''_{\ell(h)} \setminus \{v_0\}} w_2(s'', v_0) + \sum_{s' \in S'} w_2(s', v_0) \\
&\leq (q-1) \sum_{i=1}^{p} W'_i \ .
\end{aligned}$$

Therefore, $POG_2(p, q) \leq q - 1$, for any p, q with $p \geq q \geq 3$. □

Next, we prove that the upper bounds in Lemma 4 are tight by showing the following lemma.

Lemma 5 *For vertex-unweighted trees and any* $p \geq 1$, $POG_2(p, 2) \geq \min\{p, 2\}$ *and* $POG_2(p, q) \geq \min\{p, q-1\}$ *for any* $q \geq 3$.

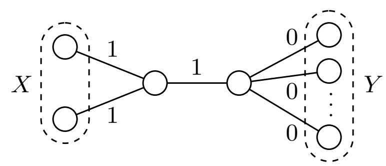


Figure 6: An instance of $POG_2(N, p, 2) = \min\{p, 2\}$.

Proof: This is proved by showing an instance (N, k, p, q) that has $POG_2(N, p, 2) = \min\{p, 2\}$ and $POG_2(N, p, q) = \min\{p, q-1\}$ for $q \geq 3$.

When $p = 1$, clearly $POG_2(N, 1, q) = 1$ for any network N and $q \geq 2$.

In the case of $q > p \geq 2$, consider the previously shown Fig. 2, but assume that the weight of the edge uv is 1 and the weights of the other edges are 0. Let $|X| = p$, $|Y| = pq - p$. Selfish player 1 may locate p sources on X and $q - p$ sources on Y, and obtain profit 1. Then the other selfish players cannot obtain any profit. Each of the optimal players $1, 2, \ldots, p$ obtains profit 1 by locating one source on X and $q - 1$ sources on Y, and so the total optimal profit is p. This network N has $POG_2(N, p, q) = p = \min\{p, q-1\}$.

For $p \geq q \geq 3$, in Fig. 2 again let the weight of the edge uv be 1 and that of the other edges be 0. Let $|X| = q - 1$ and $|Y| = pq - q + 1$. Selfish player 1 may locate $q-1$ sources on X and one source on Y, and obtain profit 1, while the other selfish players obtain no profits. Each of the optimal players $1, 2, \ldots, q-1$ $(\leq p)$ obtains profit 1 by locating one source on X and $q - 1$ sources on Y, and the other optimal players obtain no profit. The total optimal profit is $q-1$, and so this network N has $POG_2(N, p, q) = q-1 = \min\{p, q-1\}$.

For the remaining case, $p \geq 2$ and $q = 2$, we show that $POG_2(N, p, 2) = 2$ for the network N in Fig. 6. The numbers beside edges denote their weights, and $|Y| = 2p - 2$. Selfish player 1 may locate two sources on X and obtains profit 2, while the other selfish players obtain no profits. On the other hand, optimal players 1 and 2 each obtain profit 2, by locating one source on X and the other one on Y. The other optimal players obtain no profits. Hence the least selfish total profit and the optimal one are 2 and 4, respectively. Hence $POG_2(N, p, 2) = 2$ for this network N. Therefore $POG_2(N, p, 2) = \min\{p, 2\}$ for $p \geq 2$ and $q = 2$. This completes the proof of the lemma. □

Note that even if we do not assume that sources are located only on the leaves, there exist instances equivalent to those for $p \geq q \geq 3$ and for $q > p \geq 2$ in the proof of Lemma 5 by removing an arbitrary leaf from X and also one from Y.

3.2 The Case of Locating Sources on Any Vertex

In this section, we assume that players can locate sources on any vertex. This problem is weaker than the case of locating only on leaves in Sect. 3.1, since this case can be reduced to the case in Sect. 3.1 by adding a leaf to every non-leaf vertex as we discussed above. However the reverse is not always clear. In fact, when $q = p = 2$, we show that an upper bound is lower.

When $q = 1$, no player obtains profits. Moreover, when $p = 1$, $POG_2(1, q) = 1$ for $q \geq 2$ clearly. Hence we assume $q \geq 2$ and $p \geq 2$. We show the upper and lower bounds for this case in the following theorem.

Theorem 3 *When $k = 2$ and sources can be located on any vertex in vertex-unweighted trees, $POG_2(p, q)$ is as follows.*

(*I*) $POG_2(p, q) = \min\{p, q - 1\}$ *for* $q \geq 3$,

(II) $POG_2(2, 2) = \frac{4}{3}$,

(III) $2 - \frac{1}{p} \leq POG_2(p, 2) \leq 2$ *for odd* $p \geq 3$, *and*

(IV) $2 - \frac{1}{p-1} \leq POG_2(p, 2) \leq 2$ *for even* $p \geq 4$.

First, we show the lower bounds of the price of greed for this case in the following lemma.

Lemma 6 *When sources can be located on any vertex in vertex-unweighted trees,*

(1) $POG_2(p, q) \geq \min\{p, q - 1\}$ *for* $q \geq 3$,

(2) $POG_2(2, 2) \geq \frac{4}{3}$.

(3) $POG_2(p, 2) \geq 2 - \frac{1}{p}$ *for odd* $p \geq 3$, *and*

(4) $POG_2(p, 2) \geq 2 - \frac{1}{p-1}$ *for even* $p \geq 4$.

Proof: We prove the lower bounds by showing an instance (N, k, p, q).

(1) There exist instances equivalent to those for $p \geq q \geq 3$ and for $q > p \geq 2$ in the proof of Lemma 5, by removing an arbitrary leaf from X and also one from Y in Fig. 2. Thus $POG_2(p, q) \geq \min\{p, q - 1\}$ for $q \geq 3$.

(2) We show an instance with $POG_2(N, 2, 2) = \frac{4}{3}$ in Fig. 7. The numbers beside edges denote their weights. Let $|X| = 2$ and $|Y| = 2$. (Fig. 7 is used in cases (3) and (4) also, and the tree is illustrated as having many leaves.) Optimal players 1 and 2 obtain profit 2 by locating one source on X and the other on $Y \cup \{v\}$, respectively. Hence the total profit is 4. On the other hand, if selfish player 1 locates two sources on X and obtains profit 2, then selfish player 2 obtains profit 1 by locating two sources on u and v. The total profit is 3. Therefore, this instance has $POG_2(N, 2, 2) = \frac{4}{3}$.

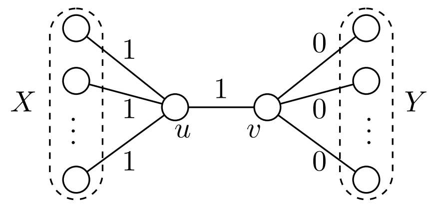


Figure 7: An instance giving the lower bounds for $p = 2$.

(3) For odd $p \geq 3$ let $|X| = p-1$ and $|Y|$ be sufficiently large in Fig. 7. Optimal player i with $i \in \{1, \ldots, p-1\}$ locates one source on X and the other source on Y, and obtains profit 2. Optimal player p obtains profit 1 by locating two sources on u and v. Hence the total profit is $2(p-1) + 1 = 2p - 1$. On the other hand, if selfish player i with $i \in \{1, \ldots, \frac{p-1}{2}\}$ locates two sources on X and obtains profit 2, then selfish player $\frac{p+1}{2}$ obtains profit 1 by locating two sources on u and v. Note that $\frac{p-1}{2}$ and $\frac{p+1}{2}$ are positive integers, since $p \geq 3$ is odd. The other selfish players cannot obtain profit. Hence the total profit is $2(\frac{p-1}{2}) + 1 = p$. Therefore, this instance has $POG_2(N, p, 2) = \frac{2p-1}{p} = 2 - \frac{1}{p}$ for odd $p \geq 3$.

(4) For even $p \geq 4$, let $|X| = p - 2$ and $|Y|$ be sufficiently large in Fig. 7. Optimal and selfish players locate sources similarly to (3). Then it is shown that this instance has $POG_2(p, 2) = 2 - \frac{1}{p-1}$ for even $p \geq 4$. □

The following lemma shows that the lower bound in Lemma 6 is tight for $q = p = 2$.

Lemma 7 *When $k = 2$ and sources are located on any vertex in vertex-unweighted trees, $POG_2(2, 2) \leq \frac{4}{3}$.*

Proof: First, if $S'_1 \cap S_1 = \emptyset$, then $W'_2 \geq W_1$, since selfish player 2 optimally locates sources on the vertices other than the two vertices in S'_1. In this case, $POG_2(N, 2, 2) = 1$, since $W'_1 \geq W'_2 \geq W_1 \geq W_2$. Moreover, if $S'_1 \cap S_2 = \emptyset$, then $W'_2 \geq W_2$ for similar reasons. By considering $W'_1 \geq W_1$, $POG_2(N, 2, 2) = 1$ in this case as well. Thus we assume that $S'_1 \cap S_1 \neq \emptyset$ and $S'_1 \cap S_2 \neq \emptyset$. Concretely, let $S'_1 = \{s_1, s_2\}$, $S_1 = \{s_1, s_3\}$, and $S_2 = \{s_2, s_4\}$.

Now we consider the case in which $W_1 + W_2 > W'_1 + W'_2$. Then the two paths between the two vertices in S_1 and those in S_2 have common edges, since if the two paths have no common edges, then the two source sets $\{s_1, s_2\}$ and $\{s_3, s_4\}$ cover the same edges as S_1 and S_2, and hence this contradicts the optimality of S_1 and S_2 or $W_1 + W_2 = W'_1 + W'_2$ holds. The common edges induce a path P. Let $u_1 \in V(P)$ be the endpoint of P that is the nearest to s_1 (see Figs. 8(a) and (b)), and let $u'_1 \in V(P)$ be the endpoint of P that is the nearest to s_2. If $u_1 \neq u'_1$ (see Fig. 8(a)), then $W_1 + W_2 = W'_1 + w_2(s_3, s_4)$ and because $W'_2 \geq w_2(s_3, s_4)$, we get $W_1 + w_2 \leq W'_1 + W'_2$, a contradiction. Thus we consider the case of $u_1 = u'_1$ (see Fig. 8(b)).

Let $u_2 \in V(P)$ be the other endpoint of P. Figure 8(c) denotes the positional relation of the vertices s_1, s_2, u_1 and u_2. Let $a, b, \ldots, e$ in Fig. 8 denote the weights of paths, e.g., the weight of the path from s_1 to u_1 is a, where the weight of a path means the total weight of the edges in the path. From the definitions of u_1 and u_2, the sources s_3 and s_4 are located on two of the three vertices u_2, v_1 and v_2 in Fig. 8(c).

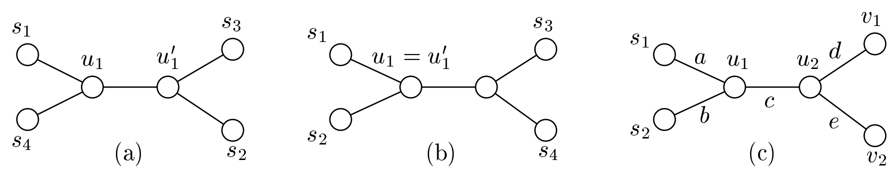


Figure 8: Proof of Lemma 7.

Since $S'_1 = \{s_1, s_2\}$, $W'_1 = a + b$. The path between s_1 and s_2 has the largest weight of all the paths. Hence, from the paths s_1v_2 and s_2v_2, we have $a \geq c + e$ and $b \geq c + e$, respectively. Thus,

$$\begin{aligned} W'_1 &= a + b \geq 2(c + e) \\ c + e &\leq \frac{1}{2} W'_1. \end{aligned} \tag{11}$$

Additionally, since selfish player 2 optimally locates sources on the vertices other than s_1 and s_2,

$$W'_2 \geq c + d \tag{12}$$

$$\text{and} \quad W'_2 \geq c + e. \tag{13}$$

Since s_3 and s_4 are located on two of the three vertices u_2, v_1, and v_2,

$$W_1 + W_2 \leq a + b + 2c + d + e.$$

From $W'_1 = a + b$ and (12),

$$\begin{aligned} W_1 + W_2 &\leq W'_1 + W'_2 + c + e \\ &= W'_1 + W'_2 + \frac{2}{3}(c + e) + \frac{1}{3}(c + e) \\ &\leq W'_1 + W'_2 + \frac{1}{3} W'_1 + \frac{1}{3} W'_2 \qquad \text{(from (11) and (13))} \\ &= \frac{4}{3}(W'_1 + W'_2). \end{aligned}$$

Therefore, $POG_2(2, 2) \leq \frac{4}{3}$. □

Now Theorem 3 follows.

Proof of Theorem 3: The lower bounds for cases (I), (II), (III) and (IV) are immediate from Lemma 6. We show the upper bound for each case.

(I) For $q \geq 3$, the upper bound $POG_2(p,q) \leq \min\{p, q-1\}$ in Lemma 4 is available, since this problem is contained in the case in Sect. 3.1. Hence the tight bound $POG_2(p,q) = \min\{p, q-1\}$ for $q \geq 3$ is obtained.

(II) For $q = p = 2$, the upper bound is immediate from Lemma 7. The tight bound $POG_2(2,2) = \frac{4}{3}$ is obtained in this case.

(III) For $p = 2$ and $p \geq 3$, the upper bound $POG_2(p,2) \leq 2$ is immediate from Lemma 2. Consequently, $2 - \frac{1}{p} \leq POG_2(p,2) \leq 2$ for odd $p \geq 3$.

(IV) Similarly to case (III), the upper bound $POG_2(p,2) \leq 2$ in Lemma 2 is available. Hence $2 - \frac{1}{p-1} \leq POG_2(p,2) \leq 2$ for even $p \geq 4$. □

4 Conclusions and Future Work

In this paper, we presented a new problem, the multi-commodity source location problem and analyzed the value of the price of greed. We showed the tight bound $POG_k(p,q) = \min\{p,q\}$ in the general case for any $k \geq 2$, $p \geq 1$ and $q \geq 1$. In addition, for a vertex-unweighted tree and $k = 2$, we showed $POG_2(p,q) = \min\{p, q-1\}$ for $q \geq 3$, and $POG_2(p,2) = \min\{p,2\}$ for any $p \geq 1$ if the players locate sources only on the leaves.

Without the assumption that sources are located only on leaves, we also show that $POG_2(p,q)$ for $q \geq 3$ is tightly bounded by $\min\{p, q-1\}$, which is the same as that under the assumption. However, $POG_2(p,2)$ for $q = 2$ is distinct. In fact, for $q = p = 2$ we obtain the tight bound $POG_2(2,2) = 4/3$. In addition, for general p, we can only give lower bounds $POG_2(p,2) \geq 2 - 1/p$ for odd $p \geq 3$ and $POG_2(p,2) \geq 2 - 1/(p-1)$ for even $p \geq 4$, while the upper bound for $p \geq 3$ and $q = 2$ is 2. We conjecture these lower bounds are tight.

Further work is to analyze the value of the price of greed and the behaviours of selfish players when each player in turn locates sources one at a time. It would also be interesting to consider the problem when all players simultaneously locate sources, with possibly several players choosing the same vertices. The profit of vertices and edges covered by several players' sources would be divided among those players in some appropriate way.

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