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On the Perspectives Opened by Right Angle Crossing Drawings

Patrizio Angelini [1] *Luca Cittadini* [1] *Giuseppe Di Battista* [1]
Walter Didimo [2] *Fabrizio Frati* [1] *Michael Kaufmann* [3]
Antonios Symvonis [4]

[1]Dipartimento di Informatica e Automazione,
Roma Tre University, Italy
[2]Dipartimento di Ingegneria Elettronica e dell'Informazione,
Perugia University, Italy
[3]Wilhelm-Schickard-Institut für Informatik,
Universität Tübingen, Germany
[4]Department of Mathematics,
National Technical University of Athens, Greece

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E-mail addresses: angelini@dia.uniroma3.it (Patrizio Angelini) ratm@dia.uniroma3.it (Luca Cittadini) gdb@dia.uniroma3.it (Giuseppe Di Battista) walter.didimo@diei.unipg.it (Walter Didimo) frati@dia.uniroma3.it (Fabrizio Frati) mk@informatik.uni-tuebingen.de (Michael Kaufmann) symvonis@math.ntua.gr (Antonios Symvonis)

Abstract

Right Angle Crossing (RAC) drawings are polyline drawings where each crossing forms four right angles. RAC drawings have been introduced because cognitive experiments provided evidence that increasing the number of crossings does not decrease the readability of a drawing if edges cross at right angles. We investigate to what extent RAC drawings can help in overcoming the limitations of widely adopted planar graph drawing conventions, providing both positive and negative results.

First, we prove that there exist acyclic planar digraphs not admitting any straight-line upward RAC drawing and that the corresponding decision problem is NP-hard. Also, we show digraphs whose straight-line upward RAC drawings require exponential area. Exploiting the techniques introduced for studying straight-line upward RAC drawings, we also show that there exist planar undirected graphs requiring quadratic area in any straight-line RAC drawing.

Second, we study whether RAC drawings allow us to draw bounded-degree graphs with lower curve complexity than the one required by more constrained drawing conventions. We prove that every graph with vertex-degree at most six (at most three) admits a RAC drawing with curve complexity two (resp. one) and with quadratic area.

Third, we consider a natural non-planar generalization of planar embedded graphs. Here we give bounds for curve complexity and area different from the ones known for planar embeddings.

1 Introduction

In graph drawing, it is commonly accepted that *crossings* and *bends* can make the layout difficult to read and experimental results show that the human performance in path-tracing tasks is negatively correlated to the number of edge crossings and to the number of bends along the edges [20, 21, 23]. However, further cognitive experiments in graph visualization show that increasing the number of crossings does not decrease the readability of the drawing if the edges cross at right angles [14, 15]. These results provide evidence for the effectiveness of *orthogonal* drawings (in which edges are chains of horizontal and vertical segments) with few bends [5, 16] and motivate the study of a new class of drawings, called *Right Angle Crossing drawings* (*RAC drawings*), introduced by Didimo, Eades, and Liotta [9]. A RAC drawing of a graph G is a polyline drawing Γ of G such that any two crossing segments are orthogonal. Figure 1 shows a RAC drawing with curve complexity two, where the *curve complexity* of Γ is the maximum number of bends along an edge of Γ. If Γ has curve complexity zero, then Γ is a *straight-line RAC drawing.*

This paper investigates RAC drawings with low curve complexity for both directed and undirected graphs.

For directed graphs, also called *digraphs*, a widely studied drawing standard is the *upward drawing* convention, where edges are monotone in the vertical direction. A digraph has an upward planar drawing if and only if it has a straight-line upward planar drawing [6]. However, not all planar digraphs have

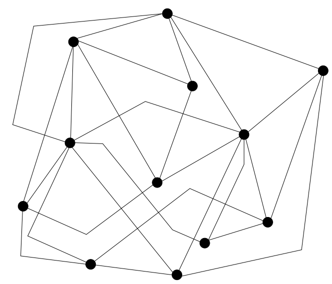


Figure 1: A RAC drawing with curve complexity two.

an upward planar drawing and straight-line upward planar drawings require exponential area for some families of digraphs [7].

We investigate *straight-line upward RAC drawings*, i.e. straight-line upward drawings with right angle crossings. In particular, it is natural to ask if every planar acyclic digraph admits an upward RAC drawing and if every digraph with an upward RAC drawing admits one with polynomial area. Both these questions have a negative answer:

- We prove that there exist acyclic planar digraphs that do not admit any straight-line upward RAC drawing and that the problem of deciding whether an acyclic planar digraph admits such a drawing is NP-hard;
- we show that there exist upward planar digraphs whose straight-line upward RAC drawings require exponential area.

Exploiting the techniques introduced for proving that straight-line upward RAC drawings of upward planar digraphs may require exponential area, we also show that there exist planar undirected graphs requiring quadratic area in any straight-line RAC drawing.

It is known [9] that any n-vertex straight-line RAC drawing of an undirected graph has at most $4n - 10$ edges, for every $n \geq 4$, and this bound is tight. Further, every graph admits a RAC drawing with at most three bends per edge, and this curve complexity is required in infinitely many cases [9]. Indeed, RAC drawings with curve complexity one and two have at most $21n$ and $150n$ edges, respectively, as shown by Arikushi and Tóth [1], who improved previous sub-quadratic area bounds by Didimo *et al.* [9]. Hence, we investigate families of graphs that can be drawn with curve complexity one or two, proving the following results:

- Every degree-6 graph admits a RAC drawing with curve complexity two;
- every degree-3 graph admits a RAC drawing with curve complexity one.

In both cases, the drawings can be computed in linear time and require quadratic area. Observe that degree-4 graphs, with the exception of the octahedron [12], admit planar orthogonal drawings with curve complexity two [18], while there exist degree-3 graphs, as for example K_4, that require two bends on one edge in any planar orthogonal drawing.

In a *fixed embedding* setting, the input graph G is given with a (non-planar) *embedding*, i.e., a circular ordering of the edges incident to each vertex and an ordering of the crossings along each edge. A RAC drawing algorithm can not change the embedding of G. For such a setting it has been proved [9] that any n-vertex graph admits a RAC drawing with $O(kn^2)$ bends per edge, where k is the maximum number of crossings between any two edges. Also, there exist graphs whose RAC drawings require $\Omega(n^2)$ bends along some edges. We study the fixed embedding setting, namely we study non-planar graphs obtained by augmenting a plane triangulation with edges inside pairs of adjacent faces; we call these graphs *kite-triangulations*:

- We prove that one bend per edge is always sufficient and sometimes necessary for a RAC drawing of a kite-triangulation;

- we show that there exist kite-triangulations requiring cubic area in any straight-line RAC drawing. Recall that every embedded planar graph admits a planar drawing with quadratic area [4, 22].

The rest of the paper is organized as follows. In Sect. 2 we introduce some definitions and preliminaries; in Sect. 3 we study straight-line upward RAC drawings of planar acyclic digraphs; in Sect. 4 we study RAC drawings of bounded-degree graphs; in Sect. 5 we study RAC drawings of kite-triangulations; finally, in Sect. 6, we conclude the paper with some open problems.

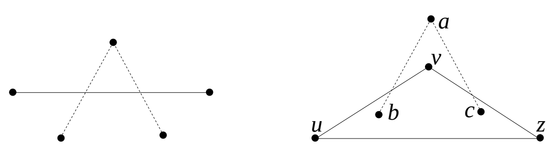
2 Preliminaries

We assume familiarity with graph drawing and planarity [5, 16]. In the following, unless otherwise specified, all considered graphs are *simple*.

The *degree* of a vertex is the number of edges incident to it. The *degree* of a graph is the maximum among the degrees of its vertices. A graph is *regular* if all its vertices have the same degree.

A *drawing* of a graph is a mapping of each vertex to a distinct point of the plane and of each edge to a Jordan curve between its endpoints. A *straight-line drawing* is such that all edges are straight-line segments. A *polyline drawing* is such that all edges are sequences of straight-line segments, where any point shared by consecutive segments of different slopes is a *bend*. The *curve complexity* of a drawing Γ is the maximum number of bends along an edge in Γ. A *grid drawing* of a graph is such that each vertex has integer coordinates. The *area* of a grid drawing is the area of the smallest rectangle with sides parallel to the axes completely enclosing the drawing. A *planar drawing* is such that no two edges intersect except, possibly, at common endpoints. A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two drawings of the same graph are *equivalent* if they determine the same circular ordering around each vertex. A *planar embedding* is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *external face*. A graph together with a planar embedding and a choice for its external face is called *plane graph*. A plane graph is a *triangulation* when all its faces are triangles. When dealing with non-planar graphs, an *embedding* of such a graph is a circular ordering of the edges incident to each vertex and a linear order of the edges crossing each edge. An *upward drawing* of a digraph is such that all edges are curves monotonically increasing in the upward direction. An *upward planar drawing* of a digraph G is a drawing of G that is both upward and planar. If G admits an upward planar drawing, then G is an *upward planar digraph*.

A *Right Angle Crossing drawing* (*RAC drawing*) of a graph G is a polyline drawing Γ of G such that any two crossing segments in Γ are orthogonal. If a RAC drawing Γ has curve complexity zero, then Γ is a *straight-line RAC drawing*. An *upward RAC drawing* of a digraph is a RAC drawing that is also upward. A *fan* in a drawing Γ is a pair of edge segments incident to the same vertex. Two segments s_1 and s_2 crossing the same segment in Γ are parallel. This leads to the following properties, illustrated in Fig. 2(a) and 2(b), and proved in [9] and [10].



(a) (b)

Figure 2: Illustrations for (a) Property 1 and for (b) Property 2.

Property 1 *In a straight-line RAC drawing no edge can cross a fan.*

Property 2 *In a straight-line RAC drawing there can not be a triangle $\triangle$ and two edges (a,b), (a,c) such that a lies outside $\triangle$ and b,c lie inside $\triangle$.*

3 Upward RAC Drawings

We now study straight-line upward RAC drawings of directed graphs. In order to achieve our results on straight-line upward RAC drawings of directed graphs, we prove some lemmata concerning undirected graphs. Consider K_4, that is, the complete graph on four vertices u, v, z, and w. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the embeddings of K_4 shown in Fig. 3(a) and 3(b), respectively.

Lemma 1 *In any straight-line drawing of K_4, its embedding is one of $\mathcal{E}_1$ and $\mathcal{E}_2$, up to a renaming of the vertices.*

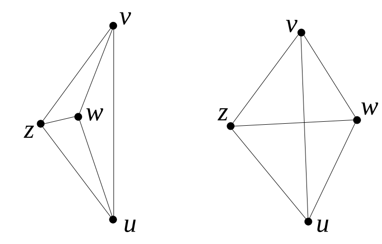


Figure 3: (a) $\mathcal{E}_1$; (b) $\mathcal{E}_2$.

Proof: Consider any straight-line drawing Γ of K_4. Either three or four vertices are on the convex hull of Γ, as otherwise there would be two overlapping edges. Observe that, since the drawing is straight-line, the edges delimiting the convex hull of Γ do not cross any edge of K_4. If exactly three vertices of K_4 are on the convex hull of Γ, then the fourth vertex is inside such a convex hull. Since the drawing is straight-line, the edges incident to the fourth vertex do not cross any edge of K_4. It follows that the embedding of K_4 is $\mathcal{E}_1$. If exactly four vertices of K_4 are on the convex hull of Γ, then the two edges between non-consecutive vertices of the convex hull cross. Since the drawing is straight-line, such edges cross exactly once. It follows that the embedding of K_4 is $\mathcal{E}_2$. □

Lemma 2 *Let G be a graph containing two vertex-disjoint copies K_4' and K_4'' of K_4. Let Γ be any straight-line RAC drawing of G. For any* 3*-cycle* (a', b', c') *of K_4', which is represented in Γ by a triangle $\triangle'$, either all the vertices of K_4'' are inside $\triangle'$ or they are all outside it.*

Proof: If at least two vertices a'' and b'' of K_4'' are inside $\triangle'$ and at least one vertex c'' is outside it, then Property 2 is violated, since vertex c'' is connected to both a'' and b''.

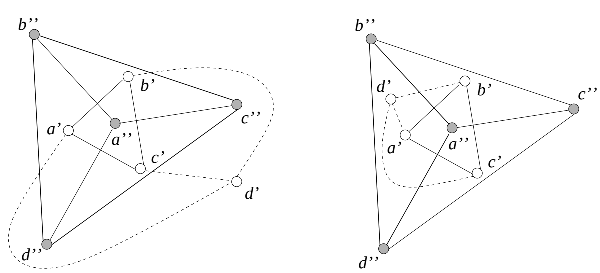


Figure 4: (a) If d' is placed outside $\triangle''$, then Property 2 is violated. (b) If d' is placed inside $\triangle''$, then Property 2 is violated.

If exactly one vertex a'' of K_4'' is inside $\triangle'$, then b'', c'', and d'' are outside it. Since the drawing is straight-line, if there is a crossing between an edge of the 3-cycle (b'', c'', d'') of K_4'' and an edge of (a', b', c'), then such an edge of (b'', c'', d'') crosses a fan composed of two edges of (a', b', c'), thus violating Property 1. It follows that $\triangle'$ is contained inside the triangle $\triangle''$ representing (b'', c'', d'') in Γ, with each of the edges (a'', b''), (a'', c''), and (a'', d'') crossing a distinct edge of $\triangle'$ with a right-angle crossing. However, in this case every possible placement of d' violates Property 2. Namely, if d' is outside $\triangle''$, then Property 2 is violated since d' is connected to a', b', and c', which are inside $\triangle''$ (see Fig. 4(a)). Further, if d' is inside $\triangle''$, then it is inside one of the faces internal to $\triangle''$, say the one containing a'; then, Property 2 is violated since d' and a' are both connected to b', which is outside the triangle representing such a face in Γ (see Fig. 4(b)). □

Lemma 3 *Let G be a graph containing two vertex-disjoint copies K_4' and K_4'' of K_4. In any RAC drawing Γ of G, no edge of K_4' crosses an edge of K_4''.*

Proof: Let Γ^* be Γ restricted to the edges of K_4' and K_4''. We show that in Γ^* there is no crossing between the edges of K_4' and the edges of K_4''. Let u', v', z', and w' be the vertices of K_4', and let u'', v'', z'', and w'' be the vertices of K_4''.

If the embedding of K_4' in Γ^* is $\mathcal{E}_1$, then assume, without loss of generality up to a renaming of the vertices, that (u', v', z') is the 3-cycle delimiting the external face of K_4' in Γ^* and hence enclosing w'. By Lemma 2, either all the vertices of K_4'' lie outside $\triangle'$ or they all lie inside it. In the former case, if there is a crossing between an edge of K_4' and an edge of K_4'', then such an edge of K_4'' crosses a fan composed of two edges of K_4', thus violating Property 1. In the latter case, the vertices of K_4'' lie in the faces of K_4' internal to $\triangle'$. By Lemma 2, all the vertices of K_4'' lie in the same internal face of K_4'. Hence, in both cases, no edge of K_4' crosses an edge of K_4''.

If the embedding of K_4' in Γ^* is $\mathcal{E}_2$, then assume, without loss of generality up to a renaming of the vertices, that (u', v', z', w') is the 4-cycle delimiting the external face of K_4' in Γ^*. Thus, the edges of K_4' delimit five connected regions $R_1, \dots, R_5$ of the plane, where R_1, R_2, R_3, and R_4 are inside (u', v', z', w'), and R_5 is outside (u', v', z', w'). We prove that all the vertices of K_4'' are inside the same region R_i. Suppose that vertices a'' and b'' exist such that $a'', b'' \in \{u'', z'', v'', w''\}$ and a'' is inside R_i and b'' is inside R_j, with $j \neq i$. For every pair of regions R_i and R_j, with $j \neq i$, a 3-cycle (a', b', c') of K_4', with $a', b', c' \in \{u', z', v', w'\}$, exists containing R_i in its interior and R_j in its exterior, or vice versa. Then, a'' is inside the triangle representing (a', b', c') and b'' is outside such a triangle, or vice versa. However, by Lemma 2, Γ^* is not a RAC drawing. Hence, all the vertices of K_4'' are inside the same region R_i. If all the vertices of K_4'' are in the same region R_i, with $1 \leq i \leq 4$, then no edge of K_4' crosses an edge of K_4''. If all the vertices of K_4'' are in R_5, then suppose that a crossing between an edge of K_4' and an edge of K_4'' exists. However, such an edge of K_4'' crosses a fan composed of two edges of K_4', thus violating Property 1. □

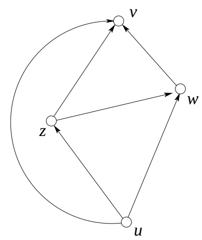


Figure 5: The upward planar digraph H obtained by acyclically orienting the edges of K_4.

Now we use the previous lemmata to prove the main results of this section. First, we introduce an upward planar digraph H, shown in Fig. 5, which is obtained by acyclically orienting the edges of K_4. Denote by u and v the only source and the only sink of H, respectively.

We get the following:

Lemma 4 *Consider a planar acyclic digraph K. Replace each edge (a, b) of K with a copy of H, by identifying vertices a and b of K with vertices u and v of H, respectively. Let K' be the resulting planar digraph. Digraph K is upward planar if and only if K' is straight-line upward RAC drawable.*

Proof: Refer to Figs. 6(a) and 6(b).

First, suppose that K admits an upward planar drawing. Then, by the results of Di Battista and Tamassia [6], K admits a straight-line upward planar drawing Γ. Consider the drawing Γ' of K' obtained from Γ by drawing each copy of H that replaces an edge (a, b) in such a way that: (i) The drawing of H is upward planar; (ii) the drawing of edge (u, v) of H in Γ' coincides with the drawing of edge (a, b) of K in Γ; and (iii) the drawing of the other vertices and edges of H is arbitrarily close to (u, v). Since Γ is a straight-line upward planar drawing, Γ' is a straight-line upward planar drawing. Hence, Γ' is a straight-line upward RAC drawing of K'.

Second, suppose that K' admits a straight-line upward RAC drawing Γ'. Consider the straight-line drawing Γ of K obtained by restricting Γ' to the edges of K, that is, obtained by removing from Γ', for every copy of H, all the vertices of H different from u and v and all the edges of H different from (u, v). As Γ' is an upward drawing of K', then Γ is an upward drawing of K. Suppose, for a contradiction, that two edges cross in Γ. If such two edges are adjacent, then they do not cross, as otherwise they overlap. If such two edges are not adjacent, then they belong to two distinct copies of H in K'. However, by Lemma 3, no two edges belonging to distinct copies of H cross in Γ', thus obtaining a contradiction. Hence, Γ is a straight-line upward planar drawing of K. □

We are ready to prove the first theorem of this section.

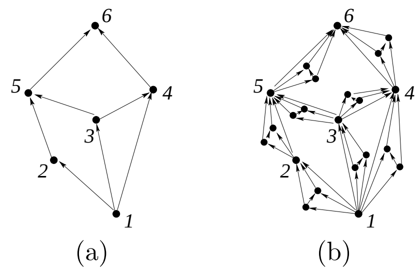


Figure 6: (a) A straight-line upward drawing of an upward planar acyclic digraph K. (b) A straight-line upward RAC drawing of the planar acyclic digraph K' obtained by replacing each edge of K with a copy of H.

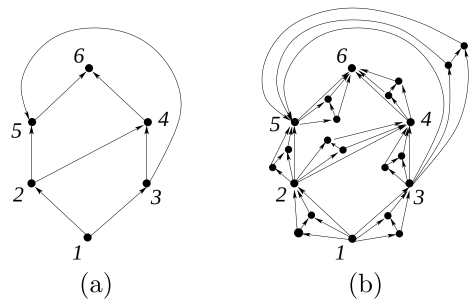


Figure 7: (a) A planar acyclic digraph G that is not upward planar. (b) The planar acyclic digraph G' obtained by replacing each edge of G with a copy of H is not straight-line upward RAC drawable.

Theorem 1 *There exist acyclic planar digraphs that do not admit any straight-line upward RAC drawing.*

Proof: Consider any planar acyclic digraph G (as the one of Fig. 7(a)) that is not upward planar. By Lemma 4, the planar acyclic digraph G' obtained by replacing each edge of G with a copy of H does not admit any straight-line upward RAC drawing (see Fig. 7(b)). □

Note that there exist planar digraphs, as the one in Fig. 8, that do not admit any straight-line upward RAC drawing, that are not constructed using gadget H, and whose size is smaller than the one of the digraph in Fig. 7(b). However, proving that they are not straight-line upward RAC drawable could result in a complex case-analysis.

Motivated by the fact that there exist acyclic planar digraphs that do not admit any straight-line upward RAC drawing, we study the time complexity of the corresponding decision problem.

We show that the problem of testing whether a digraph admits a straight-line upward RAC drawing (Upward RAC Drawability Testing) is NP-hard, by

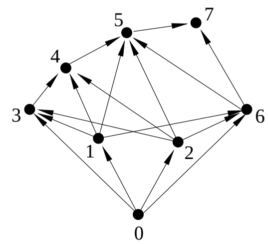


Figure 8: An 8-vertex planar digraph that does not admit any straight-line upward RAC drawing.

means of a reduction from the problem of testing whether a digraph admits a straight-line upward planar drawing (UPWARD PLANARITY TESTING), which is NP-complete [13].

Theorem 2 UPWARD RAC DRAWABILITY TESTING *is NP-hard.*

Proof: We reduce UPWARD PLANARITY TESTING to UPWARD RAC DRAWABILITY TESTING. Let G be an instance of UPWARD PLANARITY TESTING. Replace each edge (a, b) of G with a copy of H, by identifying vertices a and b of G with vertices u and v of H, respectively. Let G' be the resulting planar digraph. By Lemma 4, G is upward planar if and only if G' admits a straight-line upward RAC drawing. □

Next, we show that there exists a class of planar acyclic digraphs that require exponential area in any straight-line upward RAC drawing.

Consider the class of upward planar digraphs G_n (see Fig.9), defined by Di Battista *et al.* [7], which requires $\Omega(2^n)$ area in any straight-line upward planar drawing, under any resolution rule. Replace each edge (a, b) of G_n with a copy of H, by identifying vertices a and b of G_n with vertices u and v of H, respectively. Let G'_n be the resulting planar digraph. Observe that, assuming that G_n has n vertices, G'_n has $O(n)$ vertices since, for every edge of G_n, two new vertices are introduced in G'_n.

Theorem 3 *Any straight-line upward RAC drawing of G'_n requires $\Omega(b^n)$ area, under any resolution rule, for some constant $b > 1$.*

Proof: Suppose, for a contradiction, that, for every constant $b > 1$, G'_n admits a straight-line upward RAC drawing Γ' with $o(b^n)$ area, under some resolution rule. Consider the straight-line drawing Γ of G_n obtained by restricting Γ' to the edges of G_n, that is, obtained by removing from Γ', for every copy of H, all the vertices of H different from u and v and all the edges of H different from (u, v). As Γ' is an upward drawing of G'_n, then Γ is an upward drawing of G_n. If two edges of G_n are adjacent, then they do not cross in Γ, as otherwise they overlap. If two edges of G_n are not adjacent, then they belong to two distinct

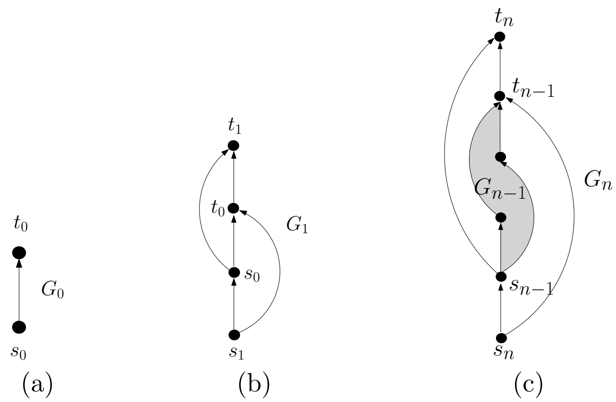


Figure 9: (a) Graph G_0. (b) Graph G_1. (c) Graph G_n.

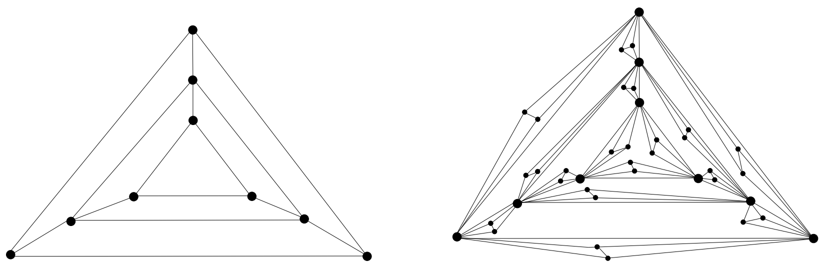
copies of H in G'_n. However, by Lemma 3, no two edges belonging to distinct copies of H cross in Γ', thus they do not cross in Γ. Hence, Γ is a straight-line upward planar drawing of G_n. Further, the area of Γ is $o(b^n)$, as the area of Γ' is $o(b^n)$, thus obtaining a contradiction and proving the theorem. □

We now turn our attention to straight-line RAC drawings of undirected graphs. We exploit the techniques introduced for straight-line upward RAC drawings to get a quadratic lower bound on the area requirements of straight-line grid RAC drawings of planar graphs.

Consider a *nested triangles graph* G, that is, a triconnected graph composed of $\frac{n}{3}$ 3-cycles nested one into the other (see Fig. 10(a)). Graph G is known to require $\Omega(n^2)$ area in any straight-line planar drawing [4]. Replace each edge (a, b) of G with a copy of K_4, by identifying vertices a and b of G with vertices u and v of K_4, respectively. Let G' be the resulting planar graph (see Fig. 10(b)). Observe that G' has $O(n)$ vertices since, for every edge of G, two new vertices are introduced in G'. We have the following.

Theorem 4 *Any straight-line grid RAC drawing of G' requires $\Omega(n^2)$ area.*

Proof: Consider any straight-line grid RAC drawing Γ' of G'. Consider the straight-line drawing Γ of G obtained by restricting Γ' to the edges of G, that is, obtained by removing from Γ', for every copy of K_4, all the vertices of K_4 different from u and v and all the edges of K_4 different from (u, v). If two edges of G are adjacent, then they do not cross, as otherwise they overlap. If two edges of G are not adjacent, then they belong to two distinct copies of K_4 in G'. However, by Lemma 3, no two edges belonging to distinct copies of K_4 cross in Γ', thus they do not cross in Γ. Hence, Γ is a straight-line planar drawing of G. It follows that the area of Γ is $\Omega(n^2)$, and the area of Γ' is $\Omega(n^2)$, as well. □



(a) (b)

Figure 10: (a) A nested triangles graph G. (b) The graph G' obtained by replacing each edge (a, b) of G with a copy of K_4.

4 RAC-Drawings of Bounded-Degree Graphs

In this section, we present algorithms for constructing RAC drawings of graphs of bounded degree. The algorithms are based on the decomposition of a regular directed multigraph into directed 2-factors. A *2-factor* of an undirected graph G is a spanning subgraph of G consisting of vertex-disjoint cycles (see also [3, pp.227]). Analogously, a *directed 2-factor* of a directed graph is a spanning subgraph consisting of vertex-disjoint directed cycles. The decomposition of a regular directed multigraph into directed 2-factors follows from a classical result for undirected graphs [19] stating that "a regular multigraph of degree $2k$ has k edge-disjoint 2-factors". A constructive proof of the following theorem was given by Eades *et al.* [11].

Theorem 5 ***(Eades,Symvonis,Whitesides [11])*** *Let $G = (V, E)$ be an n-vertex undirected graph of degree Δ and let $d = \lceil \Delta/2 \rceil$. Then, there exists a directed multi-graph $G' = (V, E')$ such that:*

1. *each vertex of G' has indegree d and outdegree d;*
2. *G is a subgraph of the underlying undirected graph of G'; and*
3. *the edges of G' can be partitioned into d edge-disjoint directed 2-factors.*

Furthermore, the directed graph G' and its d directed 2-factors can be computed in $O(\Delta^2 n)$ time.

Let u be a vertex placed at a grid point. We say that an edge e exiting u uses the *Y-port* of u (resp. the *$-Y$-port* of u) if it exits u along the $+Y$ direction (resp. along the $-Y$ direction). In an analogous way, we define the *X-port* and the *$-X$-port*. We have the following.

Theorem 6 *Every n-vertex graph with degree at most six admits a RAC drawing with curve complexity two in $O(n^2)$ area. Such a drawing can be computed in $O(n)$ time.*

Proof: Let $G = (V, E)$ be a graph of degree six. Let $G' = (V, E')$ be the directed multigraph obtained from G as in Theorem 5, and let C_1, C_2, and C_3 be the three edge-disjoint directed 2-factors of G'. We show how to obtain a RAC drawing of G'. Then, a RAC drawing of G can be obtained by removing from the drawing all the edges in $E' \setminus E$ and by ignoring the direction of the edges.

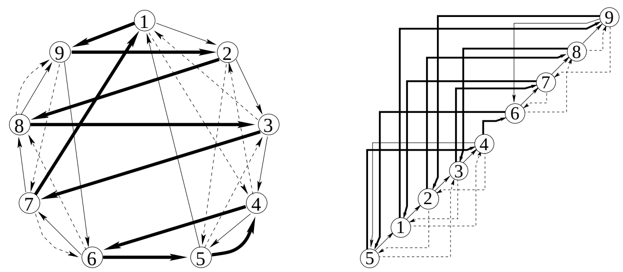


Figure 11: (a) A regular directed multigraph G' with indegree and outdegree equal to three and its directed 2-factors C_1, C_2, and C_3. The edges of C_1 are represented by solid thin lines, the edges of C_2 are represented by solid thick lines, and the edges of C_3 are represented by dashed lines. (b) The RAC drawing of G' with two bends per edge constructed by the algorithm described in the proof of Theorem 6.

The algorithm places the vertices of V on the main diagonal of an $O(n) \times O(n)$ grid, in an order determined by one of the directed 2-factors, say C_1. Most of the edges of C_1 are drawn as straight-line segments along the diagonal while the edges of C_2 and C_3 are drawn as 3-segment lines above and below the diagonal, respectively. Finally, the remaining "closing" edges of C_1 (i.e., the edges that are not drawn on the diagonal) are drawn as 2- or 3-segment lines either above or below the diagonal.

We first describe how to place the vertices of G' along the main diagonal. Arbitrarily name the cycles $c_1, c_2, \ldots, c_k$ of C_1. Consider each cycle c_i, for $1 \leq i \leq k$.

- If there exist a vertex $u \in c_i$ and an edge $(u, z) \in C_2$ or C_3 such that z belongs to a cycle c_j of C_1 with $j > i$, then let u be the *topmost vertex* of c_i and let the vertex following u in c_i be the *bottommost vertex* of c_i.
- Otherwise, if there exist a vertex $v \in c_i$ and an edge $(v, w) \in C_2$ or C_3 such that w belongs to a cycle c_j of C_1 with $j < i$, then let v be the bottommost vertex of c_i and let the vertex preceding v in c_i be the topmost vertex of c_i.
- Otherwise, all the edges of C_2 and C_3 exiting vertices of c_i are directed to vertices of c_i. In this case, let an arbitrary vertex w of c_i be the

bottommost vertex of c_i and let the vertex preceding w in c_i be the topmost vertex of c_i.

Figure 11(a) shows a regular directed multigraph G' of indegree and outdegree three and its directed 2-factors C_1, C_2, and C_3. C_1 consists of cycles $c_1 : (5, 1, 2, 3, 4, 5)$ and $c_2 : (6, 7, 8, 9, 6)$. We set 4 as the topmost vertex of c_1 since edge $(4, 6)$ of C_2 has vertex 6 of c_2 as its destination. Analogously, we set 6 as the bottommost vertex of c_2 since edge $(6, 5)$ of C_2 has vertex 5 of c_1 as its destination. Figure 11(b) shows the RAC drawing of G' with curve complexity two constructed by the algorithm described in this proof.

Then, the vertices of G' are placed on the diagonal so that each vertex of c_i is placed on the diagonal before each vertex of c_j, for each $i < j$, and so that the vertices of c_i are placed on the diagonal in the order defined by c_i, starting at the bottommost vertex of c_i and ending at the topmost vertex of c_i, for each i. When the h-th vertex of G' is placed on the diagonal, it is assigned coordinates $(16(h-1), 16(h-1))$.

Having placed the vertices on the grid, we turn our attention to drawing the edges of G'. Each edge is drawn either as a 1-segment line along the diagonal, or as a 2- or 3-segment line either above or below the diagonal. We draw the edges so that all the crossing line segments are parallel to the axes and, consequently, all the crossings are at right angles. In our drawings, every line segment s that is not parallel to the axes is incident to a vertex v_s of the graph; further, such a segment s is contained in a dedicated region within a square $Q(v_s)$ whose diagonals meet at v_s and whose side has length 16 (see Fig. 12(a)).

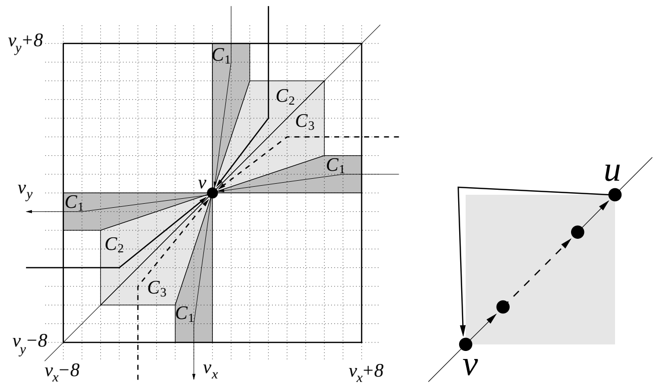
The edges of C_2 are drawn above the diagonal as follows. Consider an edge (u, v) of C_2 and let u and v be placed at grid points (u_x, u_y) and (v_x, v_y), respectively.

- If u is placed below v (i.e., $u_y < v_y$), then edge (u, v) is drawn as a 3-segment line exiting vertex u from the Y-port and being defined by bend-points $(u_x, v_y - 4)$ and $(v_x - 5, v_y - 4)$. Note that the third line segment of (u, v) is contained in the lightly-shaded region (above the diagonal) of the south-west quadrant of $Q(v)$ (see Fig. 12(a)).

- If u is placed above v (i.e., $u_y > v_y$), then edge (u, v) is drawn as a 3-segment line exiting vertex u from the $-X$-port and being defined by bend-points $(v_x + 3, u_y)$ and $(v_x + 3, v_y + 4)$. Note that, in this case, the third line segment of (u, v) is contained in the lightly-shaded region (above the diagonal) of the north-east quadrant of $Q(v)$ (see Fig. 12(a)).

It is easy to observe that the only line segments that belong to edges of C_2 and that cross other line segments are parallel to the axes, hence they cross at right angles. Namely, all the line segments that are not parallel to the axes are contained in the lightly-shaded regions shown in Fig. 12(a), and there is at most one of such line segments per region.

The edges of C_3 are drawn below the diagonal in an analogous way.



(a) (b)

Figure 12: (a) The square $Q(v)$ around a vertex v. The shaded regions contain line segments not parallel to the axes and are used to visualize the absence of crossings inside $Q(v)$. (b) Drawing the closing edge of a cycle of C_1 in Case 3.

Consider now the edges of C_1. All such edges, except those closing the cycles of C_1, are drawn as straight-line segments along the diagonal. As all the edges of C_2 (resp. C_3) are drawn above (resp. below) the diagonal, the edges of C_1 drawn along the diagonal are not involved in any edge crossing. To complete the drawing of G', we describe how to draw the edges connecting the topmost vertex to the bottommost vertex of each cycle of C_1. Consider an arbitrary cycle c_i of C_1 and let (u, v) be its closing edge. We consider three cases:

Case 1: *u was selected to be the topmost vertex of c_i due to the existence of an edge (u, z) of C_2 or C_3 such that z is above u.*

In such a case, after drawing the edges of C_2 and C_3, vertex u has not used either its $-X$-port, or its $-Y$-port, or both. Namely, u used its $-X$-port if there is an edge (u, v) of C_2 such that v is below u, and u used its $-Y$-port if there is an edge (u, v) of C_3, such that v is below u. However, since an edge (u, z) of C_2 or of C_3 exists such that z is above u, if u used both its $-X$-port and its $-Y$-port, there would be three edges exiting u in C_2 and C_3, while there are exactly two of such edges.

Assume that the $-X$-port of u is free (the case where the $-Y$-port of u is free can be treated analogously). Edge (u, v) is drawn above the diagonal as a 3-segment line exiting vertex u from the $-X$-port and being defined by bend-points $(v_x + 1, u_y)$ and $(v_x + 1, v_y + 7)$. Note that, in this case, the third line segment of (u, v) is contained in the dark-shaded region (above the diagonal) of the north-east quadrant of $Q(v)$ (see Fig. 12(a)).

Case 2: *v was selected to be the bottommost vertex of c_i due to the existence of an edge (v, w) of C_2 or C_3 such that w is below v.*

In such a case, after drawing the edges of C_2 and C_3, vertex v has not used either its X-port, or its Y-port, or both, which can be proved analogously to

Case 1.

Assume that the Y-port of v is free (the case where the X-port of v is free can be treated analogously). Edge (u, v) is drawn above the diagonal as a 3-segment line exiting vertex v from the Y-port and being defined by bend-points $(v_x, u_y - 1)$ and $(u_x - 7, u_y - 1)$. Note that, in this case, the first line segment of (u, v) is contained in the dark-shaded region (above the diagonal) of the south-west quadrant of $Q(u)$ (see Fig. 12(a)).

Case 3: *Neither Case 1 nor Case 2 applies.*

In such a case, all the edges of C_2 and C_3 exiting vertices of cycle c_i are also directed to vertices of c_i. Notice that this also implies that all the edges of C_2 and C_3 entering vertices of c_i are originated from vertices of c_i. Namely, if there were an edge (u, v) such that v is in c_i and u is not, then there would be an edge (w, z) such that w is in c_i and z is not. Hence, denoting by u and v the topmost vertex and the bottommost vertex of c_i, respectively, (observe that the bottommost vertex was chosen arbitrarily) the drawing of the edges of C_2 and C_3 incident to vertices of c_i takes place entirely within the square having points (v_x, v_y) and (u_x, u_y) as opposite corners (the shaded square in Fig. 12(b)). Hence, the closing edge can be drawn as a 2-segment line connecting u and v and being defined by bend-point $(v_x - 1, u_y + 1)$ (see Fig. 12(b)).

Given C_1, C_2, and C_3, it is easy to see that the drawing can be constructed in linear time. By Theorem 5, C_1, C_2, and C_3 can be also computed in linear time, resulting in a linear-time algorithm. Also, the produced RAC drawing lies in an $O(n^2)$ size grid. □

We now prove the following:

Theorem 7 *Every n-vertex graph with degree at most three admits a RAC drawing with curve complexity one in $O(n^2)$ area. Such a drawing can be computed in $O(n)$ time.*

Proof: Let $G = (V, E)$ be a graph of degree three. Let $G' = (V, E')$ be the directed multigraph obtained from G as in Theorem 5. Observe that G' is a regular multigraph of degree four. Let C_1 and C_2 be two edge-disjoint directed 2-factors of G'. We will show how to obtain a RAC drawing of G' such that only the edges of E and the edges of $E' \setminus E$ might partially overlap. Removing from the constructed drawing the edges of $E' \setminus E$ results into a RAC drawing of G.

We place the vertices of G' along the main diagonal of an $O(n) \times O(n)$ grid based on their order of appearance along the cycles of C_1. Consider an arbitrary cycle c_i of C_1.

- If c_i contains an edge $(u, v) \in E' \setminus E$, then we make vertices u and v be the topmost and bottommost vertex of c_i, respectively.
- Otherwise, if there exist a vertex $u \in c_i$ and an edge $(u, z) \in C_2 \cap E$ such that z belongs to a cycle c_j of C_1 with $j > i$, then let u be the topmost vertex of c_i and let the vertex following u in c_i be the bottommost vertex of c_i.

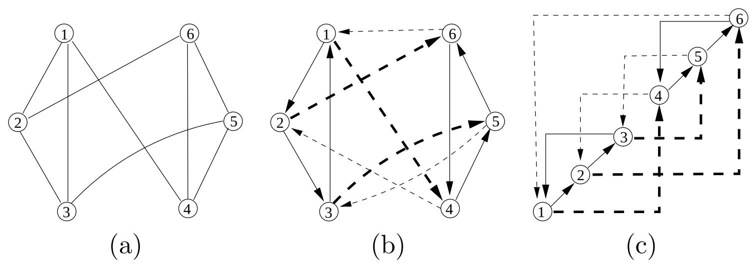


Figure 13: (a) A graph $G = (V, E)$ of degree three. (b) The regular directed multigraph $G' = (V, E')$ with indegree and outdegree equal to two obtained from G and its directed 2-factors C_1 and C_2. The edges of C_1 are represented by solid lines and the ones of C_2 by dashed lines. Edges not in G are thinner than the other edges. (c) The RAC drawing of G' with one bend per edge constructed by the algorithm described in the proof of Theorem 7.

- Otherwise, if there exist a vertex $v \in c_i$ and an edge $(v, w) \in C_2 \cap E$ such that w belongs to a cycle c_j of C_1 with $j < i$, then let v be the bottommost vertex of c_i and let the vertex preceding v in c_i be the topmost vertex of c_i.
- Otherwise, all the edges of $C_2 \cap E$ exiting vertices of c_i are also directed to vertices of c_i. In this case, let an arbitrary vertex w of c_i be the bottommost vertex of c_i and let the vertex preceding w in c_i be the topmost vertex of c_i.

Figure 13(a) shows a graph G of degree three. Figure 13(b) shows its corresponding directed graph G' and its directed 2-factors C_1 and C_2. C_1 consists of cycles $c_1 : (1, 2, 3, 1)$ and $c_2 : (4, 5, 6, 4)$. We set 3 as the topmost vertex of c_1 since edge $(3, 5)$ of C_2 has vertex 5 of c_2 as its destination. Analogously, we set 4 as the bottommost vertex of c_2 since edge $(4, 2)$ of C_2 has vertex 2 of c_1 as its destination. Figure 13(c) shows the RAC drawing of G' with curve complexity one constructed by the algorithm described in this proof. In such a drawing, edge overlaps are allowed involving at least one edge in $E' \setminus E$.

Then, the vertices of G' are placed on the diagonal so that each vertex of c_i is placed on the diagonal before each vertex of c_j, for each $i < j$, and so that the vertices of c_i are placed on the diagonal in the order defined by c_i, starting at the bottommost vertex of c_i and ending at the topmost vertex of c_i, for each i. When the h-th vertex of G' is placed on the diagonal, it is assigned coordinates $(2(h-1), 2(h-1))$.

Having placed the vertices on the grid, we turn our attention to drawing the edges of G'. Each edge is drawn either as a 1-segment line along the diagonal, or as a 2-segment line either above or below the diagonal. We draw the edges so that all the crossing line segments are parallel to the axes and, consequently, all the crossings are at right angles.

We first describe how to draw the edges of C_2. Consider an arbitrary edge (u, v) of C_2.

- If u is placed below v (i.e., $u_y < v_y$), then edge (u, v) is drawn as a 2-segment line below the diagonal, exiting vertex u from the X-port and being defined by bend-point (v_x, u_y). Such a line enters v from its $-Y$-port.
- If u is placed above v (i.e., $u_y > v_y$), then edge (u, v) is drawn as a 2-segment line above the diagonal, exiting vertex u from the $-X$-port and being defined by bend-point (v_x, u_y). Such a line enters vertex v from its Y-port.

The edges of C_2 do not overlap each other. Further, they intersect each other only at right angles, as every line segment is parallel to the axes.

Consider now the edges of C_1. All such edges, except those closing the cycles of C_1, are drawn as straight-line segments along the diagonal. As each edge of C_2 is drawn above or below the diagonal, the edges of C_1 drawn along the diagonal are not involved in any edge crossing. To complete the drawing of G', we describe how to draw the closing edge of each cycle of C_1. Consider an arbitrary cycle c_i of C_1 and let (u, v) be its closing edge. We consider four cases:

Case 1: *Edge (u, v) belongs to $E' \setminus E$.*

In this case, (u, v) is not part of G and it is not in the drawing.

Case 2: *Edge (u, v) belongs to E and u was selected to be the topmost vertex of c_i due to the existence of an edge $(u, z) \in C_2 \cap E$ such that z is above u.*

Since both edges of c_i incident to u and edge (u, z) belong to G and since there are at most three edges incident to u in G, both the $-X$-port and the $-Y$-port of u are free. Now observe that, since both edges of c_i incident to v belong to G, then at most one of the two edges of C_2 incident to v belongs to G. Hence, at most one of the Y-port and the X-port of v is used by an edge of G (the other port might be used by an edge that belongs to G' but not to G). Thus, it is always possible to draw edge (u, z) with its only bend either at point (v_x, u_y) or at point (u_x, v_y), so that it overlaps only with an edge of $E' \setminus E$.

Case 3: *Edge (u, v) belongs to E and v was selected to be the bottommost vertex of c_i due to the existence of an edge $(v, w) \in E \cap C_2$ with vertex w being placed lower on the diagonal than v.*

Analogously to the previous case, both the X-port and the Y-port of v are free and at most one of the two edges of C_2 incident to u belongs to G. Hence, at most one of the $-Y$-port and the $-X$-port of u is used by an edge of G and it is always possible to draw edge (v, w) with its only bend either at point (v_x, u_y) or at point (u_x, v_y), so that it overlaps only with an edge of $E' \setminus E$.

Case 4: *None of the above cases applies.*

In this case, all the edges of $C_2 \cap E$ exiting vertices of c_i are also directed to vertices of c_i. Notice that this also implies that all the edges of $C_2 \cap E$ entering vertices of c_i are originated from vertices of c_i. Hence, denoting by u and v the topmost vertex and the bottommost vertex of c_i, respectively, (observe that the bottommost vertex was chosen arbitrarily) the drawing of the edges of $C_2 \cap E$ incident to vertices of c_i takes place entirely within the square having points (v_x, v_y) and (u_x, u_y) as opposite corners. Hence, the closing edge (u, v) of c_i can be drawn as a 2-segment line connecting u and v and being defined by bend-point $(v_x - 1, u_y + 1)$.

Given C_1 and C_2, it is easy to see that the drawing can be constructed in linear time. By Theorem 5, C_1 and C_2 can be also computed in linear time, resulting in a linear-time algorithm. Also, the produced RAC drawing lies in an $O(n^2)$ size grid. □

5 RAC Drawings of Kite-Triangulations

In this section we study the impact of admitting orthogonal crossings on the drawability of the non-planar graphs obtained by adding edges to maximal planar graphs inside two adjacent faces, in a fixed embedding scenario. We show that such graphs always admit RAC drawings with curve complexity one and that such a curve complexity is sometimes required.

Let G' be a triangulation and let (u, z, w) and (v, z, w) be two adjacent faces of G' sharing edge (z, w). We say that $[u, v]$ is a *pair of opposite vertices* with respect to (z, w). Let $E^+ = \{[u_i, v_i] | i = 1, 2, \cdots, k\}$ be a set of pairs of opposite vertices of G', where $[u_i, v_i]$ is a pair of opposite vertices with respect to (z_i, w_i) and edge (u_i, v_i) does not belong to G'. Suppose that, for any $1 \leq i, j \leq k$ and $i \neq j$, edges (z_i, w_i) and (z_j, w_j) are not incident to the same face of G'. Let G be the embedded non-planar graph obtained by adding an edge (u_i, v_i) to G', for each pair $[u_i, v_i]$ in E^+, so that edge (u_i, v_i) crosses edge (z_i, w_i) and does not cross any other edge of G. We say that G is a *kite-triangulation* and that G' is its *underlying triangulation*.

Figure 14 shows a kite-triangulation. We get the following:

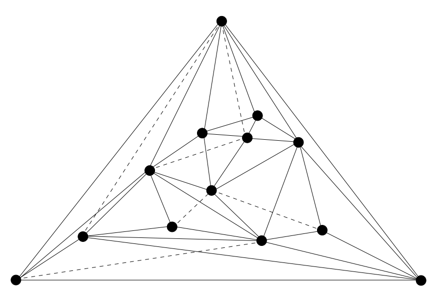


Figure 14: A kite-triangulation G. Solid lines represent the edges of the underlying triangulation G' of G. Dashed lines represent edges between pairs of opposite vertices.

Theorem 8 *Every kite-triangulation admits a RAC drawing with curve complexity one.*

Proof: Consider any kite-triangulation G and its underlying triangulation G'. Remove from G' all the edges (z_i, w_i), for $i = 1, \ldots, k$, obtaining a new planar graph G''. Since, by definition, no two edges (z_i, w_i) and (z_j, w_j), with $1 \leq i, j \leq k$ and $i \neq j$, are adjacent to the same face of G', all the faces of G'' contain at most four vertices.

Construct any straight-line planar drawing Γ'' of G''. We show how to insert in Γ'' edges (u_i, v_i) and (z_i, w_i), for each $i = 1, \ldots, k$, in order to obtain a RAC drawing Γ of G. We consider two cases, depending on whether face (u_i, w_i, v_i, z_i) is strictly convex in Γ'' or not.

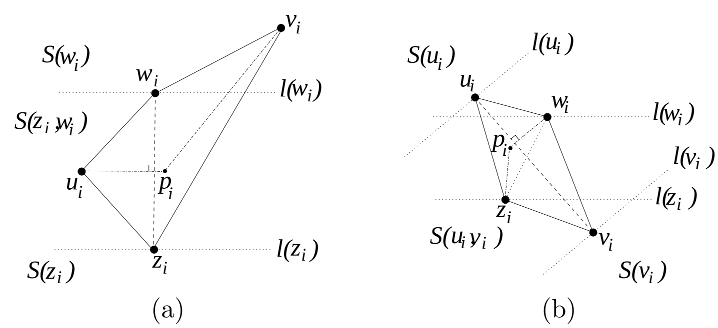


Figure 15: Drawing (u_i, v_i) and (z_i, w_i) inside (u_i, w_i, v_i, z_i), if (u_i, w_i, v_i, z_i) is strictly convex. (a) u_i lies inside $S(z_i, w_i)$; (b) both u_i and v_i lie outside $S(z_i, w_i)$.

Suppose that (u_i, w_i, v_i, z_i) is strictly convex in Γ''. Consider the straight-line segment $\overline{z_i w_i}$ and consider the lines $l(z_i)$ and $l(w_i)$ orthogonal to $\overline{z_i w_i}$ and passing through z_i and through w_i, respectively. Further, consider the following three regions of the plane: The closed half-plane $S(z_i)$ delimited by $l(z_i)$ and not containing w_i, the closed half-plane $S(w_i)$ delimited by $l(w_i)$ and not containing z_i, and the open strip $S(z_i, w_i)$ delimited by $l(z_i)$ and $l(w_i)$. If at least one out of u_i and v_i, say u_i, lies inside $S(z_i, w_i)$ (see Fig. 15(a)), then draw edge (z_i, w_i) as a straight-line segment $\overline{z_i w_i}$. Draw a straight-line segment $\overline{u_i p_i}$ starting at u_i, orthogonally crossing (z_i, w_i), and ending at a point p_i arbitrarily close to (z_i, w_i). Complete a drawing of (u_i, v_i) by drawing the straight-line segment $\overline{p_i v_i}$. If both u_i and v_i lie outside $S(z_i, w_i)$ (see Fig. 15(b)), by the strict convexity of (u_i, w_i, v_i, z_i), u_i and v_i lie one in $S(z_i)$ and one in $S(w_i)$ and segment $\overline{u_i v_i}$ intersects segment $\overline{z_i w_i}$. Hence, the open strip $S(u_i, v_i)$ delimited by the lines $l(u_i)$ and $l(v_i)$ orthogonal to $\overline{u_i v_i}$ and passing through u_i and through v_i, respectively, contains both w_i and z_i. Then, draw edge (u_i, v_i) as a straight-line segment $\overline{u_i v_i}$; draw a straight-line segment $\overline{w_i p_i}$ starting at w_i, orthogonally crossing (u_i, v_i), and ending at a point p_i arbitrarily close to (u_i, v_i); finally, complete a drawing of (w_i, z_i) by drawing the straight-line segment $\overline{p_i z_i}$.

Suppose that (u_i, w_i, v_i, z_i) is not strictly convex (see Fig. 16); more precisely,

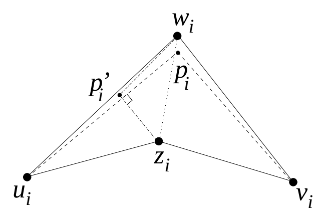


Figure 16: Drawing (u_i, v_i) and (z_i, w_i) inside (u_i, w_i, v_i, z_i), if (u_i, w_i, v_i, z_i) is strictly convex.

suppose that angle $\widehat{u_i z_i v_i} \geq 180°$, the cases in which the angle greater than or equal to 180° is incident to another vertex being analogous. Segment $\overline{z_i w_i}$ splits (u_i, w_i, v_i, z_i) into two triangles (u_i, z_i, w_i) and (v_i, z_i, w_i). Since $\widehat{u_i z_i v_i} \geq 180°$, $\widehat{u_i z_i w_i} \geq 90°$ or $\widehat{w_i z_i v_i} \geq 90°$. Suppose that $\widehat{u_i z_i w_i} \geq 90°$, the other case being analogous. Consider a point p_i inside (u_i, w_i, v_i, z_i), arbitrarily close to w_i. Draw edge (u_i, v_i) as a polygonal line composed of segments $\overline{u_i p_i}$ and $\overline{p_i v_i}$. Since $\widehat{u_i z_i w_i} \geq 90°$, the line through z_i orthogonally crossing the line through u_i and w_i crosses segment $\overline{u_i w_i}$ in an interior point. Hence, if p_i is sufficiently close to w_i, a straight-line segment $\overline{z_i p_i'}$ can be drawn starting at z_i, orthogonally crossing segment $\overline{u_i p_i}$, and ending at a point p_i' arbitrarily close to $\overline{u_i p_i}$. Complete a drawing of (w_i, z_i) by drawing the straight-line segment $\overline{p_i' w_i}$. □

Theorem 9 *There exist kite-triangulations that do not admit any straight-line RAC drawing.*

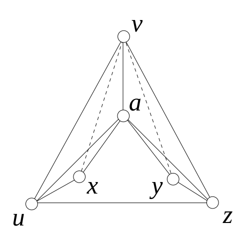


Figure 17: An embedded graph that is a subgraph of infinitely many kite-triangulations with curve complexity one in any RAC drawing.

Proof: Consider an embedded planar graph H defined as follows. Graph H has external face (u, v, z). Let a be a vertex of H creating faces (u, v, a) and (a, v, z). Let x and y be vertices of H creating faces (u, a, x) and (y, a, z), respectively, in such a way that $[v, x]$ is a pair of opposite vertices with respect to edge (a, u) and that $[v, y]$ is a pair of opposite vertices with respect to edge (a, z). See Fig. 17.

Consider any kite triangulation G containing H as a subgraph and containing edges (v, x) and (v, y), respectively crossing (a, u) and (a, z). Consider any RAC drawing Γ of G. Consider the triangle $\triangle$ representing (a, u, z). Vertex v, which lies outside $\triangle$, is connected to vertices x and y, which lie inside $\triangle$. Hence, by Property 2, Γ can not be a straight-line RAC-drawing of G. □

Planar graphs are a proper subset of straight-line RAC drawable graphs. However, while straight-line planar drawings can always be realized on a grid of quadratic size (see, e.g., [4, 22]), straight-line RAC drawings may require larger area, as shown in the following.

Theorem 10 *There exists an n-vertex kite-triangulation that requires $\Omega(n^3)$ area in any straight-line grid RAC drawing.*

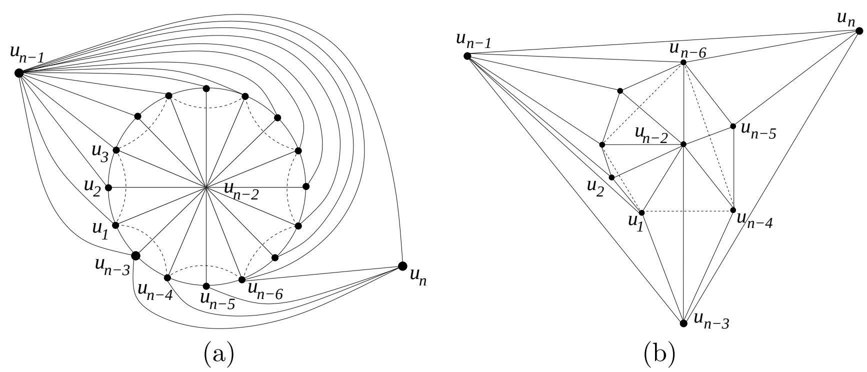


Figure 18: (a) A kite triangulation G requiring $\Omega(n^3)$ area in any straight-line grid RAC drawing. (b) A straight-line RAC drawing of G.

Proof: Consider a triangulation G' defined as follows (see Fig. 18(a)). Let $C = (u_1, u_2, \ldots, u_{n-4}, u_{n-3})$ be a simple cycle, for some odd integer n. Insert a vertex u_{n-2} inside C and connect it to u_i, with $i = 1, 2, \ldots, n-3$. Insert two vertices u_{n-1} and u_n outside C. Connect u_{n-1} to u_i, with $i = 1, 2, \ldots, n-6$, and connect u_{n-1} to u_{n-3}; connect u_n to u_{n-6}, u_{n-5}, u_{n-4}, u_{n-3}, and u_{n-1}. Let (u_{n-3}, u_{n-1}, u_n) be the external face of G'. Let G be the kite-triangulation obtained from G' by adding edges (u_i, u_{i+2}), for $i = 1, 3, 5, \ldots, n-6$, and edge (u_1, u_{n-4}), so that (u_i, u_{i+2}) crosses edge (u_{i+1}, u_{n-2}) of G', and so that (u_1, u_{n-4}) crosses edge (u_{n-3}, u_{n-2}) of G'.

In the following we prove that, in any straight-line RAC drawing of G, cycle $C' = (u_1, u_3, \ldots, u_{n-6}, u_{n-4}, u_1)$ is a strictly-convex polygon. This claim, together with the observation that G admits a straight-line RAC drawing (see Fig. 18(b)), clearly implies the theorem, since any strictly-convex polygon needs cubic area if its vertices have to be placed on a grid (see, e.g., [2]).

Suppose, for a contradiction, that there exists a straight-line RAC drawing Γ of G with an angle $u_i\widehat{u_{i+2}}u_{i+4} \geq 180°$ inside C'. Then, any two segments orthogonally crossing $\overline{u_i u_{i+2}}$ and $\overline{u_{i+2} u_{i+4}}$, respectively, meet at a point outside C', possibly at infinity, while they should meet at u_{n-2}, which is inside C'. Thus, either $\overline{u_{n-2} u_{i+1}}$ is not orthogonal to $\overline{u_i u_{i+2}}$ or $\overline{u_{n-2} u_{i+3}}$ is not orthogonal to $\overline{u_{i+2} u_{i+4}}$, hence contradicting the assumption that Γ is a RAC drawing. □

6 Conclusions and Open Problems

When a graph G does not admit any planar drawing in some desired drawing convention, requiring that all crossings form right angles can be considered as an alternative solution for the readability of a drawing of G.

In this direction, this paper has shown negative results for directed graphs that must be drawn upward with straight-line edges, and positive results for undirected graphs that must be drawn with edges bending once or twice.

We now list some open problems that are related to the results of this paper.

While recognizing upward planar digraphs is NP-hard, a characterization is known [6] stating that a digraph is upward planar if and only if it is a subgraph of a planar *st*-digraph. As we proved that recognizing straight-line upward RAC-drawable digraphs is also NP-hard, the following problem naturally arises.

Problem 1 *Is it possible to characterize digraphs admitting straight-line upward RAC drawings?*

We have proved the existence of infinitely many planar acyclic digraphs not admitting any straight-line upward RAC drawing. However, we are not aware of planar acyclic digraphs requiring more than one bend on some edges.

Problem 2 *Does every planar acyclic digraph admit an upward RAC drawing with curve complexity one (with curve complexity two)?*

There exist outerplanar digraphs that are not upward planar and that admit upward straight-line RAC drawings [17]. Studying the upward RAC drawability of outerplanar digraphs seems to be interesting.

Problem 3 *Does every outerplanar acyclic digraph admit a straight-line upward RAC drawing? What is the time complexity of deciding whether an outerplanar digraph admits a straight-line upward RAC drawing?*

Turning our attention to undirected graphs, we have shown that graphs with degree three and six admit RAC drawings with curve complexity one and two, respectively. The following is, however, still open.

Problem 4 *What are the exact bounds for the curve complexity of RAC drawings of bounded-degree graphs?*

While for directed graphs deciding upward straight-line RAC drawability is a difficult problem, the time complexity of deciding whether an undirected graph admits a straight-line RAC drawing is not yet known, and constitutes, in our opinion, the main algorithmic challenge in the area.

Problem 5 *What is the time complexity of deciding whether a graph admits a straight-line RAC drawing?*

We have shown that there exist planar graphs that require quadratic area in any straight-line RAC drawing. Of course such a bound is tight for planar graphs, as planar straight-line drawings can be constructed in quadratic area [4, 22]. However, the following two problems are worth studying:

Problem 6 *Does every planar graph admit a RAC drawing with curve complexity one or two in sub-quadratic area?*

Problem 7 *What is the area requirement of straight-line RAC drawings of straight-line RAC drawable graphs?*

Related to the last two problems, we remark that a quadratic-area lower bound for RAC drawings (possibly with bends) of general graphs has been proved by Di Giacomo *et al.* [8], and that Theorem 10 provides a cubic-area lower bound for straight-line RAC drawings of straight-line RAC drawable embedded graphs.

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