Genus Distributions of Cubic Outerplanar Graphs

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Abstract

We present a quadratic-time algorithm for computing the *genus distribution* of any 3-regular *outerplanar graph*. Although recursions and some formulas for genus distributions have previously been calculated for bouquets and for various kinds of ladders and other special families of graphs, cubic outerplanar graphs now emerge as the most general family of graphs whose genus distributions are known to be computable in polynomial time. The key algorithmic features are the syntheses of the given outerplanar graph by a sequence of *edge-amalgamations* of some of its subgraphs, in the order corresponding to the *post-order traversal* of a plane tree that we call the *inner tree*, and the coordination of that synthesis with *just-in-time* root-splitting.

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1   Introduction

Counting the embeddings of interesting graphs in designated orientable and non-orientable surfaces is a pursuit with considerable on-going activity. Various papers derive formulas with exact numbers, recursions that are useful in constructing tables, or lower bounds. In this paper, we inventory the orientable embeddings of *3-regular outerplanar graphs* according to genus, by designing a tree-based recursion algorithm. This involves new methods beyond those used for linear-like families. It leads immediately to a genus distribution algorithm for all 3-regular outerplanar graphs.

An ***outerplanar graph*** is a graph G that has an embedding in the sphere S_0, such that there is a face f_∞ whose boundary-walk contains every vertex of G. The graph can be drawn in the plane so that the face f_∞ contains the point at infinity. Then f_∞ is called the ***exterior face***, and the fixed embedding represented is called an ***outerplane embedding***.

TERMINOLOGY   A ***graph*** is taken to be connected and an ***embedding*** to be cellular (i.e., the interior of each face is homeomorphic to an open disk), unless it is explicitly declared or evident from context that something else is intended. It may have multiple edges and/or self-loops. The words ***degree*** and ***valence*** are used interchangeably.

ABBREVIATION   We abbreviate "face-boundary walk" as ***fb-walk***.

Table 1: Table of Notations

$deg(v)$	degree of vertex v
$bd(f)$	boundary of face f
S_i	orientable surface of genus i
S_G	surface in which graph G has some given embedding
$\gamma(S)$	genus of orientable surface S
$\gamma_{\min}(G)$	minimum genus of graph G
$\gamma_{\max}(G)$	maximum genus of graph G
$g_i(G)$	number of embeddings of graph G in surface S_i

The sequence $\{g_i(G) \mid i \geq 0\}$ is called the ***genus distribution*** of the graph G. We recall that two ***equivalent orientable embeddings*** of a graph G are embeddings that have the same rotation at every vertex of G. We recall also that for any (connected) graph G, the ***cycle rank*** $\beta(G)$ equals the number of edges in the edge-complement of a spanning tree. In §6 we derive an algorithm for calculating the genus distribution of any 2-connected 3-regular outerplanar graph.

Prior and contemporaneous work

Counting the embeddings of a given graph according to the genus or crosscap numbers of the surfaces is an enumerative problem first proposed by [14]. The earliest important successes, [8] and [16], were with classes of graphs that could

be defined by recursive application of a single topological operation. Subsequent results of this kind appear in [28], [25], [26], [36], [27], [5], [41], [38], and [42]. In [33], [34], and [35], Stahl made several significant theoretical observations, including the resemblance of some genus distributions to Stirling numbers of the first kind and the characterization of various graph families with known genus distributions as "linear families".

Some previous papers concerned with the related pursuit of enumerating the embeddings of a graph in a minimum-genus surface are [4], [24], [9], and [10].

There is extensive complementary work about counting maps on a designated surface by [20], [21], [22], [6], and many others. We call attention to [39], [40], [19], and [7], all of which are concerned with counting maps whose 1-skeletons are 3-regular Hamiltonian graphs such that the Hamilton cycle bounds a face. The name *chord diagrams* has been used for such graphs, which are a generalization of outerplanar graphs.

Results contained in this paper and in the following sequence of related papers were obtained within a year or so of each other: [15], [11], [12], [13], [30], [23], [32], and [31]. We call special attention to [31], which derives a quadratic-time algorithm for the genus distributions of 4-regular outerplanar graphs.

Distinguishing features of this paper

In previous papers on embedding distributions, all *roots* are static; that is, they are assigned at the outset of a computation. In general, the number of *partials* required in a *partitioned genus distribution* increases exponentially with the number of roots, and the number of associated *productions* is at least the square of that number of partials. A distinguishing feature of this paper is the introduction of *dynamic assignment of roots*, by the new technique called *root-splitting*, which is critical to the success of the algorithm. No graph occurring in the calculation ever has more than two roots, but some graphs with one root are given an additional root in precisely the necessary place at precisely the necessary time that permits continuation of the process of iterated *amalgamation*.

Another distinguishing feature of this paper is the importance of the class of graphs whose genus distributions are its objective. Outerplanar graphs occur in many graph-theoretic contexts within chromatic graph theory, topological graph theory, and elsewhere. Moreover, the 2-connected outerplanar graphs generalize to Hamiltonian planar graphs and, beyond the plane, to graphs that are 1-skeletons of chord diagrams in any surface.

Calculating the minimum genus of a graph is known to be NP-hard [37], and calculating its genus distribution is clearly at least that hard. However, knowing that the outerplanar graphs have treewidth at most 2 (see [3]) suggests that their genus distributions might be computationally accessible, since there are instances (esp. [29]) in which bounding the tree-width makes an otherwise NP-hard graph-theoretic problem solvable in polynomial time.

Organization of this paper

§2 provides some definitions that are useful in topological graph theory. §3 defines a tree in the dual of an outerplane embedding that will be used to provide an order for the operations of the algorithm by which the genus distribution is to be calculated. §4 introduces the concept of partitioning a genus distribution and the concept of productions. §5 offers an intuitive characterization of the structure of 2-connected cubic outerplanar graphs. §6 presents the algorithm for the 2-connected case. §7 analyzes the computational complexity and extends the algorithm to the 1-connected case. §8 offers some conclusions.

2   Some Definitions for Genus Distributions

This paper uses terminology and notations from topological graph theory that are consistent with [17] and [2]. We use some topological results from [30], which analyzes the genus distribution of edge-amalgamations. This paper is predominantly self-contained, but prior reading of [30] may be helpful.

An **edge-end** is a small, connected region of an edge that contains an endpoint of the edge and lies to one side of the midpoint. Thus, every edge has two edge-ends, even if it is a self-loop (i.e., with only one endpoint).

A **rotation** at a vertex is an assignment of a cyclic ordering to the edge-ends incident at that vertex. A **rotation system** for a graph is an assignment of a rotation at every vertex. There is a bijective correspondence between the set of rotation systems for a graph and the topological equivalence classes of its cellular embeddings in orientable surfaces.

The general context for any investigation of genus distribution is that the number of embeddings of a graph G is

$$\prod_{v \in V(G)} ((deg(v) - 1)!)$$

and that calculating genus distributions is NP-hard. Thus, the emphasis is on deriving genus distributions for interesting families of graphs.

Any edge in a graph may be designated as an **edge-root**. A graph with one or more edge-roots is called an **edge-rooted graph**. This paper is primarily concerned with graphs having two root edges, which are called **double-edge-rooted graphs**. A graph G with a single edge-root d may be denoted by (G, d), and by (G, d, e) if it has a second edge-root e.

3   Inner Tree for a Cubic Outerplanar Graph

A **plane tree** is a rooted tree such that at each vertex, there is a linear ordering of the children. This corresponds to a planar drawing of the tree, such that in each generation of descendents, the children occur left-to-right in the prescribed

linear order. Alternatively, a plane tree can be specified by a pair $\langle T, \rho \rangle$ consisting of a tree T and a rotation system ρ for that tree, plus the designation of a root vertex v and a principal descendant of the root v.

We observe that every 2-connected cubic outerplane embedding $G \to \mathbb{R}^2$ can be obtained by adding non-intersecting inner chords to a cycle in the plane. The Jordan curve theorem implies that the dual of a 2-connected cubic outerplane embedding is obtainable by joining each vertex of a *tree* (lying inside $bd(f_\infty)$) one or more times to the vertex in the exterior region. More precisely, the number of times a dual vertex inside $bd(f_\infty)$ is joined to the vertex in the exterior region equals the number of edges in which its corresponding primal region meets the exterior region f_∞.

By selecting an arbitrary vertex of the tree as its root and selecting an arbitrary child of the root as its leftmost child, we make the tree into a plane tree, which we call an ***inner tree*** of the outerplane embedding of G. We recall (e.g., from [1] or [18]) that the ***post-order*** for a plane tree is obtained from a traversal of the fb-walk for its only face, starting with the edge from the root to its leftmost child. Figure 1 shows the post-order for an inner tree of an outerplane embedding. To construct the post-order, each vertex is enqueued the *last time* that it occurs along this walk. Alternatively, one might traverse the fb-walk in the opposite direction and push a vertex onto a stack the *first time* that it occurs.
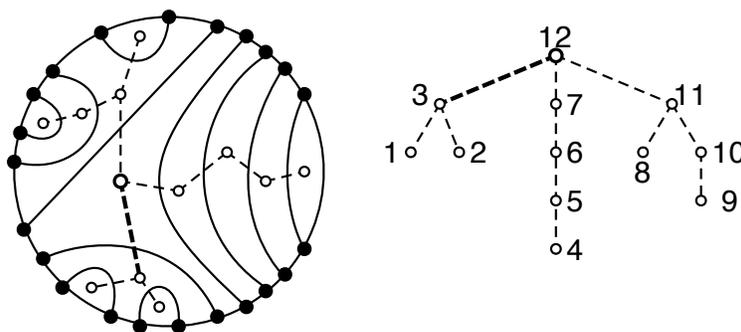
Figure 1: Outerplanar graph (left); post-order for its inner tree (right).

In §3 and §4, we restrict our attention to calculating the genus distribution of 2-connected cubic outerplanar graphs. The result of subdividing one edge of each of two disjoint graphs and then running a new edge between the two new vertices is a form of ***bar-amalgamation***, terminology introduced by [14]. In §7, we use the bar-amalgamation operation to extend our algorithm from the 2-connected case to all cubic outerplanar graphs.

4   Partials and Productions

The ***edge-amalgamation*** of a pair of edge-rooted graphs (G, c) and (H, d) is the graph obtained from their disjoint union by merging the root-edges c and d. We use an asterisk to denote the operation:

$$(G, c) * (H, d) \;=\; X$$

Of course, there are two ways to merge edges c and d, depending on which end of d is matched to which end of c. In what follows, it is assumed that an edge-amalgamation is only one of these two ways, not both. We further assume throughout this paper, as in [30], that root-edges have two 2-valent endpoints. It follows that there are four rotation systems for the graph $X = (G, c) * (H, d)$ that are consistent with a pair of prescribed rotation systems for (G, c) and (H, d). Figure 2 illustrates an edge-amalgamation and the four consistent rotation systems for the resulting graph.

(G,d)   (H,e)
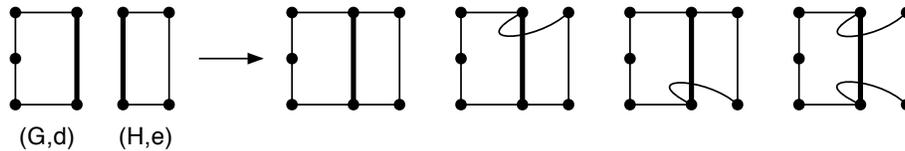
Figure 2: Edge-amalgamation and the four resultant rotation systems.

The distribution of genera for this set of four embeddings depends only on the genus $\gamma(S_G)$, the genus $\gamma(S_H)$, and the respective numbers of faces in which the two root-edges c and d lie. Accordingly, we partition the embeddings of a single-edge-rooted graph (G, c) in a surface of genus i into the subset of ***type-d_i embeddings***, in which root-edge c lies on two distinct fb-walks, and the subset of ***type-s_i embeddings***, in which root-edge c occurs twice on the same fb-walk. Moreover, we define

$$
\begin{aligned}
d_i(G, c) &= \text{the number of embeddings of type-}d_i\text{, and} \\
s_i(G, c) &= \text{the number of embeddings of type-}s_i.
\end{aligned}
$$

Thus,

$$g_i(G, c) \;=\; d_i(G, c) + s_i(G, c)$$

The numbers $d_i(G, c)$ and $s_i(G, c)$ are called ***single-root partials***. The sequences $\{d_i(G, c) \mid i \geq 0\}$ and $\{s_i(G, c) \mid i \geq 0\}$ are called ***(single-root) partial genus distributions***.

NOTATION   We may simply write d_i and s_i, when it is clear from context to which graph they apply.

REMARK   More generally, with a root-edge whose endpoints may have higher valence, there would be more partials, corresponding to a larger number of possible configurations of fb-walks at the root.

A ***production for an edge-amalgamation***

$$(G, c) * (H, d) \ = \ X$$

of single-edge-rooted graphs is a rule of the form

$$p_i(G, c) * q_j(H, d) \longrightarrow \alpha_{i+j}\, g_{i+j}(X) + \alpha_{i+j+1}\, g_{i+j+1}(X)$$

where, p_i and q_j are partials, and where α_{i+j} and α_{i+j+1} are integers. It means that amalgamation of a type-p_i embedding of graph G and a type-q_j embedding of graph H induces a set of α_{i+j} genus-$(i + j)$ embeddings of X and α_{i+j+1} genus-$(i + j + 1)$ embeddings of X. We often write such a rule in the form

$$p_i * q_j \ \longrightarrow \ \alpha_{i+j}\, g_{i+j} \ + \ \alpha_{i+j+1}\, g_{i+j+1}$$

A double-edge-rooted graph (H, d, e) has many more partials than a single-edge-rooted graph. The art of calculating genus distributions for recursively defined families involves careful selection of the building blocks, so as to avoid their having an unwieldy number of non-zero-valued double-root partials. The two non-zero-valued double-root partials we need here, where our building blocks are cycle graphs — and thus both endpoints of both root-edges d and e are 2-valent — are as follows:

- The value of the ***double-root partial*** $dd''_i(H, d, e)$ is the number of embeddings of graph H in the surface S_i such that root-edge d lies on two distinct fb-walks, and such that there is an occurrence of root-edge e on each of these fb-walks.

- The value of the ***double-root partial*** $ss^1_i(H, d, e)$ is the number of embeddings of graph H in the surface S_i such that both occurrences of root-edge d lie on the same fb-walk, and such that when that fb-walk is broken into two *strands* by deleting the occurrences of edge d, one of these strands contains both occurrences of root-edge e.

A ***production for an edge-amalgamation of a single-edge-rooted graph to a double-edge-rooted graph***

$$(G, c) * (H, d, e) \ = \ (X, e)$$

such that edges c and d are merged and e becomes the root of the resulting graph is a rule

$$
\begin{aligned}
p_i(G, c) * q_j(H, d, e) \ \longrightarrow \ & \alpha_{i+j}\, d_{i+j}(X, e) \ + \ \alpha_{i+j+1}\, d_{i+j+1}(X, e) \\
& + \beta_{i+j}\, s_{i+j}(X, e) \ + \ \beta_{i+j+1}\, s_{i+j+1}(X, e)
\end{aligned}
$$

where p_i and q_j are partials, and where α_{i+j}, α_{i+j+1}, β_{i+j} and β_{i+j+1} are integers. It means that the amalgamation of a type-p_i embedding of graph G with a type-q_j embedding of graph H induces a set of α_{i+j} type-d_{i+j} embeddings, α_{i+j+1} type-d_{i+j+1} embeddings, β_{i+j} type-s_{i+j} embeddings, and β_{i+j+1} type-s_{i+j+1} embeddings of X. We often write such a rule in the form

$$p_i * q_j \ \longrightarrow \ \alpha_{i+j}\, d_{i+j} \ + \ \alpha_{i+j+1}\, d_{i+j+1} \ + \ \beta_{i+j}\, s_{i+j} \ + \ \beta_{i+j+1}\, s_{i+j+1}$$

REMARK If two graphs are pasted on root-edges whose endpoints have valence larger than 2, there may be additional terms on the right side of the production.

Theorem 4.1 *Let $(X, f) = (G, d) * (H, e, f)$ be an edge-amalgamation where the endpoints of root-edge d are 2-valent in G and the endpoints of root-edges e and f are 2-valent in H. Then the genus distribution of (X, f) conforms to the following productions:*

$$d_i(G, d) * dd_j''(H, e, f) \longrightarrow 2d_{i+j}(X, f) + 2s_{i+j+1}(X, f) \tag{1}$$
$$s_i(G, d) * dd_j''(H, e, f) \longrightarrow 4d_{i+j}(X, f) \tag{2}$$
$$d_i(G, d) * ss_j^1(H, e, f) \longrightarrow 4s_{i+j}(X, f) \tag{3}$$
$$s_i(G, d) * ss_j^1(H, e, f) \longrightarrow 4s_{i+j}(X, f) \tag{4}$$

Proof: See Theorems 3.1 and 3.2 of [30]. □

In the next subsection, we apply Theorem 4.1 to the example of closed-end ladders, with some attention devoted to the time required for recursive calculation of their genus distributions.

Calculating the genus distribution of an edge-rooted ladder L_n

Example 4.1 We recall from [8] that the **closed-end ladder** L_n is defined to be the result of doubling both the leftmost and rightmost edges of the cartesian product of the path-graph P_n with K_2. Thus, there are n interior rungs and two outer rungs. See Figure 3.

For convenience, we define L_0 to be the cycle graph C_4 with roots on two non-adjacent edges, i.e., a ladder with no interior rungs. On the other ladders, we trisect one of the edges at the end of the ladder, and we regard the resulting middle segment as the root-edge.
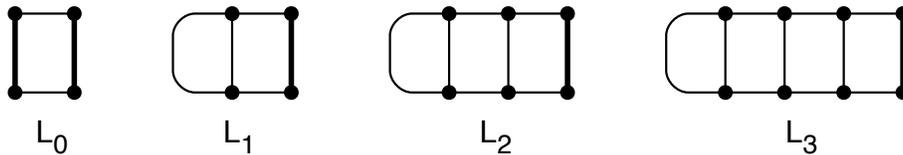
Figure 3: Some closed-end ladders with root-edges.

A basis for an inductive calculation is that the ladder L_0 has the single-root partitioned genus distribution

$$d_0(L_0, w) = 1$$

and the double-root partitioned genus distribution

$$dd_0''(L_0, x, y) = 1$$

The single-rooted ladder L_j is representable as the edge-amalgamation of a copy of L_{j-1} to a double-rooted copy of L_0. For instance, using Production (1), we can calculate the single-root partitioned genus distribution of the ladder L_1:

$$d_0(L_1, x) = 2 \quad s_1(L_1, x) = 2$$

Next, using Production (1) and Production (2), we can calculate the single-root partitioned genus distribution of the ladder L_2:

$$d_0(L_2, x) \;=\; 4 \quad d_1(L_2, x) \;=\; 8 \quad s_1(L_2, x) \;=\; 4$$

To obtain the single-root partitioned genus distribution of the ladder L_j from the single-root distribution for L_{j-1} and the double-root distribution for L_0, Productions (1) and (2) are sufficient. We continue applying these rules until we obtain a partitioned genus distribution for L_n.

Proposition 4.2 *The number of multiplications needed to calculate either the single-edge-root or the double-edge-root partitioned genus distribution of the ladder L_n is in $O(n^2)$.*

Proof: Since the cycle rank of the ladder L_{k-1} is k, we have

$$\gamma_{\max}(L_{k-1}) \;\leq\; \left\lfloor \frac{k}{2} \right\rfloor$$

Since there are only two single-edge-root partials, i.e., d_i and s_i, for each genus g_i, it follows that the number of non-zero single-edge-root partials of the ladder L_{k-1} is at most $2\lfloor \frac{k}{2} \rfloor$, and thus, at most k. Since L_0 has only one non-zero partial, it follows that the number of combinations of a partial of L_{n-1} with a partial of L_0 needed to calculate the partials of L_k is at most k. Since each production for a single-edge-root partial has at most four terms on its right side, the total number of multiplications required is at most $4k$. Accordingly, in calculating the single-edge-root partial genus distribution of L_n, starting from the partial genus distribution of L_0, the number of multiplications is at most

$$\sum_{k=1}^{n} 4k \;=\; 4\binom{n+1}{2} \;\leq\; 4n^2$$

There are 16 double-edge-root partials given in [30] (see Table 5). Thus, the number of non-zero double-edge-root partials of the ladder L_{k-1} is at most $8k$. By a similar analysis, it follows that in calculating the double-edge-root partial genus distribution of L_n, starting from the partial genus distribution of L_0, the number of multiplications is at most

$$\sum_{k=1}^{n} 32k \;=\; 32\binom{n+1}{2} \;\leq\; 32n^2$$

$\square$

REMARK A closed formula for the genus distribution of unrooted closed-end ladders is given by [8]. However, since the present objective is obtaining genus distributions for a much more extensive family of graphs, we regard closed formulas for various rooted varieties of ladders as peripheral.

5   Characterizing Cubic Outerplanar Graphs

In this section, we define a special kind of outerplanar graph, called a *star-ladder graph*, and we show that every 2-connected cubic outerplanar graph can be regarded as a tree of star-ladder graphs. This is helpful in understanding the algorithm that will be used to calculate the genus distribution.

Star-ladder graphs

For an r-tuple of non-negative integers $U = (k_1, k_2, \ldots, k_r)$ the **star-ladder** with **signature** U is the graph SL_U obtained from the cycle graph C_{2r}, with consecutive edges labeled $e_1, e_2, \ldots, e_{2r}$ as follows:

1. Subdivide one endrung of each of the closed-end ladders

$$L_{k_1}, L_{k_2}, \ldots, L_{k_r}$$

   into three parts and take the middle third as the root-edge.

2. For $i = 1, \ldots, r$, amalgamate L_{k_i} across its newly created root edge to edge e_{2i} of the cycle C_{2r}.

We regard a cycle graph as a degenerate star-ladder corresponding to the empty tuple.

Each of the closed-end ladders is regarded as a **ray** of the star and the cycle as the **hub**. On each ray, one of the end-rungs farthest from hub is regarded as the **tip** of the ray. The star-ladder $SL_{(3,2,1)}$ is shown in Figure 4.
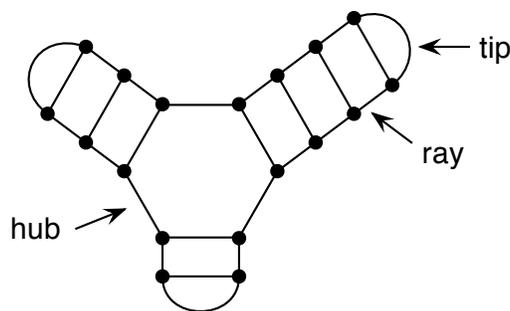
Figure 4: The star-ladder $SL_{(3,2,1)}$.

REMARK   We observe that two 3-regular star-ladders are isomorphic if the signature of one can be obtained by a rotation and/or a reversal of the signature of the other. However, placing a root-edge at the tip of one of the rays of a star-ladder may not be equivalent to placing it at the tip of another ray. Similarly, two different locations on the hub may be inequivalent.

Trees of star-ladders

We seek to characterize a cubic outerplanar graph in an intuitive manner. Toward that objective, we define a family $\mathcal{T}L$ of graphs called **trees of star-ladders** recursively:

- Every graph that is homeomorphic to a star-ladder graph is in $\mathcal{T}L$.

- Let (G_1, e_1) and (G_2, e_2) be in $\mathcal{T}L$, with each of the edge-roots e_1 and e_2 either at the tip of a ray or on a hub, and with both endpoints of both edges e_1 and e_2 2-valent. (Obtaining such an edge for amalgamating can be achieved by trisecting an edge and using the middle third.) Then the edge-amalgamated graph $(G_1, e_1) * (G_2, e_2)$ is in $\mathcal{T}L$.

Theorem 5.1 *Every 2-connected cubic outerplanar graph is homeomorphic to a graph in the family $\mathcal{T}L$.*

Proof: Let G be a 2-connected cubic outerplanar graph. If there is only one chord, then G is isomorphic to the star-ladder $SL_{(0)}$ and, therefore, in $\mathcal{T}L$. As an induction hypothesis, assume that this theorem is true when there are up to $m-1$ chords. Next, suppose that the graph G is 2-connected cubic outerplanar with m chords, and let e be any chord. As illustrated in Figure 5, it is possible to represent the graph G as an edge-amalgamation $G = (G_1, e_1) * (G_2, e_2)$ of two graphs, such that edge e corresponds to the image of e_1 and e_2.
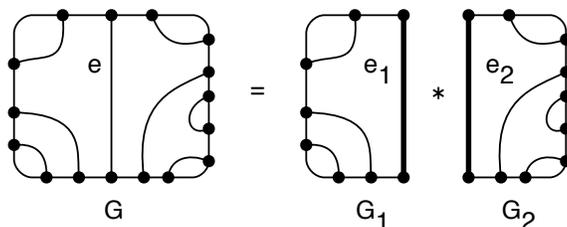
Figure 5: Splitting an outerplane embedding of G on the chord e.

Splitting an outerplane embedding of G on the chord e induces outerplane embeddings of the graphs G_1 and G_2. Since G_1 and G_2 each have fewer chords than G, it follows by the induction hypothesis that they are both in $\mathcal{T}L$. Let e_1 and e_2 be the images of edge e in graphs G_1 and G_2, respectively. If each of them is either on a hub or at the tip of a ray, then the recursive construction of the generalized star-ladders implies immediately that the edge-amalgamation $G = (G_1, e_1) * (G_2, e_2)$ is in $\mathcal{T}L$.

It is helpful here to call a 4-cycle in a closed-end ladder a "hole" (i.e., suggesting a place in the ladder where one might insert a foot while climbing the ladder). If, alternatively, the edge e_i (where i is 1 or 2) lies within a side-edge of some ladder in G_i, then let H be a hole in that ladder that contains the edge e_1. We may regard the hole H as a hub and proceed. □

Theorem 5.2 *Every graph in the family $\mathcal{T}L$ is homeomorphic to a 2-connected cubic outerplanar graph.*

Proof: Let G be a graph in $\mathcal{T}L$. If no amalgamations are needed in the recursive construction of G, then G is a star-ladder, and, thus, outerplanar. Assume, by way of induction, that this theorem is true when G is constructable with $m-1$ or fewer amalgamations. Next suppose that graph G is obtained by a sequence of m amalgamations of graphs in $\mathcal{T}L$. Consider the mth amalgamation $G = (G_1, e_1) * (G_2, e_2)$ in such a sequence. By the induction hypothesis, both G_1 and G_2 are homeomorphic to 2-connected cubic outerplanar graphs. Since their respective root-edges e_1 and e_2 each lie either on a hub or at the tip of a ray, they lie on Hamiltonian cycles C_1 and C_2 that bound the outer face of respective planar embeddings of G_1 or G_2. When these two planar embeddings are amalgamated across e_1 and e_2, the edges e_1 and e_2 merge into a single edge e, and the subgraph $(C_1 \cup C_2) - e$ is a Hamiltonian cycle in the resulting planar embedding, and it bounds the outer face. Moreover, since each vertex of the graph resulting from such an amalgamation is either 2-valent or 3-valent, the graph is homeomorphic to a cubic graph. $\qquad\square$

6   Algorithm for a Cubic Outerplanar Graph

To calculate the genus distribution of a 2-connected cubic outerplanar graph, we introduce a new technique. The necessity arises in that constructing an outerplane embedding G with n chords from cycle graphs involves n edge-amalgamations. As we may observe in Figure 5, for example, sometimes there are more than two chords on the boundary of a single region, and each of them must be an active root-edge at some point in the process of iterative edge-amalgamation used to construct G. In general, the number of partials increases so rapidly with the number of root-edges on a graph, that having a graph with more than two root-edges as an amalgamand would make the number of productions required for calcution of the genus distribution formidably large. Accordingly, we now introduce a just-in-time technique that avoids having more than two root-edges on any amalgamand at any time during the process. It is designed to be used in conjunction with the post-order traversal.

<div style="text-align:center">

Splitting a single root-edge into two root-edges

</div>

Let (G, a) be an edge-rooted graph such that both endpoints of edge a are 2-valent. By **_splitting the root-edge_** a, we mean trisecting edge a and regarding the two "outer" segments as root-edges of the resulting graph. (We may call one of these outer segments a.) See Figure 6.

Theorem 6.1 *Let (G, a, b) be a double-edge-rooted graph such that both endpoints of root-edges a and b are 2-valent, and such that there is a path from edge a to edge b along which every internal vertex is 2-valent. Then for every non-negative integer i,*
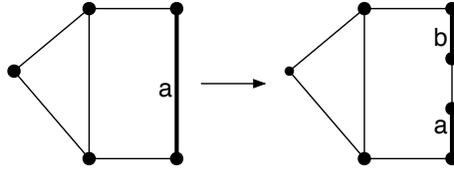
Figure 6: Splitting a root-edge a into root-edges a and b.

$$dd_i''(G, a, b) = d_i(G, a) \tag{5}$$
$$ss_i^1(G, a, b) = s_i(G, a) \tag{6}$$

Moreover, every other double-root partial of (G, a, b) is zero-valued.

Proof: The partial $d_i(G, a)$ counts embeddings in which the occurrences of root-edge a lie on separate fb-walks. We observe that, under the given premises, one occurrence of edge b lies on one of these two fb-walks, and the other occurrence of edge b lies on the other fb-walk. Thus, we have $dd_i''(G, a, b) = d_i(G, a)$, as illustrated at the left of Figure 7.

Figure 7: Transforming a single-root partial into a double-root partial.

The partial $s_i(G, a)$ counts embeddings in which both occurrences of root-edge a lie on the same fb-walk. We observe that, under the given premises, both occurrences of edge b lie on that same fb-walk and that a single strand resulting from the removal of a from that fb-walk contains both occurrences of edge b. Thus, $ss_i^1(G, a, b) = s_i(G, a)$, as illustrated at the right of Figure 7.

It follows from the definition of the single-root partials that

$$g_i(G) = d_i(G, a) + s_i(G, a)$$

It follows, further, that

$$dd_i''(G, a, b) + ss_i^1(G, a, b) = g_i(G)$$

Thus, the other double-root partials for genus i are zero-valued. This holds for every non-negative integer i. $\square$

Calculating the genus distribution of a cubic outerplanar graph

We precede the statement of the algorithm by applying its sequence of amalgamations to the cubic outerplanar graph illustrated in Figure 1. Figure 8 shows the complete set of amalgamands used to construct that graph.
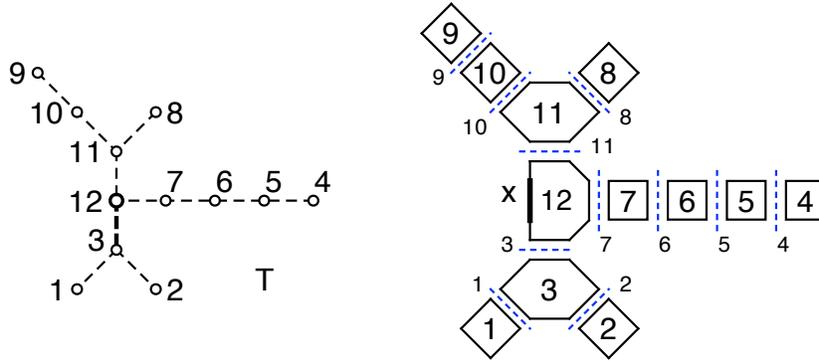
Figure 8: Order of the amalgamation steps.

Example 6.1   Let $A_1, A_2, \ldots, A_{12}$ be the boundaries of the twelve non-outer faces. The order of amalgamation is as follows:

1. Amalgamating (A_1, c_1) and (A_3, c_1, c_2) and then splitting the root-edge c_2 yields a graph we call (G_2, c_2, c_3).

2. Amalgamating (A_2, c_2) and (G_2, c_2, c_3) yields a graph we call (G_3, c_3).

3. Amalgamating (G_3, c_3) and (A_{12}, c_3, c_7) and then splitting the root-edge c_7 yields a graph we call (G_{12}, c_7, c_{11}).

4. Iteratively amalgamating $(((A_4, c_4) * (A_5, c_4, c_5)) * (A_6, c_5, c_6)) * (A_7, c_6, c_7)$ yields a graph we call (G_7, c_7).

5. Amalgamating $(G_7, c_7) * (G_{12}, c_7, c_{11})$ and then splitting the root-edge c_{11} yields a graph we call (G'_{12}, c_{11}, x).

6. Amalgamating (A_8, c_8) and (A_{11}, c_8, c_{10}) and then splitting the root-edge c_{10} yields a graph we call (G_{11}, c_{10}, c_{11}).

7. Amalgamating (A_9, c_9) and (A_{10}, c_9, c_{10}) yields a graph we call (G_{10}, c_{10}).

8. Amalgamating (G_{10}, c_{10}) and (G_{11}, c_{10}, c_{11}) yields a graph we call (G'_{11}, c_{11}).

9. Amalgamating (G'_{11}, c_{11}) and (G'_{12}, c_{11}, x) yields the graph we call (G, x).

By following these steps, we obtain the following genus distribution (see Table 2) for the 22-vertex cubic outerplanar graph of Figure 1.

REMARK   In Example 6.1, after an amalgamation $(H_i, c_i) * (H_j, c_i, c_k)$, notice that we split the surviving root-edge c_k whenever more children are to be pasted to the resulting graph before it is pasted to its parent.

Table 2: Genus distribution of the outerplanar graph of Figure 1.

genus i	0	1	2	3	4	5
g_i	2048	55296	458752	1482752	1671168	524288

Example 6.2 This smaller example involves less computation. Consider the star-ladder $(SL_{(0,1,1)}, x)$, where the root-edge x is at the tip of the ray corresponding to the entry 0 in the signature. Table 3 gives the single-root partitioned genus distribution.

Table 3: Partitioned genus distribution of the star-ladder $SL_{(0,1,1)}$.

genus i	0	1	2	3
d_i	32	320	384	0
s_i	0	32	128	128
g_i	32	352	512	128

The following algorithm synthesizes an edge-rooted graph (G, x) in the family $\mathcal{CO}$ from cycle graphs. During the sequence of edge-amalgamations, **no amalgamand ever has more than two root-edges,** so the algorithm can calculate the genus distribution at each step, using Theorem 4.1 and sometimes Theorem 6.1. **This is why we use the post-order of a plane tree.** The input to the algorithm includes a specification (by rotation system) of a plane embedding of G.

Preliminaries:
   1. Designate an inner tree T for the given outerplane embedding of G, such that the global root-edge x of G lies on the boundary of the face containing the root-vertex of T, and such that the leftmost subtree traversed in the post-order for T lies immediately counterclockwise after x.

   2. Label the vertices of the inner tree T as $v_1, v_2, \ldots, v_n$, the faces of the outerplane embedding of G as $f_1, f_2, \ldots, f_n$, and the chords relative to the designated outer cycle as $c_1, c_2, \ldots, c_{n-1}$, according to their order of occurrence in the post-order of T.

Main Loop: For $j = 1, \ldots, n-1$
   1. Perform the amalgamation corresponding to chord c_j.

   2. Use Theorem 4.1 to calculate the single-edge-root genus distribution for the resulting graph.

   3. If more children are to be pasted to the result before it is pasted to its parent, then split the surviving root-edge and use Theorem 6.1 to calculate the resulting double-edge-root genus distribution.

   4. Continue with next j

Final Step: Label the surviving root-edge x.

7   Computational Complexity of the Algorithm

We would like to proceed in a fashion similar to our derviation in Proposition 4.2 of a quadratic-time upper bound for the genus distribution of closed-end ladders. However, that previous derivation is based on having all the second amalgamands in a path-based iterative sequence isomorphic to the same graph. In this tree-based sequence of amalgamations of cycle graphs, both amalgamands may grow in size as the process progresses. Accordingly, we need a new principle to obtain the quadratic upper bound for the total computation time.

Lemma 1 *Let both (H, c) and (K, d, e) be homeomorphic to 2-connected cubic outerplanar graphs, and such that both endpoints of the three root-edges are 2-valent. Then the number of multiplications needed to calculate the genus distribution of $(H, c) * (K, d, e)$ is at most $80\, \gamma_{\max}(H)\, \gamma_{\max}(K)$, and the number of additions needed is at most $160\, \gamma_{\max}(H)\, \gamma_{\max}(K)$.*

Proof: For each $i = 0, \ldots, \gamma_{\max}(H)$, the number of single-root partials is 2 (i.e., d_i and s_i), and for each $j = 0, \ldots, \gamma_{\max}(K)$, the number of double-root partials is 10 (see §2 of [30]). Thus, the number of pairs $(p_i(H, c),\, q_j(K, d, e))$ of such partials is

$$
\begin{aligned}
& 20\,(\gamma_{\max}(H) + 1)\,(\gamma_{\max}(K) + 1) \\
\leq\ & 20\,\gamma_{\max}(H)\,\gamma_{\max}(K) \,+\, 20\,\gamma_{\max}(H) \,+\, 20\,\gamma_{\max}(K) \,+\, 20 \\
\leq\ & 80\,\gamma_{\max}(H)\,\gamma_{\max}(K)
\end{aligned}
$$

The last inequality uses the facts that $\gamma_{\max}(H) \geq 1$ and $\gamma_{\max}(K) \geq 1$. Each pair $(p_i(H, c),\, q_j(K, d, e))$ requires only one multiplication of coefficients. Moreover, each pair is subject to a single production, which yields at most two terms, by Theorems 3.1, 3.2, and 3.3 of [30]. Thus, the number of terms in all is at most

$$
160\,\gamma_{\max}(H)\,\gamma_{\max}(K)
$$

In calculating the non-zero single-root partials of the genus distribution of $(H, c) * (K, d, e)$ from all these terms, each term is added into only one of the partials. Hence, the total number of additions cannot exceed the number of terms. ☐

Theorem 7.1 *Let (G, x) be an n-vertex graph in $\mathcal{T}L$ (i.e, homeomorphic to a 2-connected cubic outerplanar graph) with root-edge x on the outer cycle, and such that both endpoints of x are 2-valent. Then the time needed to calculate its single-edge-root partitioned genus distribution is in $O(n^2)$.*

Proof: Since the number of edges of a planar graph is linear in the number of vertices, the dual graph of a planar embedding of G can be constructed in linear time, using the familiar Heffter-Edmonds algorithm (e.g., see [17]). Thus, the preliminary steps of the algorithm can be executed in linear time. We can smooth out all the 2-valent vertices except the endpoints of root-edge x; we call the resulting graph G.

In analyzing the Main Loop, we consider an inner tree whose number of vertices is $\beta = \beta(G)$, the cycle rank of G, corresponding to $\beta - 1$ amalgamation steps. The smallest possible case is when G is isomorphic to the dipole $\overline{D}_3$, which is the graph obtained by joining two vertices with three edges and then trisecting one of the edges and allowing the middle part to serve as the root x. Then $n = 4$, so we have

$$d_0(G, x) = 2 \quad \text{and} \quad s_1(G, x) = 2$$

as the only non-zero partials. When iteratively calculating the non-zero partials for the graph resulting from each amalgamation, the number of arithmetic operations required is less than a fixed multiple M of the product of the numbers of vertices in the amalgamands, by Lemma 1.

We denote the cycle graphs that bound the non-outer faces of the outerplane embedding of G and their lengths by

$$A_1, A_2, \ldots, A_\beta \qquad \text{and} \qquad k_1, k_2, \ldots, k_\beta$$

respectively. We let

$$N = \sum_{j=1}^{\beta} k_j$$

Clearly, $N \leq 3n$, since N is the length of the face-sizes – excluding the outer face, so it is less than $2|E_G|$, which equals the sum of the vertex degrees, which is exceeded by $3n$, because the maximum degree is 3. (Indeed, $N \leq 2n$.)

We denote the amalgamands of the jth amalgamation as G_{ℓ_j} and G_{r_j}. The maximum genus of each of these amalgamands is less than the sum of the lengths of the cycle graphs from which that amalgamand is formed. It follows from Lemma 1 that the number of arithmetic operations required to calculate the partitioned genus distribution of the result from the partitioned genus distributions of the amalgamands is dominated by

$$M \sum_{i:A_i \subset G_{\ell_j}} k_i \sum_{i:A_i \subset G_{r_j}} k_i$$

where M is a multiplicative constant. It follows that the total number of arithmetic operations needed to calculate the genus distribution of G (via $\beta - 1$ edge-amalgamations) is at most

$$\sum_{j=1}^{\beta-1} M \sum_{i:A_j \subset G_{\ell_j}} k_i \sum_{i:A_j \subset G_{r_j}} k_i = M \sum_{j=1}^{\beta} k_j (k_1 + k_2 + \cdots + k_{j-1})$$

(equality holds since two distinct cycle graphs are merged only once)

$$\leq M(k_1 + \cdots + k_\beta)(k_1 + \cdots + k_\beta)$$
$$= MN^2 \leq 9Mn^2$$

Thus, the computation time for the partitioned genus distribution of G is in $O(n^2)$. $\qquad \square$

It is often convenient to represent the genus distribution of a graph G by the polynomial

$$P_G(x) \,=\, g_0(G) + g_1(G)x + g_2(G)x^2 + \cdots + g_{\gamma_{\max}(G)}x^{\gamma_{\max}(G)}$$

which may be called the **genus distribution polynomial**.

Corollary 7.2 *Let (G, x) be an n-vertex cubic outerplanar graph. Then the time needed to calculate its single-edge-root partitioned genus distribution is in $O(n^2)$.*

Proof: It is easily shown that every cubic outerplanar graph is obtainable as an iterated bar-amagamation of 2-connected cubic outerplanar graphs. Suppose that G is so obtained from 2-connected cubic outerplanar graphs G_1, G_2, ..., G_p of n_1, n_2, ..., n_q vertices, respectively. Then their respective genus distribution polynomials $P_1(x)$, $P_2(x)$, ..., $P_q(x)$ have degrees less than n_1, n_2, ..., n_q, respectively. By Theorem 5 of [14], the genus distribution of a bar-amalgamation of two graphs is a multiple (in this case, by the number 4) of the convolution of the genus distributions of those two graphs. It follows that the genus distribution polynomial of the graph G equals 4^{q-1} times the product

$$P_1(x) \cdot P_2(x) \cdot \cdots \cdot P_q(x)$$

It takes at most $n_1 n_2$ integer multiplications to obtain the polynomial product $P_1(x) \cdot P_2(x)$, which has degree less than $n_1 + n_2$. It then takes at most $(n_1 + n_2)n_3$ integer multiplications to obtain the polynomial product $P_1(x) \cdot P_2(x) \cdot P_3(x)$, which has degree less than $n_1 + n_2 + n_3$. And so on. Accordingly, the total number of integer multiplications needed to calculate the genus distribution polynomial of G is less than

$$n_1 n_2 \,+\, (n_1 + n_2)n_3 \,+\, \cdots \,+\, (n_1 + \cdots + n_{q-1})n_q$$

which is less than n^2. $\square$

8   Conclusions

We have demonstrated that the genus distribution of a cubic outerplanar graph can be obtained in $O(n^2)$-time by synthesizing that graph via iterative amalgamation of cycle graphs and applying productions with each amalgamation. This indicates the power of the advances in theoretical development provided by [15], [30], and related papers.

Whereas previous approaches to counting embeddings have been developed primarily for linear sequences of graphs, the new approach developed here is applicable to families that can be creatively characterized as tree-like, rather than linear. In going beyond linearly synthesized families, it is computationally convenient to limit the number of roots of all (connected) graphs occurring in the calculation to no more than two, lest there be a large increase in the

number of partials and in the number of productions. This requires blending new just-in-time techniques such as root-splitting into the algorithm.

The number of partials needed for a partitioned genus distribution of a rooted graph grows exponentially in the number of roots. On a hub in the calculations here, half the edges serve as roots at some time during the calculation. The new technique of root-splitting introduced here is coordinated with a post-order tree-traversal to avoid at any time having more than two roots on any graph occurring in the calculation.

Research Problems.

1. It is proved in [8] that the genus distributions of closed-end ladders are unimodal. Furthermore, there are no known examples of genus distributions that are not unimodal. Determine whether the genus distribution of every cubic outerplanar graph is unimodal.

2. We observe that a Hamiltonian cycle in a 3-regular plane graph partitions the other edges into inner chords and outer chords. This suggests the problem of developing a polynomial-time algorithm to calculate the genus distribution of any cubic planar Hamiltonian graph.

3. Another possible direction for extension of these results is to derive genus distributions for toroidal chord diagrams and for chord diagrams of higher genus.

Acknowledgements

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