Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 17, no. 3, pp. 221–244 (2013)
DOI: 10.7155/jgaa.00292

On the Page Number of Upward Planar Directed Acyclic Graphs

Fabrizio Frati[1] *Radoslav Fulek*[2] *Andres J. Ruiz-Vargas*[3]

[1]School of Information Technologies
University of Sydney, Australia
[2]Department of Applied Mathematics
Charles University, Prague, Czech Republic
[3]School of Basic Sciences
École Polytechnique Fédérale de Lausanne, Switzerland

Abstract

In this paper we study the page number of upward planar directed acyclic graphs. We prove that: (1) the page number of any n-vertex upward planar triangulation G whose every maximal 4-connected component has page number k is at most $\min\{O(k \log n), O(2^k)\}$; (2) every upward planar triangulation G with $o(\frac{n}{\log n})$ diameter has $o(n)$ page number; and (3) every upward planar triangulation has a vertex ordering with $o(n)$ page number if and only if every upward planar triangulation whose maximum degree is $O(\sqrt{n})$ does.

Submitted:	Accepted:	Final:	Published:
January 2013	March 2013	April 2013	May 2013
	Article type:	Communicated by:	
	Regular paper	G. Liotta	

Work partially supported by the Swiss National Science Foundation, grant no. 200021-125287/1, and by the ESF project 10-EuroGIGA-OP-003 "Graph Drawings and Representations". A preliminary version of this paper is to appear at GD '11 [8].

E-mail addresses: brillo@it.usyd.edu.au (Fabrizio Frati) radoslav.fulek@gmail.com (Radoslav Fulek) andres.ruizvargas@epfl.ch (Andres J. Ruiz-Vargas)

1 Introduction

Let σ be a total ordering of the vertex set V of a graph $G{=}(V,E)$. Two edges (u,v) and (w,z) in E *cross* if $u <_\sigma w <_\sigma v <_\sigma z$. A *k-page book embedding* of G is a total ordering σ of V and a partition of E into subsets $E_1, E_2, \ldots, E_k$, called *pages*, such that no two edges in the same set E_i cross. The *page number* of G is the minimum k such that G admits a k-page book embedding.

Book embeddings (first introduced by Kainen [16] and by Ollmann [20]) find applications in several contexts, such as VLSI design, fault-tolerant processing, sorting networks, and parallel matrix multiplication (see, e.g., [4, 12, 21, 22]). Henceforth, they have been widely studied from a theoretical point of view; namely, the literature is rich of combinatorial and algorithmic contributions on the page number of various classes of graphs (see, e.g., [2, 7, 9, 10, 11, 18, 19]). We remark here a famous result of Yannakakis [24] stating that any planar graph has page number at most four.

Heath *et al.* [14, 15] extended the notions of book embedding and page number to directed acyclic graphs (*DAGs* for short) in a very natural way: Given a DAG $G{=}(V,E)$, book embedding and page number of G are defined as for undirected graphs, except that the total ordering of V is now required to be a *linear extension* of the partial order of V induced by E. That is, if G contains an edge from a vertex u to a vertex v, then $u <_\sigma v$ in any feasible total ordering σ of V. The authors of [14, 15] showed that DAGs with page number equal to one can be characterized and recognized efficiently; however, they proved that, in general, determining the page number of a DAG is NP-complete.

The main problem raised by Heath *et al.* and studied in, e.g., [1, 6, 13, 14, 15], is whether every *upward planar DAG* admits a book embedding in few pages. An upward planar DAG is a DAG that admits a drawing which is simultaneously *upward*, *i.e.*, each edge is represented by a curve monotonically increasing in the y-direction, and *planar*, *i.e.*, no two edges cross. Upward planar DAGs are the natural counterpart of planar graphs in the context of directed graphs. Notice that there exist DAGs which admit a planar non-upward embedding and that require $\Omega(|V|)$ pages in any book embedding (see [13, 15] and Fig. 1). No upper bound better than the trivial $O(|V|)$ and no lower bound better than the trivial $\Omega(1)$ are known for the page number of upward planar DAGs. It is however known that *directed trees* have page number one [15], that *unicyclic DAGs* have page number two [15], and that *series-parallel DAGs* have page number two [1, 6].

In this paper we study the page number of upward planar DAGs. Before stating our results we need some background.

First, it is known that every upward planar DAG G can be augmented to an *upward planar triangulation* G' [5]. That is, edges can be added to G so that the resulting graph G' is still an upward planar DAG and every face of G' is delimited by a 3-cycle. Thus, in order to establish tight bounds on the page number of upward planar DAGs, it suffices to look at upward planar triangulations, as the page number of a subgraph G of a graph G' is at most the page number of G'. In the following, unless otherwise specified, all the considered graphs are

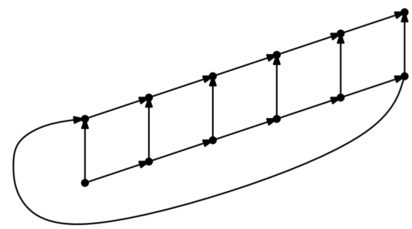


Figure 1: A DAG with page number $|V|/2$.

upward planar triangulations.

Second, consider a total ordering σ of V. A *twist* is a set of pairwise crossing edges, *i.e.*, a set $\{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$ of edges such that $u_1 <_\sigma u_2 <_\sigma \cdots <_\sigma u_k <_\sigma v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k$. It is straightforward that the page number of a graph G is lower bounded by the minimum over all vertex orderings σ of the maximum size of a twist in σ. Moreover, a function of the maximum size of a twist in a vertex ordering upper bounds the page number of an n-vertex graph G, as stated in the following two lemmata.

Lemma 1 *[3] Let σ be a vertex ordering of an n-vertex graph G. Suppose that the maximum twist of σ has size k. Then G admits a book embedding with vertex ordering σ and with $O(k \log n)$ pages.*

Lemma 2 *[17] Let σ be a vertex ordering of an n-vertex graph G. Suppose that the maximum twist of σ has size k. Then G admits a book embedding with vertex ordering σ and with $O(2^k)$ pages.*

Thus, in order to get upper bounds for the page number of a graph, it often suffices to construct vertex orderings with small maximum twist size.

In this paper we consider the relationship between the page number of an n-vertex upward planar triangulation G and three important graph parameters of G: The connectivity, the diameter, and the degree. We show the following results.

- In Sect. 3, we prove that an upward planar triangulation G admits a vertex ordering with maximum twist size $O(f(n))$ if and only if every maximal 4-connected component of G does. As a corollary, maximal upward planar 3-trees have constant page number. It is easy to prove that any n-vertex *series-parallel DAG* [1, 6] can be augmented to a maximal upward planar 3-tree with $O(n)$ vertices. Thus, our result extends the largest known class of upward planar DAGs with constant page number.

- In Sect. 4, we prove that every upward planar triangulation G has a vertex ordering whose maximum twist size is a function of the *diameter* of G, that is, of the length of the longest directed path in G. As a corollary, every upward planar triangulation whose diameter is $o(n/\log n)$ admits a book embedding in $o(n)$ pages. Such a result pairs the easy observation that upward planar triangulations with $n - o(n)$ diameter have $o(n)$ page number.

- In Sect. 5, we show that every upward planar triangulation has a vertex ordering with $o(n)$ page number if and only if every upward planar triangulation whose maximum degree is $O(\sqrt{n})$ does.

2 Definitions

A *directed graph* is a graph with direction on the edges. The *underlying graph* of a directed graph G is the undirected graph obtained from G by removing the directions on its edges. We denote by (u, v) an edge directed from a vertex u, which is called the *origin* of (u, v), to a vertex v, which is called the *destination* of (u, v); edge (u, v) is *incoming* v and *outgoing* u. A *source* (resp. *sink*) is a vertex with no incoming edge (resp. with no outgoing edge). A *directed cycle* is a directed graph whose underlying graph is a cycle and containing no source and no sink. A *directed acyclic graph* (*DAG* for short) is a directed graph containing no directed cycle. A *directed path* is a directed graph whose underlying graph is a path and containing exactly one source and one sink. The *diameter* of a directed graph is the number of vertices in its longest directed path.

A *drawing* of a directed graph is a mapping of each vertex to a point in the plane and of each edge to a Jordan curve between its end-points. A drawing is *upward* if each edge (u, v) is a curve monotonically increasing in the y-direction and it is *planar* if no two edges intersect except, possibly, at common end-points. A drawing is *upward planar* if it is both upward and planar. An *upward planar graph* is a graph that admits an upward planar drawing. A planar drawing of a graph partitions the plane into connected regions, called *faces*. The unbounded face is the *outer face*, all the other faces are *internal faces*. Two upward planar drawings of an upward planar DAG are *equivalent* if they determine the same clockwise ordering of the edges around each vertex. An *embedding* of an upward planar DAG is an equivalence class of upward planar drawings. An *embedded upward planar graph* is an upward planar DAG together with an embedding. Consider an embedded upward planar graph G with exactly one source s. Then, the *leftmost path* of G is the path $(u_1, \dots, u_k)$ defined as follows: $u_1 = s$; for $i = 2, \dots, k$, u_i is the neighbor of u_{i-1} such that (u_{i-1}, u_i) is the first edge in the clockwise order of the edges outgoing u_{i-1}; u_k is a sink. The *rightmost path* of G is defined analogously.

An *upward planar triangulation* is an upward planar graph whose underlying graph is a maximal planar graph. Consider any two upward planar drawings Γ_1 and Γ_2 of an upward planar triangulation G. Then, either Γ_1 and Γ_2 are equivalent, or the clockwise ordering of the edges around each vertex in Γ_1 is exactly the opposite of the one in Γ_2. The outer face of an upward planar drawing Γ of an upward planar triangulation G is delimited by a cycle composed of three edges (u, v), (u, z), and (v, z). Then, u, v, and z are called *bottom vertex*, *middle vertex*, and *top vertex* of Γ, respectively. Consider the two embeddings $\mathcal{E}_1$ and $\mathcal{E}_2$ of an upward planar triangulation G. Then, the bottom, middle, and top vertex of $\mathcal{E}_1$ coincide with the bottom, middle, and top vertex of $\mathcal{E}_2$, respectively. Hence such vertices are simply called the *bottom vertex of* G, the *middle vertex of* G, and the *top vertex of* G, respectively.

A total vertex ordering σ of a DAG G is *upward* if G has no edge (u, v) such that $v <_\sigma u$. The upward vertex orderings are all and only the vertex orderings that are feasible for a book embedding of a DAG. We say that an upward vertex ordering σ *induces* a twist of size k if G contains edges $(u_1, v_1), \dots, (u_k, v_k)$ such that $u_1 <_\sigma \dots <_\sigma u_k <_\sigma v_1 <_\sigma \dots, v_k$. The *maximum twist size* of an upward vertex ordering σ is the maximum number of edges in a twist induced by σ. Two edges (u_1, v_1) and (u_2, v_2) *are nested* in σ if $u_1 <_\sigma u_2 <_\sigma v_2 <_\sigma v_1$. Two edges (u_1, v_1) and (u_2, v_2) *cross* in σ if $u_1 <_\sigma u_2 <_\sigma v_1 <_\sigma v_2$.

An undirected graph is k*-connected* if the removal of any $k-1$ vertices leaves the graph connected. A directed graph is k*-connected* if its underlying graph is. A *maximal* k*-connected component* of a graph G is a subgraph G' of G such that G' is k-connected and no subgraph G'' of G with $G' \subset G''$ is k-connected. A *separating triangle* C in a graph G is a 3-cycle such that the removal of the vertices of C from G disconnects G. A separating triangle C in a graph G is *maximal* if G has no separating triangle C' such that C is internal to C'.

The *degree of a vertex* is the number of edges incident to it. The *degree of a graph* is the maximum among the degrees of its vertices. A DAG is *Hamiltonian* if it contains a directed path passing through all its vertices. An Hamiltonian DAG G has exactly one upward total vertex ordering. Moreover, if G is upward planar, then it has page number at most 2. A *plane* 3*-tree* is a maximal plane graph that can be constructed as follows. Let G_3 be a 3-cycle embedded in the plane. A plane 3-tree with n vertices is a plane graph that can be constructed from a plane graph G_{n-1} with $n-1$ vertices by inserting a vertex inside an internal face of G_{n-1} and by connecting such a vertex to the three vertices incident to the face. A *planar* 3*-tree* is a planar graph that can be embedded as a plane 3-tree. An *upward plane* 3*-tree* is an upward planar DAG whose underlying graph is a plane 3-tree.

3 Page Number and Connectivity

In this section we study the relationship between the page number of an upward planar DAG and the page number of its maximal 4-connected components. We prove the following:

Theorem 1 *Let $f(n)$ be any function such that $f(n) \in \Omega(1)$ and $f(n) \in O(n)$. Consider any n-vertex upward planar triangulation G and suppose that every maximal 4-connected component of G has an upward vertex ordering with maximum twist size at most $f(n)$. Then G has an upward vertex ordering with maximum twist size $O(f(n))$.*

First, we define a rooted tree $T = (V', E')$, whose nodes correspond to subgraphs of $G{=}(V, E)$, which reflects the structure of separating triangles in G. The tree T appeared already in the work of [23], where it is called the 4-block tree. Tree T is recursively defined as follows (see Fig. 2). The root r of T corresponds to $G'(r) = G$. Suppose that a node a of T corresponds to a subgraph $G'(a)$ of G. If $G'(a)$ contains no separating triangle, then a is a leaf of T. Otherwise, consider every maximal separating triangle (u, v, z) of $G'(a)$; then, insert a node b in T as a child of a, such that $G'(b)$ is the subgraph of $G'(a)$ induced by the vertices internal to or on the border of cycle (u, v, z). For each node $a \in T$, denote as $V'(a)$ and $E'(a)$ the vertex set and the edge set of $G'(a)$. Further, for each node $a \in T$, let $G(a) = (V(a), E(a))$ denote the subgraph of $G'(a)$ induced by all the vertices which are not internal to any separating triangle of $G'(a)$. Note that $G(a)$ is 4-connected for every $a \in V'$.

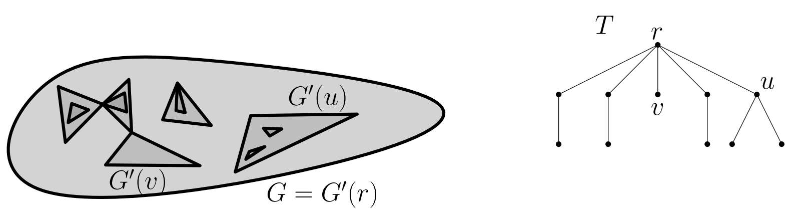


Figure 2: Tree T capturing the structure of the separating triangles in G.

We now define a total ordering $o(V)$ of V and we later prove that the maximum twist size of $o(V)$ is $O(f(n))$. Ordering $o(V)$ is constructed by induction on T.

In the base case a is a leaf; then let $o(V'(a))$ be any total ordering of $V'(a)$ such that the maximum twist size of $o(V'(a))$ is $f(n)$. Such an ordering exists by hypothesis, since $G'(a)$ is 4-connected.

In the inductive case, let $a_1, \dots, a_m$ be the children of a in T, where total orderings $o(V'(a_1)), \dots, o(V'(a_m))$ of $V'(a_1), \dots, V'(a_m)$, respectively, have already been computed. Compute a total ordering $o(V(a))$ of $V(a)$ such that the maximum twist size of $o(V(a))$ is $f(n)$. Again, such an ordering exists by hypothesis, since $G(a)$ is 4-connected. Next, we merge $o(V'(a_1)), \dots, o(V'(a_m))$ with $o(V(a))$. In order to do this, we define the operation of *merging an ordering* $o(V_2)$ *into an ordering* $o(V_1)$, that takes as input two total vertex orderings $o(V_1)$ and $o(V_2)$ such that V_1 and V_2 share a single vertex v, and outputs a single total vertex ordering $o(V_1 \cup V_2)$ of $V_1 \cup V_2$ such that $o(V_1 \cup V_2)$ coincides with $o(V_i)$ when restricted to the vertices in V_i, for $i = 1, 2$, and such that every vertex of V_1 that precedes v in $o(V_1)$ (resp. follows v in $o(V_1)$) precedes all the vertices of V_2 in $o(V)$ (resp. follows all the vertices of V_2 in $o(V)$). Denote by $b(H)$, by $m(H)$, and by $t(H)$ the bottom vertex, the middle vertex, and the top vertex of an upward triangulation H, respectively. Then, ordering $o(V'(a))$ is defined as follows: Let $o_1 = o(V(a))$ and let o_{i+1} be the ordering obtained by merging $o(V'(a_i)) \setminus \{b(G'(a_i)), t(G'(a_i))\}$ into o_i, for $i = 1, \dots, m$;

then $o(V'(a)) = o_{m+1}$. Observe that $o(V'(a))$ is an upward vertex ordering because $o(V(a)), o(V'(a_1)), \dots, o(V'(a_m))$ are and because of the definition of the merging operation.

We now prove that the size of the maximum twist induced by $o(V)$ is $O(f(n))$. Let $M = \{e_1{=}(u_1, v_1), \dots, e_k{=}(u_k, v_k)\}$ denote any maximal twist induced by $o(V)$. We have the following:

Claim 1 *Let a be a node of T. Let a_1 and a_2 be two distinct children of a. There is no pair of distinct edges $(u_i, v_i), (u_j, v_j)$ in M such that $(u_i, v_i) \in E'(a_1)$, $(u_j, v_j) \in E'(a_2)$, and $\{u_i, v_i, u_j, v_j\} \cap V(a) = \emptyset$.*

Proof: Let (u^1, v^1, z^1) and (u^2, v^2, z^2) be the separating triangles of $G'(a)$ that delimit the outer faces of $G'(a_1)$ and $G'(a_2)$, where v^i is the middle vertex of $G'(a_i)$, for $i = 1, 2$. If $v^1 \neq v^2$, then, by the construction of $o(V)$, all internal vertices of $G'(a_1)$ precede all internal vertices of $G'(a_2)$ or vice versa, thus e_i and e_j do not both belong to M. Otherwise, $v^1 = v^2$. Then, again by the construction of $o(V)$, e_i and e_j are nested, thus they do not both belong to M. □

Let r be the root of T. We assume that G is "minimal", that is, we assume that there exists no child a of r such that all the edges in M belong to $G'(a)$. Indeed, if such a child exists, graph $G{=}G'(r)$ can be replaced by $G'(a)$, and the bound on the size of M can be achieved by arguing on $G'(a)$ rather than on $G'(r)$. Denote by M_i, with $i = 0, 1, 2$, the subset of M that contains all the edges having i endpoints in $V(r)$. Observe that $|M| = |M_0| + |M_1| + |M_2|$, hence it suffices to prove that $|M_i| \in O(f(n))$, for $i = 0, 1, 2$, in order to prove the theorem. By hypothesis and since $G(r)$ is 4-connected, we have $|M_2| \leq f(n)$. We now deal with the edges in M_1.

Claim 2 $|M_1| \in O(f(n))$.

Proof: First, we argue that M_1 contains at most one edge e such that an end-vertex of e is the middle vertex of an upward planar triangulation $G'(a)$, for some child a of r. Indeed, by the vertex ordering's construction, any two such edges, say e_a and e_b, are either incident to the same vertex or are such that both end-vertices of e_a come before both end-vertices of e_b in $o(V'(a))$. Thus, it is enough to bound the number of edges in M_1 whose end-vertex in $V(r)$ is the bottom vertex or the top vertex of an upward planar triangulation $G'(a)$, where a is a child of r.

Let M_1^b (resp. M_1^t) be the subset of the edges in M_1 whose end-vertex in $V(r)$ is the bottom vertex (resp. the top vertex) of an upward planar triangulation $G'(a)$, where a is a child of r. Observe that, by the above observation, $|M| \leq |M_1^b| + |M_1^t| + 1$. In the following we bound $|M_1^b|$ (the bound for $|M_1^t|$ can be obtained analogously).

Consider any edge $(u, v) \in M_1^b$, where $u \in V(r)$. We define a *corresponding edge* of (u, v) in $G(r)$ as follows. Let $a_{u,v}$ be the child of r such that $G'(a_{u,v})$ contains edge (u, v). Further, denote by $m_{u,v}$ the middle vertex of $G'(a_{u,v})$.

Then, $(u, m_{u,v})$ is the corresponding edge of (u, v) in $G(r)$. Observe that edge $(u, m_{u,v})$ exists and belongs to $E(r)$. Now consider the multi-set E_1^b of the corresponding edges, that is $E_1^b = \{(u, m_{u,v}) | (u, v) \in M_1^b\}$. First, we have that, for each vertex w in $V(r)$, there exist at most two edges (z, w) in E_1^b, since each vertex in $V(r)$ is the middle vertex of at most two upward planar triangulations $G'(a_i)$, where a_i is a child of r, and since $G'(a_i)$ has at most one edge in M_1^b. If there exist two edges (z_1, w) and (z_2, w) in E_1^b, then remove one of them. Then, after such deletions, $|E_1^b| \geq |M_1^b|/2$.

Next, we prove that each vertex in $V(r)$ is an end-vertex of at most two edges in E_1^b. Namely, consider any two edges (u_1, v_1) and (u_2, v_2) in E_1^b. Then, $v_1 \neq v_2$ because of the deletions performed on E_1^b, and $u_1 \neq u_2$ as otherwise the corresponding edges in M_1^b would share a vertex, contradicting the assumption that M is a twist; thus, each vertex in $V(r)$ is the source of at most one edge in E_1^b and the sink of at most one edge in E_1^b. Since the degree of graph $(V(r), E_1^b)$ is two, there exists a subset E^* of E_1^b such that the degree of graph $(V(r), E^*)$ is one and $|E^*| \geq |E_1^b|/3$.

Finally, we have that every two edges in E^* cross. Namely, if they do not, then by the vertex ordering's construction the corresponding edges in M_1^b would not cross either, thus contradicting the assumption that M is a twist.

Since $E^* \subseteq E(r)$ and the maximum size of a twist of edges in $E(r)$ is $f(n)$, given that $G(r)$ is 4-connected, it follows that $E^* \leq f(n)$. Using $|E^*| \geq |E_1^b|/3$ and $|E_1^b| \geq |M_1^b|/2$, we get $|M_1^b| \leq 6f(n)$. Such an inequality, together with the analogous bound $|M_1^t| \leq 6f(n)$ and with $|M| \leq |M_1^b| + |M_1^t| + 1$, proves the theorem. □

We now proceed by bounding the size of M_0.

Claim 3 $|M_0| \in O(f(n))$.

Proof: By Claim 1, all the edges in M_0 belong to a graph $G'(a)$, for a certain descendant a of r. Let us choose a so that the length of the path from a to r is maximized. That is, a is the node of T farthest from r containing all the edges of M_0. Let w be the middle vertex of the separating triangle (u, v, w) delimiting $G'(a)$. Let a' denote the child of r which is an ancestor of a or that coincides with a. Let w' be the middle vertex of the separating triangle (u', v', w') delimiting $G'(a')$.

For any edge $(y, z) \in M_0$, we have that (y, z) "nests around w'", that is, y precedes w' and w' precedes z in $o(V)$. Indeed, if both y and z precede w' in $o(V)$ (or if they both follow w' in $o(V)$), then only the edges in $G'(a')$ can possibly cross (y, z), by the construction of $o(V)$, thus contradicting the minimality of r.

If $w \neq w'$, then $|M_0| \leq 3$, since only the edges incident to u, v, and w can nest around w' and hence belong to M_0. Otherwise we have $w' = w$ (see Fig. 3). Consider graph $G'(a)$; partition the edges in M_0 into two subsets, namely M_0' contains all the edges of M_0 having at least one end-vertex in $V(a)$ and M_0'' contains all the edges of M_0 having no end-vertex in $V(a)$. By definition of a and by Claim 1, $|M_0'| > 0$, as otherwise there would exist a child of a containing

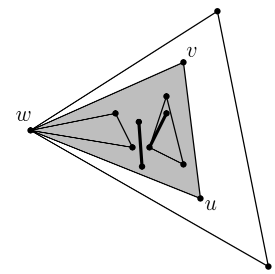


Figure 3: Graph $G'(a)$. The thick edges belong to M_0.

all the edges of M_0. However, by Claim 2 applied to $G'(a)$ and by the hypothesis of the theorem, we have $|M'_0| \in O(f(n))$. Moreover, every edge in M''_0 is in a separating triangle of $G'(a)$ having w as middle vertex; however, any such edge is nested inside any edge of M'_0; thus, since $|M'_0| > 0$, we have $|M''_0| = 0$ and hence $|M_0| \in O(f(n))$, which concludes the proof. □

Since $|M_i| \in O(f(n))$, for $i = 0, 1, 2$, it follows that $|M| \in O(f(n))$, thus proving Theorem 1. By Lemmata 1 and 2, we have the following:

Corollary 1 *If every n-vertex upward planar 4-connected triangulation has $o(\frac{n}{\log n})$ page number, then every n-vertex upward planar triangulation has $o(n)$ page number.*

Corollary 2 *Every upward planar 3-tree has $O(1)$ page number.*

4 Page Number and Diameter

In this section we study the relationship between the page number of an upward planar DAG and its diameter D. We show that upward planar DAGs with small diameter have sub-linear page number. Notice that such a result pairs the observation that graphs with diameter $n - o(n)$ have sub-linear page number as well, given that upward planar Hamiltonian DAGs have page number two. We have the following:

Theorem 2 *Every n-vertex upward planar triangulation whose diameter is at most D admits an upward vertex ordering whose maximum twist size $t(n)$ is a function satisfying $t(n) \leq aD + t(\frac{n}{2}) + b$, for some constants a and b.*

We will prove the statement for a family of upward planar DAGs that is strictly larger than the family of upward planar triangulations. Namely, we call *upward cactus* an embedded upward planar DAG G having exactly one source $s(G)$ and such that every internal face is delimited by a 3-cycle. See Fig. 4. Observe that an upward planar triangulation is an upward cactus.

Consider an upward cactus G. We call *monotone path* any directed path $P = (u_1, \ldots, u_k)$ from $s(G)$ to a sink of G. Consider an upward planar drawing Γ of G in which u_k is the vertex with highest y-coordinate. Observe that such a drawing Γ always exists because G is an upward cactus. Then, we define the *left side of* P as the subgraph of G induced by all the vertices which are to the left of the Jordan curve representing P in Γ. The *right side of* P is defined analogously. Observe that the vertices of P, the vertices of the left side of P, and the vertices of the right side of P form a partition of the vertices of G. We have the following:

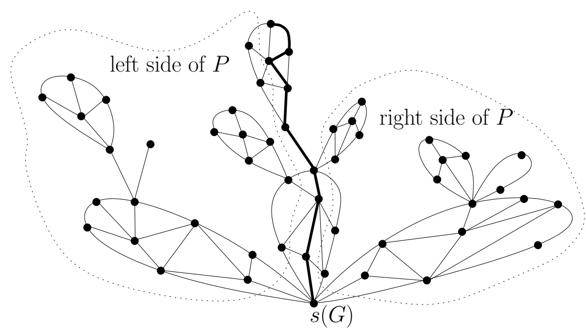


Figure 4: An upward cactus G. The thick edges represent a monotone path P.

Claim 4 *In every n-vertex upward cactus there exists a monotone path P such that both the left side of P and the right side of P have less than $\frac{n}{2}$ vertices.*

Proof: We construct a sequence of monotone paths $P_1, P_2, \ldots, P_h$ and prove that $P = P_i$ satisfies the statement for a certain $1 \leq i \leq h$. Path $P_1 = (u_1^1, \ldots, u_{k_1}^1)$ is the leftmost path of G. Clearly, the left side of P_1 contains no vertex. Then, two cases are possible. Namely, either the right side of P_1 has less than $\frac{n}{2}$ vertices, and in such a case $P = P_1$ is the desired path, or the right side of P_1 has at least $\frac{n}{2}$ vertices. Suppose that the left side of $P_{i-1} = (u_1^{i-1}, \ldots, u_k^{i-1})$ has $l < \frac{n}{2}$ vertices and that the right side of P_{i-1} has $r \geq \frac{n}{2}$ vertices, for a certain $i \geq 2$.

We distinguish three cases (see Fig. 5).

- Case 1: There exists a vertex v such that (u_j^{i-1}, v) follows $(u_j^{i-1}, u_{j+1}^{i-1})$ in the clockwise order of the edges outgoing u_j^{i-1} and (v, u_{j+1}^{i-1}) follows $(u_j^{i-1}, u_{j+1}^{i-1})$ in the counter-clockwise order of the edges incoming u_{j+1}^{i-1}. Observe that $(u_j^{i-1}, v, u_{j+1}^{i-1})$ is an internal face of G. Then, $P_i = (u_1^{i-1}, \ldots, u_j^{i-1}, v, u_{j+1}^{i-1}, \ldots, u_k^{i-1})$; observe that P_i is a monotone path since P_{i-1} is. The left side of P_i contains exactly the same set of $l < \frac{n}{2}$ vertices that the left side of P_{i-1} contains; moreover, the right side of P_i contains $r - 1$ vertices. Hence, either $r - 1 < \frac{n}{2}$, and in such a case $P = P_i$ is the desired path, or we construct a new path P_{i+1}.

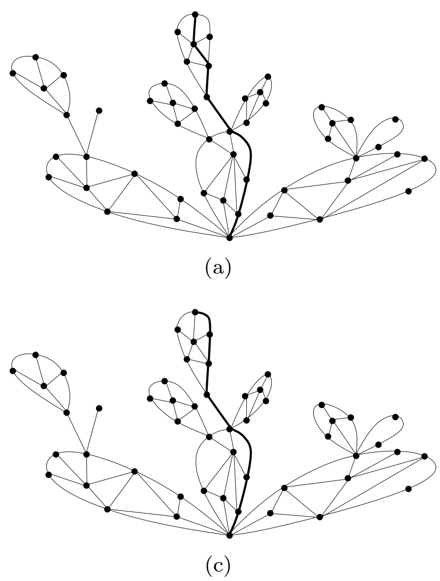


Figure 5: Case 1 applies to the monotone path in (a), yielding the monotone path in (b). Case 2 applies to the monotone path in (b), yielding the monotone path in (c). Case 3 applies to the monotone path in (c), yielding the monotone path in (d).

- Case 2: G contains an edge $(u_j^{i-1}, u_{j+2}^{i-1})$ that follows $(u_j^{i-1}, u_{j+1}^{i-1})$ in the clockwise order of the edges outgoing u_j^{i-1} and that follows $(u_{j+1}^{i-1}, u_{j+2}^{i-1})$ in the counter-clockwise order of the edges incoming u_{j+2}^{i-1}. Observe that $(u_j^{i-1}, u_{j+1}^{i-1}, u_{j+2}^{i-1})$ is an internal face of G. Then, $P_i = (u_1^{i-1}, \dots, u_j^{i-1}, u_{j+2}^{i-1}, \dots, u_k^{i-1})$; observe that P_i is a monotone path since P_{i-1} is. The right side of P_i contains exactly the same set of $r \geq \frac{n}{2}$ vertices that the right side of P_{i-1} contains; given that P_i contains at least two vertices, the left side of P_i contains less than $\frac{n}{2}$ vertices. Then, we construct a new path P_{i+1}.

- Case 3: Suppose that neither Case 1 nor Case 2 applies. Suppose, for a contradiction, that no vertex u_j^{i-1}, with $1 \leq j \leq k-1$, has an outgoing edge following $(u_j^{i-1}, u_{j+1}^{i-1})$ in the clockwise order of the edges outgoing u_j^{i-1}. Observe that $s(G)$ and u_k^{i-1} have no incoming edge and no outgoing edge, as they are a source and a sink, respectively. Hence, if any vertex u_j^{i-1}, with $2 \leq j \leq k$, has an incoming edge following $(u_{j-1}^{i-1}, u_j^{i-1})$ in the counter-clockwise order of the edges incoming u_j^{i-1}, then G would contain at least two sources, a contradiction; otherwise no vertex u_j^{i-1} has incoming or outgoing edges to the right of P_{i-1}, contradicting the hypothesis that $r \geq \frac{n}{2}$.

It follows that there exists a vertex u_j^{i-1} that has an outgoing edge (u_j^{i-1}, v) following $(u_j^{i-1}, u_{j+1}^{i-1})$ in the clockwise order of the edges outgoing u_j^{i-1} and assume that j is the maximum index such that u_j^{i-1} satisfies such a property. Consider the leftmost path $P_l(v)$ starting at v. Then, path $P_i = (u_1^{i-1}, \dots, u_j^{i-1}, v) \cup P_l(v)$.

We claim that every vertex that is in the right side of P_{i-1} is also in the right side of P_i, except for the vertices of $P_l(v)$ that now belong to P_i. Consider any vertex w in the right side of P_{i-1}. Since G has a unique source, then there exists a vertex u_y^{i-1} of P_{i-1} such that G has a directed path $P_{u_y^{i-1},w}$ from u_y^{i-1} to w, for some $1 \leq y \leq k-1$. Suppose that y is the maximum index satisfying such a property. Then, three cases are possible: (i) if $y < j$, then $P_{u_y^{i-1},w}$ is entirely in the right side of P_i, as path P_i does not share any vertex other than u_y^{i-1} with $P_{u_y^{i-1},w}$, given the maximality of y; (ii) if $y > j$, then the maximality of j would be contradicted; (iii) if $y = j$, then suppose, for a contradiction, that w is not in the right side of P_i and consider the last vertex z shared by $P_{u_y^{i-1},w}$ and P_i (observe that such a vertex always exists since such paths share vertex u_y^{i-1}); if $z = u_y^{i-1}$, then edge (u_j^{i-1}, v) would not follow $(u_j^{i-1}, u_{j+1}^{i-1})$ in the clockwise order of the edges outgoing u_j^{i-1}, a contradiction, while if $z \in P_l(v)$, then $P_l(v)$ would not be the leftmost path starting at v, a contradiction.

Since every vertex that is in the right side of P_{i-1} is either in the right side of P_i or in P_i, since $r \geq \frac{n}{2}$, and since $s(G)$ is not in the right side of P_{i-1} and is not in the left side of P_i, it follows that the number of vertices in the left side of P_i is at most $n - \frac{n}{2} - 1 < \frac{n}{2}$. Hence, either the right side of P_i contains less than $\frac{n}{2}$ vertices, and in such a case $P = P_i$ is the desired path, or we construct a new path P_{i+1}.

Eventually, the considered path P_h coincides with the rightmost path of G. The right side of such a path has no vertex. It follows that there exists a path satisfying $P = P_i$ satisfying the statement of the theorem. □

We now prove the statement of the theorem for every n-vertex upward cactus G with diameter at most D. The proof is by induction on n. If $n \leq 3$, then in any upward vertex ordering of G the maximum twist size is 1, hence $t(3) \leq b$, for any $b \geq 1$, thus proving the base case.

Suppose that $n > 3$. By Claim 4, there exists a monotone path P in G such that both the left side of P and the right side of P have less than $\frac{n}{2}$ vertices. We now associate each vertex in the left side of P and each vertex in the right side of P to a vertex of P. Namely, we associate a vertex v in the left side of P to the vertex u_i of P such that there exists a directed path from u_i to v and such that, for every $j > i$, there exists no directed path from u_j to v. Observe that, for every vertex v in the left side of P, there exists a directed path from

$s(G)$ to v, since G has a unique source, hence v is associated to exactly one vertex of P. Then, we call *left bag of* u_i the set of vertices in the left side of P which are associated to u_i, for each $i = 1, \ldots, k$. Vertices in the right side of P are associated to vertices of P analogously, thus analogously defining the *right bag of* u_i, for each $i = 1, \ldots, k$. We have the following:

Claim 5 *The subgraph G_i^L of G induced by the left bag of u_i and by u_i is an upward cactus, for every $i = 1, \ldots, k$.*

Proof: Every internal face of G_i^L is delimited by a 3-cycle since every internal face of G is. Moreover, since by definition there exists a directed path from u_i to every vertex of G_i^L different from u_i, it follows that G_i^L has a unique source. □

An analogous claim holds for the subgraph G_i^R of G induced by the right bag of u_i and by u_i.

Next, we construct an upward vertex ordering of G. This is done as follows. First, inductively construct an upward vertex ordering σ_i^L of G_i^L and an upward vertex ordering σ_i^R of G_i^R, for $i = 1, \ldots, k$, such that the maximum twist size of each of σ_i^R and σ_i^L is $t(\frac{n}{2})$. This is possible since G_i^L and G_i^R are upward cacti, by Claim 5, and they have less than $\frac{n}{2}$ vertices, by Claim 4. Observe that u_i is the first vertex both in σ_i^L and in σ_i^R, given that it is the only source of both G_i^L and G_i^R. Then, denote by σ_i the vertex ordering of $G_i^L \cup G_i^R$ which is obtained by concatenating σ_i^L and $\sigma_i^R \setminus \{u_i\}$. Finally a vertex ordering σ of G is obtained by concatenating $\sigma_1, \sigma_2, \ldots, \sigma_k$.

Claim 6 *σ is an upward vertex ordering.*

Proof: Suppose, for a contradiction, that G has an edge (u, v) such that v comes before u in σ.

If u and v both belong to P, then $v = u_i$ and $u = u_j$, with $j > i$. However, this implies that G contains a directed cycle $(u_i, u_{i+1}, \ldots, u_j, u_i)$, a contradiction to the fact that G is a DAG.

If u belongs to P, say $u = u_i$, and v is in the left side of P or in the right side of P, then there exists a directed path from u_i to v (namely such a path is edge (u, v)), hence v is associated to a vertex u_j, with $j \geq i$, and hence v appears in σ_j, with $j \geq i$. Since $u = u_i$ is the first vertex of σ_i, v does not precede u in σ, a contradiction.

If v belongs to P, say $v = u_i$, and u is in the left side of P or in the right side of P, then observe that u is associated to a vertex u_j, with $j \geq i$, as otherwise u would not follow u_i in σ. Hence, there exists a directed path $P_{u_j,u}$ from u_j to u. However, this implies that G contains a directed cycle $(u_i, u_{i+1}, \ldots, u_j) \cup P_{u_j,u} \cup (u, u_i)$, a contradiction to the fact that G is a DAG.

If u is in the left side of P and v is in the right side of P (or vice versa), then edge (u, v) crosses P, a contradiction to the upward planarity of G.

If u and v both are in the left side of P or both are in the right side of P, then we further distinguish two cases. If u and v are both associated to the same vertex u_i, then they both belong to G_i^L or they both belong to G_i^R, hence u comes before v in σ since σ_i^L and σ_i^R are upward vertex orderings, a contradiction. If v is associated to a vertex u_i and u is associated to a vertex $u_j \neq u_i$, then $j > i$, as otherwise u would come before v in σ. It follows that there exists a directed path $P_{u_j,u}$ from u_j to u and hence a directed path $P_{u_j,u} \cup (u,v)$ from u_j to v. By construction, v is associated to a vertex u_k, with $k \geq j > i$, a contradiction. □

Next, we prove that the maximum twist size $t(n)$ of σ is at most $aD+t(\frac{n}{2})+b$, for some constants a and b.

First, observe that the edges that have both end-vertices in P create twists of size at most two, since the graph induced by the vertices of P is upward planar Hamiltonian.

Second, we discuss the size of a twist composed of *intra-bag* edges, which are edges whose both end-vertices are associated to the same vertex of P. Consider any edge e_i^L of G_i^L and any edge e_i^R of G_i^R. Such edges do not cross. Namely, if such edges are both incident to u_i, then they do not cross by definition. If e_i^R is not incident to u_i, then both end-vertices of e_i^R come after both end-vertices of e_i^L, by construction, hence such edges do not cross. Moreover, if e_i^R is incident to u_i and e_i^L is not, then e_i^L is nested inside e_i^R, by construction, hence such edges do not cross. It follows that the maximum size of a twist of intra-bag edges is equal to the maximum twist size of σ restricted to the vertices in G_i^a for some $a \in \{L,R\}$ and some $1 \leq i \leq k$. By Claim 5, graph G_i^a is an upward cactus. Moreover, by Claim 4, G_i^a has at most $\frac{n}{2}$ vertices, hence the maximum size of a twist of intra-bag edges is at most $t(\frac{n}{2})$.

Third, we discuss the maximum size of a twist composed of *inter-bag* edges, which are edges whose end-vertices are associated to distinct vertices of P. We show that the maximum size of a twist composed of inter-bag edges in the left side of P is $2D$. An analogous proof shows that the maximum size of a twist composed of inter-bag edges in the right side of P is also $2D$.

Consider any two inter-bag edges (w_1,w_2) and (w_3,w_4) in the left side of P. Suppose that (w_1,w_2) and (w_3,w_4) cross in σ. Denote by u_{j_1}, u_{j_2}, u_{j_3}, and u_{j_4}, such that $u_{j_1} < u_{j_2}$ and $u_{j_3} < u_{j_4}$, the vertices of P vertices w_1, w_2, w_3, and w_4 have been assigned to, respectively. The following claim asserts that any two inter-bag edges (w_1,w_2) and (w_3,w_4) that cross in σ either have their sources assigned to the same vertex of P, or have their destinations assigned to the same vertex of P, or the source of one of them and the destination of the other of them are assigned to the same vertex of P.

Claim 7 *At least one of the following holds:* $j_1 = j_3 < j_2, j_4$*, or* $j_1 < j_2 = j_3 < j_4$*, or* $j_3 < j_4 = j_1 < j_2$*, or* $j_1, j_3 < j_2 = j_4$*.*

Proof: First, assume that $j_1 = j_3$. Then, since (w_1,w_2) and (w_3,w_4) are inter-bag edges, $j_2 > j_1$ and $j_4 > j_3$ hold, hence $j_1 = j_3 < j_2, j_4$ holds.

Second, assume that $j_1 < j_3$. Observe that $j_2 > j_1$ and $j_4 > j_3$ given that (w_1,w_2) and (w_3,w_4) are inter-bag edges. Then, observe that $j_2 \geq j_3$, otherwise both w_1 and w_2 come before both w_3 and w_4, and hence edges (w_1,w_2) and

(w_3, w_4) do not cross in σ, a contradiction. Moreover, $j_2 \leq j_4$, as otherwise edge (w_3, w_4) is nested inside edge (w_1, w_2). Suppose that $j_3 < j_2 < j_4$ and see Fig. 6. Consider the four directed paths $P_{u_{j_1},w_1}$, $P_{u_{j_2},w_2}$, $P_{u_{j_3},w_3}$, and $P_{u_{j_4},w_4}$ from u_{j_1} to w_1, from u_{j_2} to w_2, from u_{j_3} to w_3, and from u_{j_4} to w_4, respectively. Such paths exist (since w_i is assigned to u_{j_i}, for $i = 1, \ldots, 4$); moreover, they do not share vertices, as if they do, then some of vertices u_{j_1}, u_{j_2}, u_{j_3}, and u_{j_4} would coincide, by the construction of the assignment of vertices in the left side of P to the vertices of P, contradicting the hypothesis that $j_1 < j_3 < j_2 < j_4$. Then, path $P_{u_{j_1},w_1} \cup (w_1, w_2) \cup P_{u_{j_2},w_2}$ crosses path $P_{u_{j_3},w_3} \cup (w_3, w_4) \cup P_{u_{j_4},w_4}$, a contradiction to the upward planarity of G. It follows that $j_1 < j_3 < j_2 < j_4$ does not hold, hence either $j_1 < j_2 = j_3 < j_4$ holds or $j_1 < j_3 < j_2 = j_4$ holds.

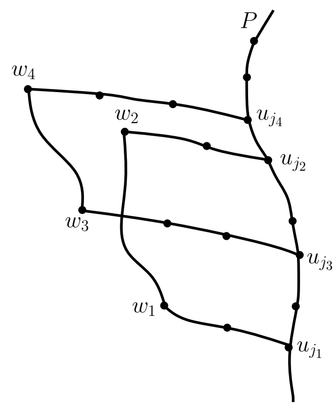


Figure 6: If $j_1 < j_3 < j_2 < j_4$, then paths $P_{u_{j_1},w_1} \cup (w_1, w_2) \cup P_{u_{j_2},w_2}$ and $P_{u_{j_3},w_3} \cup (w_3, w_4) \cup P_{u_{j_4},w_4}$ cross.

Third, assume that $j_1 > j_3$. Then, analogously to the previous case, it can be shown that either $j_3 < j_4 = j_1 < j_2$ holds or $j_3 < j_1 < j_2 = j_4$ holds. □

Hence, if there are more than $2D$ inter-bag edges pairwise crossing in the left side of P, then either there are more than D inter-bag edges pairwise crossing in the left side of P such that the origins of such edges have all been assigned to the same vertex of P, or there are more than D inter-bag edges pairwise crossing in the left side of P such that the destinations of such edges have all been assigned to the same vertex of P. In the following, we discuss such two cases.

Claim 8 *Suppose that G contains inter-bag edges $(v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)$ in the left side of P, where $v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_1 <_\sigma w_2 <_\sigma \cdots <_\sigma w_k$ and where all the vertices w_i have been assigned to the same vertex u_l of P, for $i = 1, \ldots, k$, or all the vertices v_i have been assigned to the same vertex u_l of P, for $i = 1, \ldots, k$. Then, there exists a directed path starting at u_l and passing through $w_1, w_2, \ldots, w_k$.*

Proof: We prove the statement in the case in which all the vertices w_i have been assigned to the same vertex u_l of P, the case in which they have all the vertices v_i have been assigned to the same vertex u_l of P being analogous.

A directed path P_1 starting at u_l and ending at w_1 exists since w_1 is assigned to u_l. Observe that such a path does not pass through any of $w_2, \dots, w_k$, as such vertices follow w_1 in σ. Suppose that a directed path P_{u_l,w_i} from u_l to w_i, passing through $w_1, w_2, \dots, w_{i-1}$, and not passing through any of $w_{i+1}, w_{i+2}, \dots, w_k$ has been found, for some $i \in \{1, \dots, k-1\}$. We show how to construct a directed path $P_{u_l,w_{i+1}}$ from u_l to w_{i+1}, passing through $w_1, w_2, \dots, w_i$, and not passing through any of $w_{i+2}, w_{i+3}, \dots, w_k$. Eventually, such a construction will lead to the desired path from u_l to w_k passing through $w_1, w_2, \dots, w_{k-1}$. In order to construct $P_{u_l,w_{i+1}}$, it suffices to show that there exists a directed path $P_{w_i,w_{i+1}}$ from w_i to w_{i+1}, not passing through any of $w_1, w_2, \dots, w_{i-1}$ and not passing through any of $w_{i+2}, w_{i+3}, \dots, w_k$. Path $P_{u_l,w_{i+1}}$ is then the concatenation of P_{u_l,w_i} and $P_{w_i,w_{i+1}}$.

Consider any directed path $P_{u_l,w_{i+1}}$ from u_l to w_{i+1}. Such a path exists since w_{i+1} is assigned to u_l.

If $P_{u_l,w_{i+1}}$ passes through w_i, then consider the sub-path $P_{w_i,w_{i+1}}$ of $P_{u_l,w_{i+1}}$ starting at w_i and ending at w_{i+1}. Such a path does not pass through any of $w_1, w_2, \dots, w_{i-1}$, as such vertices precede w_i in σ, and does not pass through any of $w_{i+2}, w_{i+3}, \dots, w_k$, as such vertices follow w_{i+1} in σ. Hence, $P_{w_i,w_{i+1}}$ is the desired path.

If $P_{u_l,w_{i+1}}$ does not pass through w_i, then let u_m be the vertex of P vertex v_{i+1} is assigned to. Observe that $m < l$. Let $P_{u_m,v_{i+1}}$ be a directed path from u_m to v_{i+1}. Such a path exists since vertex v_{i+1} is assigned to u_m. Then, consider the graph G' whose outer face is delimited by $P_{u_l,w_{i+1}}$, by edge (v_{i+1}, w_{i+1}), by path $P_{u_m,v_{i+1}}$, and by the sub-path $(u_m, \dots, u_l)$ of P. See Fig. 7. Observe that, since every internal face of G is internally-triangulated and since the cycle delimiting the outer face of G' has exactly one sink, then G' has exactly one sink, namely w_{i+1}. Then, it suffices to prove that w_i is in G'. Namely, if w_i is in G', consider any maximal directed path $P_{w_i,w_{i+1}}$ in G' starting at w_i. Since w_{i+1} is the only sink of G', $P_{w_i,w_{i+1}}$ ends at w_{i+1}. Moreover, $P_{w_i,w_{i+1}}$ does not pass through any of $w_1, w_2, \dots, w_{i-1}$, as such vertices precede w_i in σ, and does not pass through any of $w_{i+2}, w_{i+3}, \dots, w_k$, as such vertices follow w_{i+1} in σ.

We prove that w_i is in G'. Suppose, for a contradiction, that w_i is not in G'. Then, let u_p be the vertex of P vertex v_i is assigned to. Observe that $p < l$. Let P_{u_p,v_i} be a directed path from u_p to v_i. Such a path exists since vertex v_i is assigned to u_p. Then, consider the graph G'' whose outer face is delimited by P_{u_l,w_i}, by edge (v_i, w_i), by path P_{u_p,v_i}, and by the sub-path $(u_p, \dots, u_l)$ of P. Observe that, since every internal face of G is internally-triangulated and since the cycle delimiting the outer face of G'' has exactly one sink, then G'' has exactly one sink, namely w_i. Moreover, by the upward planarity of G, edge (v_i, w_i) crosses neither $P_{u_l,w_{i+1}}$ nor edge (v_{i+1}, w_{i+1}). It follows that G'' contains w_{i+1}. Then, consider any maximal directed path P_{w_{i+1},w_i} in G'' starting at w_{i+1}. Since w_i is the only sink of G'', P_{w_{i+1},w_i} ends at w_i, thus contradicting the fact that w_{i+1} follows w_i in σ.

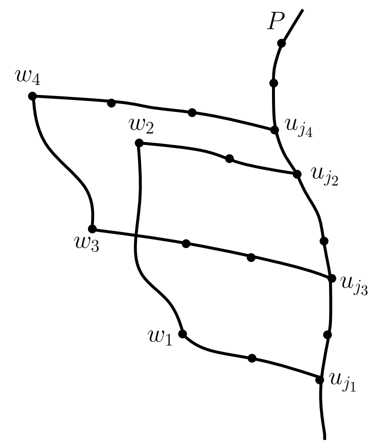


Figure 7: Graph G'.

It follows that w_i is in G', hence there exists a directed path $P_{w_i,w_{i+1}}$ from w_i to w_{i+1}, not passing through any of $w_1, w_2, \ldots, w_{i-1}$ and not passing through any of $w_{i+2}, w_{i+3}, \ldots, w_k$, thus proving the claim. □

Since by hypothesis any directed path contains at most D vertices, then, by Claim 8, the maximum size of a twist of inter-bag edges sharing their destinations in the left side of P is at most D and the maximum size of a twist of inter-bag edges sharing their origins in the left side of P is at most D. Hence, by Claim 7, the maximum size of a twist of inter-bag edges in the left side of P is at most $2D$ and the maximum size of a twist of inter-bag edges is at most $4D$. Since every edge of G is either an edge having both end-vertices in P, or is an intra-bag edge, or is an inter-bag edge, it follows that the maximum size of a twist in σ is $t(n) = 2 + t(\frac{n}{2}) + 4D$, thus proving Theorem 2.

By Lemma 1, we have the following:

Corollary 3 *Every n-vertex upward planar triangulation whose diameter is $o(\frac{n}{\log n})$ has $o(n)$ page number.*

5 Page Number and Degree

In this section we discuss the relationship between the page number of a graph and its degree. We prove the following theorem.

Theorem 3 *Let $f(n)$ be any function such that $f(n) \in \Omega(\sqrt{n})$ and $f(n) \in O(n)$. Suppose that every n-vertex upward planar triangulation whose degree is $O(f(n))$ admits a book embedding with $O(g(n))$ pages, for some function $g(n) \in \Omega(1)$ and $g(n) \in O(n)$. Then, every n-vertex upward planar triangulation admits a book embedding with $O(g(n) + \frac{n}{f(n)})$ pages.*

Consider any n-vertex upward planar triangulation G. We transform G into an $O(n)$-vertex upward planar triangulation G' with degree $O(f(n))$ as follows. Fix any constant $c > 0$ and denote by $u_1, \dots, u_k$ any ordering of the vertices of G whose degree is greater than $cf(n)$.

For $i = 1, \dots, k$, consider vertex u_i. Suppose that u_i is an internal vertex of G, the case in which u_i is an external vertex being analogous. Since it is an upward planar triangulation, G has exactly two faces (v_1, v_2, u_i) and (v_3, v_4, u_i) incident to u_i such that edges (v_1, u_i) and (v_4, u_i) are incoming u_i and such that edges (u_i, v_2) and (u_i, v_3) are outgoing u_i. Assume, w.l.o.g., that (v_1, u_i), (u_i, v_2), (u_i, v_3), and (v_4, u_i) appear in this clockwise order around u_i. Denote by $w_1 = v_2, w_2, \dots, w_{x-1}, w_x = v_3, w'_1 = v_4, w'_2, \dots, w'_{y-1}, w'_y = v_1$ the clockwise order of the neighbors of u_i (see Fig. 8(a)). Remove u_i and its incident edges from G. Let $M = \lceil \frac{x}{f(n)-1} \rceil$ and $N = \lceil \frac{y}{f(n)-1} \rceil$. Insert $M + N + 2$ vertices $z_1, \dots, z_{M+N+2}$ in G inside the cycle of the neighbors of u_i. Insert an edge from z_j to z_{j+1}, for $j = 1, \dots, M$, insert an edge from z_{j+1} to z_j, for $j = M+1, \dots, M+N+1$, and insert edges from z_{M+2} to $z_1, \dots, z_M$ and from $z_{M+3}, \dots, z_{M+N+2}$ to z_1. Insert edges from v_1 to z_1, from z_1 to v_2, from v_4 to z_{M+2}, and from z_{M+2} to v_3. Insert edges from z_j to $w_{(j-2)(f(n)-1)+1}, w_{(j-2)(f(n)-1)+2}, \dots, w_{(j-1)(f(n)-1)}$, for $j = 2, \dots, M+1$; insert edges from $w'_{(j-2)(f(n)-1)+1}, w'_{(j-2)(f(n)-1)+2}, \dots, w'_{(j-1)(f(n)-1)}$ to z_{M+j}, for $j = 3, \dots, N+2$. See Fig. 8(b).

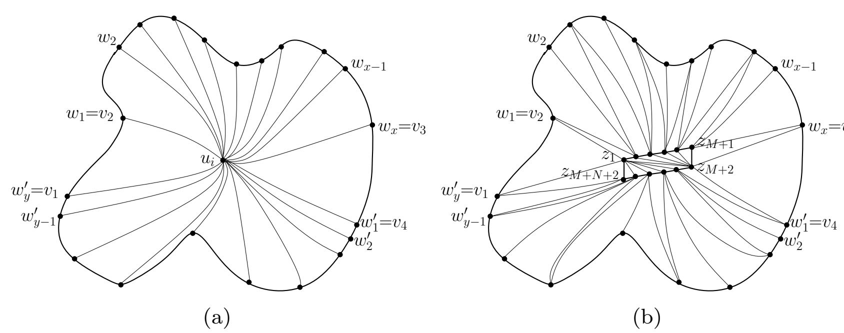


Figure 8: (a) Neighbors of a high-degree vertex u_i. (b) Replacing u_i with lower-degree vertices, assuming $f(n) = 3$.

It is easy to see that the triangulation G' obtained from G after all vertices $u_1, \dots, u_k$ have been considered is upward planar. We have the following.

Claim 9 *G' has $O(n)$ vertices and $O(f(n))$ degree. Moreover, for every upward vertex ordering σ' of G', there exists an upward vertex ordering σ of G such that σ and σ' restricted to the vertices that are both in G and in G' coincide.*

Proof: First, we prove that G' has $O(n)$ vertices. When vertex u_i is removed, $O(\frac{n}{f(n)})$ vertices are introduced, for $i = 1, \dots, k$. Since $k = O(\frac{n}{f(n)})$, then the number of vertices of G' not in G is $O(\frac{n^2}{(f(n))^2})$. Since $f(n) = \Omega(\sqrt{n})$, then G' has $O(n)$ vertices.

Second, we prove that the degree of every vertex in G' is $O(f(n))$. Consider a vertex v that belongs to G before vertex u_i is removed from G. Two cases are possible. In the first case v is not incident to u_i, and then v does not get any new neighbors from the modifications that are performed on G when u_i is removed; in the second case v is incident to u_i, and then v gets at most two new neighbors and loses one, namely u_i. It follows that the number of edges incident to v in G' is at most the number of edges incident to v when it first appears in G plus k, where $k = O(\frac{n}{f(n)})$. Observe that if v also belongs to the original triangulation G, then it has degree $O(f(n))$, given that is not in $u_1, \dots, u_k$; otherwise, v is inserted in G when vertex u_i is deleted, for a certain $1 \leq i \leq k$. The degree of v after its insertion is $O(f(n))$, since such a vertex is connected to $O(f(n))$ neighbors of u_i and to $O(\frac{n}{f(n)}) = O(f(n))$ newly inserted vertices. It follows that the degree of G' is $O(f(n))$.

Third, we consider any upward vertex ordering σ' of G', and we show how to obtain an upward vertex σ of G such that σ and σ' restricted to the vertices that are both in G and in G' coincide. We construct σ from σ' by inserting u_i and by removing the vertices which have been introduced in G to replace u_i, for $i = k, k-1, \dots, 1$. In order to show that u_i can be inserted in σ' yielding an upward vertex ordering of the current triangulation, it suffices to show that all the vertices $w_1, \dots, w_x$ come after all the vertices $w'_1, \dots, w'_y$ in σ'. Namely, in such a case, vertex u_i can be inserted in σ' at any position after all of $w'_1, \dots, w'_y$ and before all of $w_1, \dots, w_x$. Observe that, because of edges (z_j, z_{j+1}), with $j = 1, \dots, M$, all the vertices z_a come after z_1 in σ', for $a = 2, \dots, M+1$; since every vertex w_b, with $b = 1, \dots, x$ has an incoming edge from a vertex z_a, for some $a = 1, \dots, M+1$, it follows that all the vertices $w_1, \dots, w_x$ come after z_1 in σ'. Analogously, all the vertices $w'_1, \dots, w'_y$ come before z_{M+2} in σ'. Finally, because of edges (z_{M+2}, z_1), all the vertices $w_1, \dots, w_x$ come after all the vertices $w'_1, \dots, w'_y$ in σ'. □

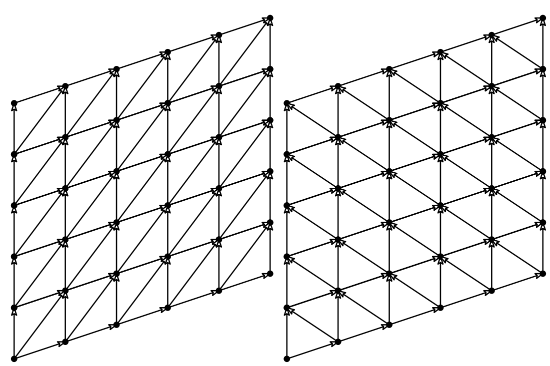
We now describe how to compute a book embedding of G in $O(g(n) + \frac{n}{f(n)})$ pages. First, construct the upward planar triangulation G' as above. Second, construct a book embedding of G' into $O(g(n))$ pages. Such a book embedding exists by hypothesis, since G' has $O(n)$ vertices and $O(f(n))$ degree (by Claim 9). Denote by σ' the total ordering of the vertices of G' in the constructed book embedding. Construct any total ordering σ of the vertices of G such that σ and σ' restricted to the vertices that are both in G and in G' coincide. Such an ordering exists (and can be easily constructed) by Claim 9. The edges of G can be assigned to pages as follows: $O(g(n))$ pages suffice to accommodate all the edges that are both in G and in G'; moreover, one page can be used to accommodate all the edges incident to vertex u_i, for $i = 1, \dots, k \in O(\frac{n}{f(n)})$. It follows that G has a book embedding in $O(g(n) + \frac{n}{f(n)})$ pages, thus proving Theorem 3.

Corollary 4 *Every n-vertex upward planar triangulation has $o(n)$ page number* *if and only if every n-vertex upward planar triangulation with degree $O(\sqrt{n})$ has $o(n)$ page number.*

6 Conclusions

In this paper we studied the relationship between the page number of an upward planar triangulation G and three important parameters of G: The connectivity, the diameter, and the degree. It would be interesting, in our opinion, to understand whether the statements of Theorems 1 and 2 can be referred to the page number rather than to the maximum twist size. That is: (1) Is it true that any upward planar triangulation G has page number $O(k)$ if and only if every maximal 4-connected subgraph of G has page number $O(k)$? (2) Is it true that any n-vertex upward planar triangulation G with diameter D has page number $p(n)$ satisfying $p(n) = p(\frac{n}{2}) + aD + b$, for some constants a and b?

Since improving the $O(n)$ upper bound for the page number of upward planar DAGs seems to be a hard nut to crack, it is natural to look for a lower bound, which would be provided by an upward planar triangulation with superconstant page number. In light of Theorem 1, it is enough to consider 4-connected triangulations; moreover, Theorem 2 suggests that we should better consider triangulations whose diameter is not too small. Thus, an upward planar internally-triangulated mesh seems to be a good candidate for such a lower bound. However, in the following we show that the page number of the two regular triangulations of a mesh (depicted in Fig. 9) is constant.



(a) (b)

Figure 9: Two ways how to internally triangulate a mesh.

We provide a total ordering of the vertices of the internally-triangulated mesh depicted in Fig. 9(a) with constant maximum twist size. Such a total ordering is shown in Fig.10(a) and defined as follows. First, we identify the vertices of the $n \times n$ mesh with the elements of the integer lattice $[0; n-1] \times [0; n-1] \subset \mathbb{Z}^2$ in the natural way. Second, we partition the vertices of the lattice (and hence the vertices of the mesh) into the sets $L_i = \{(x, y) \in \mathbb{Z}^2 | \; 2i \leq x+y \leq 2i+1; 0 \leq x, y \leq n-1\}$, with $i = 0, \dots, n-1$. Third, we order the elements in each set L_i:

$$(2i,0),(2i+1,0),(2i-1,1),(2i,1),\dots,(0,2i),(1,2i),(0,2i+1), \quad \text{if } i \text{ is } \textit{even};$$
$$(0,2i),(0,2i+1),(1,2i-1),(1,2i),\dots,(2i,0),(2i,1),(2i+1,0), \quad \text{if } i \text{ is } \textit{odd}.$$

Finally, we get a total ordering of the vertices of the $n \times n$ mesh by concatenating the above orders so that all the elements in L_i precede all the elements in L_j whenever $i < j$.

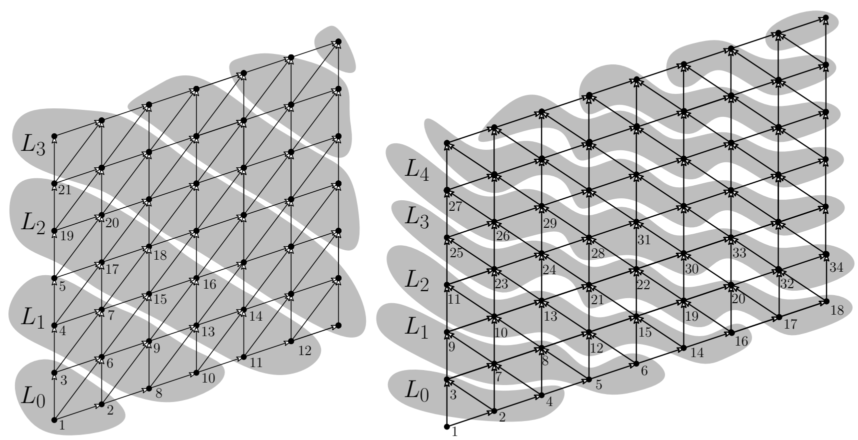


Figure 10: Orderings of the vertices in the triangulated grids yielding a constant page number.

We now provide a total ordering of the vertices of the internally-triangulated mesh depicted in Fig. 9(b) with constant maximum twist size. Such a total ordering is shown in Fig.10(b) and defined as follows. Similarly to the previous case, we associate the elements in the mesh with a suitable subset of the integer lattice. Then, we define a partition of the elements in the infinite integer lattice whose coordinates are both non-negative into sets $L'_i = \{(2i+1,0),(2i+2,0),(2i-1,1),(2i,1),\dots,(1,i),(2,i),(0,i+1)\}$, for $i = -1, 0, \dots$. We order the elements in each L'_i as follows:

$$(1,i),(0,i+1),(3,i-1),(2,i),\dots(2i+1,0),(2i,1),(2i+2,0), \quad \text{if } i \text{ is } \textit{even};$$
$$(2i+1,0),(2i+2,0),(2i-1,1),(2i,1),\dots,(1,i),(2,i),(0,i+1), \quad \text{if } i \text{ is } \textit{odd}.$$

The ordering of the elements in each set L'_i defines a total ordering of the vertices of the mesh associated with such elements. Similarly to the previous case, a total ordering of the vertices of the mesh is then obtained by imposing that all the elements in L'_i precede all the elements in L'_j whenever $i < j$.

We now sketch the reason why the described total orderings of the vertices of the meshes do not create twists of large size. We will argue about the mesh in

Fig. 9(a), the argument for the mesh in Fig. 9(b) being analogous. First, observe that the removal of the vertices in L_i and of their incident edges disconnects the mesh, for each $1 \leq i \leq n-2$. Since the ordering of the vertices of the mesh is such that all the elements in L_i precede all the elements in L_j whenever $i < j$, we get that all the edges in any twist are incident to vertices in the same set L_i, for some $1 \leq i \leq n$. The edges connecting two vertices in L_i cannot participate in a large twist as the end-vertices of any such edge differ by at most three positions in the ordering. On the other hand, the end-vertices of the edges connecting vertices in L_i to vertices in L_{i+1} can be arbitrarily far from each other in the constructed orderings. However, if the vertices in L_i are ordered "from left to right" then those in L_{i+1} are ordered "from right to left", and vice versa. Thus, most of the pairs of edges connecting vertices in L_i with vertices in L_{i+1} are nested, hence they do not create twists of large size.

The way how we construct the orderings for the two above internally-triangulated meshes suggests a general strategy to order the vertices of any upward planar DAG G that might lead to vertex orderings with small maximum twist size: First, partition the set of vertices of G into subsets $S_0, \dots, S_k$ such that the vertices in S_i are connected only to vertices in S_{i-1} (if such a set exists), to vertices in S_i, and to vertices in S_{i+1} (if such a set exists). Second, order the vertices in each set S_i "from left to right" if i is even and "from right to left" if i is odd. Finally, concatenate the orders of S_i's in such a way that all the vertices in S_i precede all the vertices in S_j whenever $i < j$. Even though in many cases, especially when the structure of G is regular, adapting this strategy is fairly simply, for a general upward planar DAG this does not seem to be the case.

Determining whether every n-vertex upward planar DAG has $o(n)$ page number and whether there exist upward planar DAGs with $\omega(1)$ page number remain among the most important problems in the theory of linear graph layouts.

Acknowledgements

The first author would like to thank Patrizio Angelini, Giuseppe Di Battista, and Stefano Saraulli for very useful discussions.

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