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Universal Point Sets for Drawing Planar Graphs with Circular Arcs

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Abstract

We prove that there exists a set S of n points in the plane such that every n-vertex planar graph G admits a planar drawing in which every vertex of G is placed on a distinct point of S and every edge of G is drawn as a circular arc.

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1 Introduction

It is a classic result of graph theory that every planar graph has a planar *straight-line drawing*, that is, a drawing where vertices are mapped to points in the plane and edges to straight-line segments connecting the corresponding points (achieved independently by Wagner, Fáry, and Stein). Tutte [21] presented the first algorithm, the *barycentric method*, that produces such drawings. Unfortunately, the barycentric method can produce edges whose lengths are exponentially far from each other. Therefore, Rosenstiehl and Tarjan [19] asked whether every planar graph has a planar straight-line drawing where vertices lie on an integer grid of polynomial size. De Fraysseix, Pach, and Pollack [7] and, independently, Schnyder [20] answered this question in the affirmative. Their (very different) methods yield drawings of n-vertex planar graphs on a grid of size $\Theta(n) \times \Theta(n)$, and there are graphs (the so-called "nested triangle graphs") that require this grid size [11].

Later, it was apparently Mohar (according to Pach [8]) who generalized the grid question to the following problem: What is the smallest value $f(n)$ of a *universal point set* for planar straight-line drawings of n-vertex planar graphs, that is, the smallest size (as a function of n) of a point set S such that every n-vertex planar graph G admits a planar straight-line drawing in which the vertices of G are mapped to points in S? The question is listed as problem #45 in the Open Problems Project [8]. Despite more than twenty years of research efforts, the best known lower bound for the value of $f(n)$ is linear in n [6, 18], while only an $O(n^2)$ upper bound is known, as first established by de Fraysseix et al. [7] and Schnyder [20]. Very recently, Bannister et al. [2] showed a universal point set with $n^2/4 - \Theta(n)$ points for planar straight-line drawings of n-vertex planar graphs. Universal point sets for planar straight-line drawings of planar graphs require more than n points whenever $n \geqslant 15$ [5]. Universal point sets with $o(n^2)$ points have been proved to exist for planar straight-line drawings of several subclasses of planar graphs, including simply-nested planar graphs [1, 2], planar 3-trees [15], and graphs of bounded pathwidth [2].

Universal point sets have also been studied with respect to different drawing standards. For example, Everett et al. [14] showed that there exist sets of n points that are universal for planar poly-line drawings with one bend per edge of n-vertex planar graphs. On the other hand, if bends are required to be placed on the point set, universal point sets of size $O(n^2/\log n)$ exist for drawings with one bend per edge, of size $O(n \log n)$ for drawings with two bends per edge, and of size $O(n)$ for drawings with three bends per edge [12].

However, smooth curves may be easier for the eye to follow and more aesthetic than poly-lines. Graph drawing researchers have long observed that poly-lines may be made smooth by replacing each bend with a smooth curve tangent to the two adjacent line segments [9, 16]. Bekos et al. [3] formalized this observation by considering smooth curves made of line segments and circular arcs; they define the *curve complexity* of such a curve to be the number of segments and arcs it contains. A poly-line drawing with s segments per edge may be transformed into a smooth drawing with curve complexity at most $2s - 1$, but Bekos et al. [3] observed that in many cases the curve complexity can be made smaller than this bound. For instance, replacing poly-lines by curves in the construction of Everett et al. [14] would give rise to a drawing with curve complexity 3, but in fact every set of n collinear points is universal for smooth piecewise-circular drawings with curve complexity 2, as can be derived from the existence of topological book embeddings of planar graphs [3, 10, 17]. A *monotone topological book embedding* of a graph is a drawing of that graph such that the vertices lie on a horizontal line, called the *spine*, and the edges are represented by non-crossing curves, monotonically increasing in the direction of the spine. Di Giacomo et al. [10] and, independently, Giordano et al. [17] showed that every planar graph has a monotone topological book embedding where each edge crosses the spine exactly once and is represented by the union of two semi-circles that lie below and above the spine (see Figure 2).

The difficulty of constructing a universal point set of linear size for straight-line drawings, the aesthetical properties of smooth curves, the recent developments on drawing planar graphs with circular arcs (see, for example, [4, 13]), and the existence of universal sets of n points for planar drawings of planar graphs with curve complexity 2 [14] naturally give rise to the question of whether there exists a universal set of n points for drawings of planar graphs with curve complexity 1, that is, for planar drawings in which every edge is drawn as a single circular arc. In this paper, we answer this question in the affirmative.

We prove the existence of a set S of n points on the parabola $\mathcal{P}$ of equation $y = -x^2$ such that every n-vertex planar graph G can be drawn with the vertices mapped to S and the edges mapped to non-crossing circular arcs. Our proof is constructive and allows us to specify the planar embedding[1] of G. We draw G in two steps, in the same spirit as Everett et al. [14]. In the first step, we construct a monotone topological book embedding of G. In the second step, we map the vertices of G to the points in S in the same order as they appear on the spine of the book embedding.

2 Circular Arcs Between Points on a Parabola

In this section, we investigate geometric properties of circular-arc drawings whose vertices lie on the parabola $\mathcal{P}$. Let $\mathcal{P}^+$ be the part of $\mathcal{P}$ to the right of the y-axis, that is, $\mathcal{P}^+ = \{(x, y) \colon x \geqslant 0, y = -x^2\}$.

In the following, when we say that a point is *to the left* of another point, we mean that the x-coordinate of the former is smaller than that of the latter. We say that an arc is to the left of a point q, when the horizontal line through q intersects the arc and all the intersection points are to the left of q. We define similarly *to the right*, *above*, and *below*, and we naturally extend these definitions to non-crossing pairs of arcs.

[1] A *planar embedding* of a planar graph is a representation of the graph in which its vertices are identified to distinct points in the plane and its edges are associated to simple arcs that do not intersect except at common vertices.

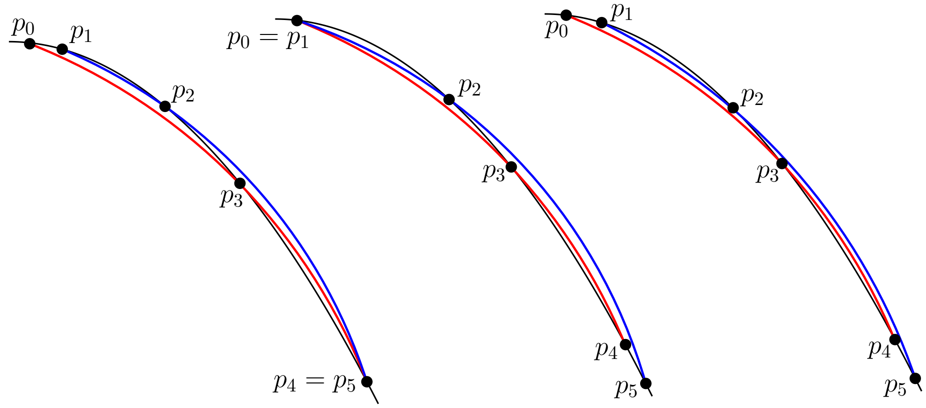

Figure 1: Three configurations of relative position of the circular arcs $C_{0,3,4}$ (red) and $C_{1,2,5}$ (blue) defined by six points $p_0, \ldots, p_5$ lying in that order on $\mathcal{P}^+$ (black). For readability, the figure is not to scale.

We denote by $\mathcal{C}(p, q, r)$ the circle through three points p, q, and r. Also, we denote by $C_{p,q,r}$ the circular arc in $\mathcal{C}(p, q, r)$ delimited by p and r and containing q; in the sequel, a variant of this notation will also be used: $C_{i,j,k}$, for three points p_i, p_j, and p_k. For any point p, we denote by x_p and y_p its x- and y-coordinates.

We start by stating a property of parabolas and circles.

Lemma 1 *For every three points p, q, and r on $\mathcal{P}^+$ with $x_p < x_q < x_r$, the circle $\mathcal{C}(p, q, r)$ intersects the parabola $\mathcal{P}$ in p, q, r and in a point of x-coordinate $-x_p - x_q - x_r$. Furthermore, the circular arc $C_{p,q,r}$ is below $\mathcal{P}$ between p and q, and above $\mathcal{P}$ between q and r.*

Proof: The equation of $\mathcal{C}(p, q, r)$ is

$$\begin{vmatrix} x_p & -x_p^2 & x_p^2 + x_p^4 & 1 \\ x_q & -x_q^2 & x_q^2 + x_q^4 & 1 \\ x_r & -x_r^2 & x_r^2 + x_r^4 & 1 \\ x & y & x^2 + y^2 & 1 \end{vmatrix} = 0.$$

Substituting y by $-x^2$ gives

$$(x_p - x_q)(x_p - x_r)(x_q - x_r)(x - x_p)(x - x_q)(x - x_r)(x + x_p + x_q + x_r) = 0,$$

which yields the first claim.

Let s denote the point on $\mathcal{P}$ with x-coordinate $-x_p - x_q - x_r$. It is straightforward to see that the arc of $\mathcal{C}(p, q, r)$ between s and r and not containing p is below $\mathcal{P}$. The second claim follows. $\square$

Consider six points $p_0 = (x_0, y_0), \ldots, p_5 = (x_5, y_5)$ on $\mathcal{P}^+$, with $x_0 \leqslant x_1 < x_2 < x_3 < x_4 \leqslant x_5$. Also, consider the following two circular arcs (see Figure 1):

$C_{0,3,4}$ (red) goes through the ordered points p_0, p_3, p_4 and $C_{1,2,5}$ (blue) goes through p_1, p_2, p_5.

Arcs $C_{0,3,4}$ and $C_{1,2,5}$ may not be x-monotone: Consider, for instance, the limit case in which p_0 and p_3 lie at the origin and $x_4 > 1$. Then the circle $\mathcal{C}(p_0, p_3, p_4)$ supporting $C_{0,3,4}$ has its center $(0, -r)$ on the y-axis and radius $r > 1$. The rightmost point $(r, -r)$ of that circle lies above $\mathcal{P}$ (since $-r > -r^2$) and thus on the arc $C_{0,3,4}$ by Lemma 1.

Arcs $C_{0,3,4}$ and $C_{1,2,5}$ are, however, y-monotone, as proved in the following lemma.

Lemma 2 *Arcs $C_{0,3,4}$ and $C_{1,2,5}$ are y-monotone.*

Proof: We prove the statement for $C_{0,3,4}$; the argument for $C_{1,2,5}$ is similar. By Lemma 1, p_0 lies on the right half-circle of $\mathcal{C}(p_0, p_3, p_4)$. Further, by assumption, $x_0 < x_3 < x_4$. Hence, p_0, p_3, and p_4 all lie on the right half-circle of $\mathcal{C}(p_0, p_3, p_4)$ and the statement follows. $\square$

We will prove in the following three lemmata that the arcs $C_{0,3,4}$ and $C_{1,2,5}$ do not intersect each other, except possibly at common endpoints, if $x_0 \geqslant 1$ and if $x_i \geqslant 2x_{i-1}$ for $i = 3, 4$. We consider in these lemmata three cases depending on whether these arcs share one of their endpoints. Refer to Figure 1.

Lemma 3 *If $p_0 \neq p_1$, $p_4 = p_5$, and $x_3 \geqslant x_1 + x_2$, the two circular arcs $C_{0,3,4}$ and $C_{1,2,5}$ intersect only at $p_4 = p_5$.*

Proof: Let $q_{0,3,4}$ be the fourth intersection point between $\mathcal{C}(p_0, p_3, p_4)$ and $\mathcal{P}$, and define similarly $q_{1,2,5}$. By Lemma 1, $q_{0,3,4}$ and $q_{1,2,5}$ have x-coordinates $-x_0 - x_3 - x_4$ and $-x_1 - x_2 - x_5$, respectively. It follows that $q_{0,3,4}$ coincides with or is to the left of $q_{1,2,5}$, because $x_3 \geqslant x_1 + x_2$, $x_4 = x_5$, and $x_0 \geqslant 0$.

Furthermore, by Lemma 1, the arc of $\mathcal{C}(p_0, p_3, p_4)$ between $q_{0,3,4}$ and p_0 and not containing p_4 is above $\mathcal{P}$, and similarly for the arc of $\mathcal{C}(p_1, p_2, p_5)$ between $q_{1,2,5}$ and p_1 and not containing p_5. These two arcs are above $\mathcal{P}$ and their endpoints alternate on $\mathcal{P}$, thus they intersect. It follows that the two circles $\mathcal{C}(p_0, p_3, p_4)$ and $\mathcal{C}(p_1, p_2, p_5)$ intersect in that point and in $p_4 = p_5$. Hence, the arcs $C_{0,3,4}$ and $C_{1,2,5}$ intersect only at $p_4 = p_5$. $\square$

Lemma 4 *If $p_0 = p_1$, $p_4 \neq p_5$, $x_0 \geqslant 1$, $x_3 \geqslant 2x_2$, and $x_4 \geqslant x_0 + x_3$, the two circular arcs $C_{0,3,4}$ and $C_{1,2,5}$ intersect only at $p_0 = p_1$.*

Proof: Refer to the middle configuration in Figure 1. We start by stating the following.

Claim 1 *$C_{1,2,5}$ is to the right of $C_{0,3,4}$ in a neighborhood of p_0.*

We first argue that Claim 1 implies Lemma 4. Suppose, for a contradiction, that $C_{1,2,5}$ is to the right of $C_{0,3,4}$ in a neighborhood of p_0 and that $C_{0,3,4}$ and $C_{1,2,5}$ intersect in a point q other than p_0. Since $C_{1,2,5}$ is above $\mathcal{P}$ in a neighborhood of p_5, and $C_{1,2,5}$ does not intersect $\mathcal{P}$ between p_4 and p_5 (by

Lemma 1), we have that $C_{1,2,5}$ is to the right of p_4. On the other hand, $C_{0,3,4}$ and $C_{1,2,5}$ intersect in no point other than q and p_0. Hence, since $C_{0,3,4}$ and $C_{1,2,5}$ intersect at q, their horizontal ordering changes in a neighborhood of q and thus $C_{1,2,5}$ is to the left of $C_{0,3,4}$ in a neighborhood of p_0, a contradiction.

In order to prove Lemma 4, it remains to prove Claim 1. We can assume without loss of generality that p_5 is at infinity, which means that $C_{1,2,5}$ is the straight ray from $p_0 = p_1$ through p_2. Indeed, for any point p'_5 that lies on $\mathcal{P}$ to the right of p_5, point p'_5 lies outside $\mathcal{C}(p_1, p_2, p_5)$, by Lemma 1. Furthermore, since p'_5 lies below p_1 and p_2, the arc through p_1, p_2, and p'_5 (in that order) lies to the left of $C_{1,2,5}$ between p_1 and p_2. Hence, if the (blue) arc $C_{1,2,5}$ is to the left of $C_{0,3,4}$ in a neighborhood of p_0, it remains to the left if p_5 moves to infinity.

Now, we prove that the tangents at $p_0 = p_1$ of $C_{0,3,4}$ and $C_{1,2,5}$ never coincide. With the above assumption, this is equivalent to showing that the normal to $C_{0,3,4}$ at p_0 is never orthogonal to the segment p_1p_2. Straightforward computations (though tedious by hand) give that the dot product of that normal and p_1p_2 is equal to

$$(x_4 - x_3)\,(x_4 - x_0)\,(x_3 - x_0)\,(x_2 - x_0)\cdot$$
$$\Big((x_3 - x_2)\,x_4^2 + (x_3 - x_2)\,(x_0 + x_3)\,x_4 + \big((x_0^2 - 1 - x_3x_0 - x_3^2)\,x_2 + x_0^3 + x_0\big)\Big).$$

The first four factors never vanish. We show that the last factor, seen as a polynomial in x_4, has no root larger than $x_0 + x_3$ (it can be shown that this polynomial has a positive root). For that purpose, we make the change of variable $x_4 = t + x_0 + x_3$ which maps the interval $(x_0 + x_3, +\infty)$ of x_4 to the interval $(0, +\infty)$ of t and maps the above degree-2 polynomial in x_4 to

$$\begin{aligned}&(x_3 - x_2)\,t^2 + 3\,(x_3 - x_2)\,(x_0 + x_3)\,t\\&-(1 + x_0^2 + 5x_0x_3 + 3x_3^2)\,x_2 + x_0 + 4x_0x_3^2 + x_0^3 + 2x_3^3 + 2x_0^2x_3\end{aligned}$$

whose first and second coefficients are positive and whose last coefficient is positive for any $x_2 \in [x_0, x_3/2]$ since it is linear in x_2 and takes value $x_3\,(3x_0 + 2x_3)\,(x_3 - x_0)$ at x_0 and value $\frac{1}{2}x_3\,(-1 + x_3^2 + 3x_0^2 + 3x_0x_3) + x_0 + x_0^3$ at $x_3/2$, which is positive since $x_0 \geqslant 1$. Note that the last coefficient is negative when $x_2 = x_3$ which is why we consider x_2 in the range $[x_0, x_3/2]$. Hence, if $x_3 \geqslant 2x_2$, all coefficients of this polynomial are positive, which implies that it has no positive roots. This, in turn, means that the initial degree-2 polynomial in x_4 has no root larger than $x_0 + x_3$.

Hence, there is no position of the points $p_0 = p_1, p_2 \ldots, p_5$ such that $x_3 \geqslant 2x_2$, $x_4 \geqslant x_0 + x_3$, and such that the tangent to $C_{0,3,4}$ is collinear with p_0p_2. Furthermore, at the limit case where $p_2 = p_0$, segment p_0p_2 is tangent to $\mathcal{P}$, and $C_{0,3,4}$ is below and to the left of that tangent in a neighborhood of p_0, by Lemma 1. Thus, $C_{0,3,4}$ is to the left of segment p_1p_2 in a neighborhood of p_0, and hence to the left of $C_{1,2,5}$ in a neighborhood of p_0. This proves Claim 1 and hence Lemma 4. $\square$

We are now ready to prove the following.

Lemma 5 *If $p_0, \ldots, p_5$ are pairwise disjoint, $x_0 \geqslant 1$, and $x_i \geqslant 2x_{i-1}$ for $i = 3, 4$, the two circular arcs $C_{0,3,4}$ and $C_{1,2,5}$ do not intersect.*

Proof: We refer to the right configuration in Figure 1. Unless specified otherwise, an arc p_ip_j refers to the arc from p_i to p_j on the arc $C_{0,3,4}$ or $C_{1,2,5}$ that supports both p_i and p_j. We first prove that the arcs p_2p_5 and p_3p_4 do not intersect. For any point q on $\mathcal{P}$ between p_4 and p_5, the arc p_3q on the circular arc through p_0, p_3, q lies above the concatenation of the arcs p_3p_4 of $C_{0,3,4}$ and p_4q of $\mathcal{P}$ (since the circular arcs p_3q and p_3p_4 lie above $\mathcal{P}$, by Lemma 1, and $\mathcal{C}(p_0, p_3, p_4)$ and $\mathcal{C}(p_0, p_3, q)$ intersect only at p_0 and p_3). It follows that if arc p_3p_4 intersects arc p_2p_5, then arc p_3q also intersects arc p_2p_5 for any position of q between p_4 and p_5 on $\mathcal{P}$. This implies that, for the limit case where $q = p_5$, arc $C_{1,2,5}$ and the circular arc through p_0, p_3, and $q = p_5$ intersect in some point other than $q = p_5$, which is not the case by Lemma 3.

We now prove, similarly, that the arcs p_0p_3 and p_1p_2 do not intersect. For any point q on $\mathcal{P}$ between p_0 and p_1, the arc qp_2 on the circular arc through q, p_2, p_5 lies below the concatenation of the arcs qp_1 of $\mathcal{P}$ and p_1p_2 of $C_{1,2,5}$. It follows that if arc p_1p_2 intersects arc p_0p_3, then arc qp_2 also intersects arc p_0p_3 for any position of q between p_0 and p_1 on $\mathcal{P}$. This implies that, for the limit case where $q = p_0$, arc $C_{0,3,4}$ and the circular arc through $q = p_0$, p_2, and p_5 intersect in some point other than $q = p_0$, which is not the case by Lemma 4.

Finally, arcs p_1p_2 of $C_{1,2,5}$ and p_3p_4 of $C_{0,3,4}$ do not intersect because they lie on different sides of $\mathcal{P}$ and similarly for arcs p_0p_3 of $C_{0,3,4}$ and p_2p_5 of $C_{1,2,5}$. Hence, the two arcs $C_{0,3,4}$ or $C_{1,2,5}$ do not intersect. □

3 Universal Point Set for Circular Arc Drawings

In this section, we construct a set of n points on $\mathcal{P}$ and, by using the lemmata of the previous section, we prove that it is universal for planar circular arc drawings of n-vertex planar graphs.

Consider n^2 points[2] $q_0, \ldots, q_{n^2-1}$ on the parabolic arc $\mathcal{P}^+$ such that $x_0 \geqslant 1$ and $x_i \geqslant 2x_{i-1}$ for $i = 1, \ldots, n^2 - 1$ and consider as a universal point set the n points $p_i = q_{ni}$ for $i = 0, \ldots, n-1$. The points that belong to $q_0, \ldots, q_{n^2-1}$ but are not in the universal point set are called *helper points.*

Theorem 1 *Every n-vertex planar graph can be drawn with the vertices on $p_0, \ldots, p_{n-1}$ and with the edges drawn as circular arcs that do not intersect except at common endpoints.*

Proof: Let G be a planar graph with n vertices. Construct a monotone topological book embedding Γ of G in which each edge has exactly one spine crossing [10, 17]. Denote by $w_0, \ldots, w_{n-1}$ the order of the vertices of G on the spine in Γ. We substitute every spine crossing with a *dummy* vertex and denote by Γ' the resulting embedded graph. The relative position of any two edges in Γ is as depicted in Figure 2 (in which two edges may share their endpoints).

[2]We consider n^2 points for simplicity but we do not actually use the last $n-1$ of them.

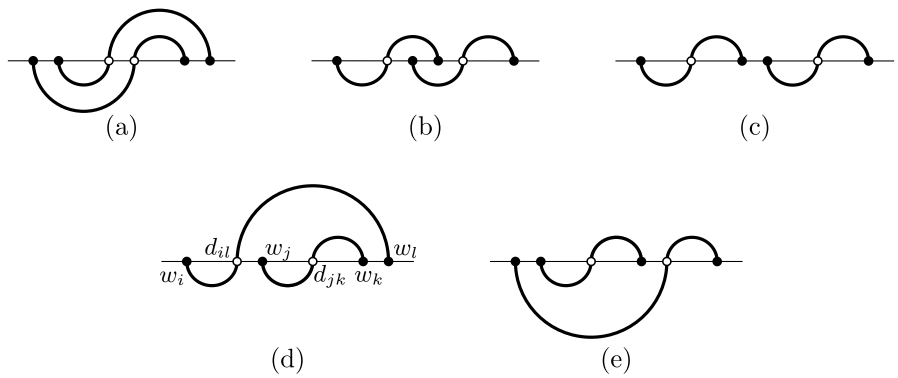

Figure 2: Relative positions of two edges in a monotone topological book embedding.

For $0 \leqslant i \leqslant n-1$, we map vertex w_i to point p_i. Furthermore, for each $0 \leqslant i \leqslant n-2$, we map the dummy vertices that lie in between w_i and w_{i+1} on the spine in Γ' to distinct helper points in between p_i and p_{i+1}, so that the order of the dummy vertices on $\mathcal{P}$ is the same as on the spine in Γ'. (We postpone the proof that there are enough helper points to map the dummy vertices.) We finally draw every edge (w_i, w_j) of G containing a dummy vertex d_l as a circular arc that is delimited by p_i and p_j, and that passes through the helper point onto which vertex d_l has been mapped. We prove that the resulting drawing is planar. In the following, we say that the circular-arc drawings of two edges in Γ or Γ' *do not intersect* if they do not intersect except at common endpoints.

By Lemmata 3, 4, and 5, the circular-arc drawings of any two edges whose relative positions in Γ are as depicted in Figure 2(a) do not intersect.

For the pairs of edges whose relative positions in Γ are as depicted in Figures 2(b) and 2(c), it is straightforward to check that their circular-arc drawings do not intersect: any two edges in Γ' are either separated by the spine or by a vertical line, hence their circular-arc drawings are either separated by $\mathcal{P}$ or by a horizontal line (since by Lemma 2 the circular-arc drawings are y-monotone).

Consider two edges (w_i, w_l) and (w_j, w_k) whose relative position in Γ is as depicted in Figure 2(d) and consider the corresponding four edges in Γ' (the argument for the pairs of edges as in Figure 2(e) is analogous). Denote by d_{il} and d_{jk} the dummy vertices of these edges, and by q_{il} and q_{jk} the helper points on $\mathcal{P}$ onto which they are mapped. Edges (d_{il}, w_l) and (w_j, d_{jk}) are separated by the spine in Γ', hence their circular-arc drawings do not intersect since they are separated by $\mathcal{P}$. The same argument holds for edges (w_i, d_{il}) and (d_{jk}, w_k). Further, edges (w_i, d_{il}) and (w_j, d_{jk}) are separated by a vertical line in Γ', hence their circular-arc drawings do not intersect since they are separated by a horizontal line (since by Lemma 2 the circular-arc drawings are y-monotone).

Hence, it suffices to prove that the circular-arc drawings of the edges (d_{il}, w_l) and (d_{jk}, w_k) do not intersect. The circular-arc drawing of (d_{jk}, w_k) is the arc

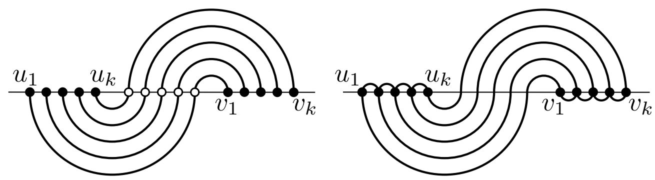

Figure 3: (a) k edges of a monotone topological book embedding that defines k consecutive dummy vertices (spine crossings). (b) Augmented outerplanar graph.

$q_{jk}p_k$ on $\mathcal{C}(p_j, q_{jk}, p_k)$. Roughly speaking, this arc inflates if p_j moves left on $\mathcal{P}$, by Lemma 1. More formally, for any point r on $\mathcal{P}^+$ that is left of p_j, the arc $q_{jk}p_k$ on $\mathcal{C}(r, q_{jk}, p_k)$ lies above the arc $q_{jk}p_k$ on $\mathcal{C}(p_j, q_{jk}, p_k)$. Hence, if the former arc $q_{jk}p_k$ does not intersect the circular-arc drawing of (d_{il}, w_l), neither does the latter. By considering $r = p_i$, these arcs do not intersect by Lemma 4 if $p_k \neq p_l$ and, if $p_k = p_l$, they also do not intersect since their supporting circles are distinct ($q_{il} \neq q_{jk}$) and intersect in the two points $r = p_i$ and $p_k = p_l$.

It remains to show that there are enough helper points to map the dummy vertices. There are $n-1$ helper points $q_{ni+1}, \ldots, q_{n(i+1)-1}$ between each pair of points $p_i = q_{ni}$ and $p_{i+1} = q_{n(i+1)}$. It thus suffices to prove that there are at most $n-1$ dummy vertices in between v_i and v_{i+1} along the spine in Γ'.

Let $(u_1, v_1), \ldots, (u_k, v_k)$ be k edges in the book embedding Γ that define consecutive dummy vertices on the spine (possibly $u_i = u_{i+1}$, for any $1 \leqslant i \leqslant k-1$; also, possibly $v_i = v_{i+1}$, for any $1 \leqslant i \leqslant k-1$; however, $u_i = u_{i+1}$ and $v_i = v_{i+1}$ do not hold simultaneously, for any $1 \leqslant i \leqslant k-1$). If no vertex w_i lies in between these dummy vertices on the spine in Γ, the k edges are such that $u_1, \ldots, u_k, v_1, \ldots, v_k$ are ordered from left to right on the spine in Γ; see Figure 3(a). Now, consider the graph that consists of these edges plus the edges $(u_i, u_{i+1}), (v_i, v_{i+1})$, for $i = 1, \ldots, k-1$; see Figure 3(b). This graph is outerplanar. It has at most n vertices and, thus, at most $n-3$ chords. On the other hand, it has exactly $k-2$ chords: $(u_2, v_2), \ldots, (u_{k-1}, v_{k-1})$. This implies that $k-2 \leqslant n-3$. Hence $k \leqslant n-1$, which concludes the proof. □

In conclusion of this section, we observe that Theorem 1 also holds for a *plane* graph, that is, a planar graph provided with a planar embedding. In the above proof, we compute a monotone topological book embedding of the given plane graph using an algorithm by Giordano et al. [17]. Their algorithm preserves the embedding of the given graph, and so does the rest of our construction.

4 Conclusion

We proved the existence of a universal point set with n points for planar circular arc drawings of planar graphs. The universal point set we constructed has an area of $2^{O(n^2)}$. It would be interesting, also for practical visualization purposes, to construct a universal point set with n points for planar circular arc drawings of planar graphs within polynomial area. In this direction, we remark that (relaxing the requirement that the set has exactly n points) a universal point set with $O(n)$ points and within $2^{O(n)}$ area for planar circular arc drawings of planar graphs is $Q = \{q_0, \ldots, q_{4n-7}\}$ (as defined in Section 3). To construct a planar circular-arc drawing of a planar graph G on Q, it suffices to map the n vertices and the up to $3n-6$ dummy vertices of a monotone topological book embedding of G to the points of Q in the order they appear in the book embedding. The geometric lemmata of Section 2 ensure that the resulting drawing is planar.

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