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Algorithm and Hardness Results for Outer-connected Dominating Set in Graphs

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Abstract

A set $D \subseteq V$ of a graph $G = (V, E)$ is called an outer-connected dominating set of G if for all $v \in V$, $|N_G[v] \cap D| \geq 1$, and the induced subgraph of G on $V \setminus D$ is connected. The Minimum Outer-connected Domination problem is to find an outer-connected dominating set of minimum cardinality of the input graph G. Given a positive integer k and a graph $G = (V, E)$, the Outer-connected Domination Decision problem is to decide whether G has an outer-connected dominating set of cardinality at most k. The Outer-connected Domination Decision problem is known to be NP-complete for bipartite graphs. In this paper, we strengthen this NP-completeness result by showing that the Outer-connected Domination Decision problem remains NP-complete for perfect elimination bipartite graphs. On the positive side, we propose a linear-time algorithm for computing a minimum outer-connected dominating set of a chain graph, a subclass of bipartite graphs. We show that the Outer-connected Domination Decision problem can be solved in linear-time for graphs of bounded tree-width. We propose a $\Delta(G)$-approximation algorithm for the Minimum Outer-connected Domination problem, where $\Delta(G)$ is the maximum degree of G. On the negative side, we prove that the Minimum Outer-connected Domination problem cannot be approximated within a factor of $(1-\varepsilon)\ln|V|$ for any $\varepsilon > 0$, unless NP $\subseteq$ DTIME$(|V|^{O(\log\log|V|)})$. We also show that the Minimum Outer-connected Domination problem is APX-complete for graphs with bounded degree 4 and for bipartite graphs with bounded degree 7.

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1 Introduction

A vertex v of a graph $G = (V, E)$ is said to *dominate* a vertex w if either $v = w$ or $vw \in E$. A set of vertices D is a *dominating set* of G if every vertex of G is dominated by at least one vertex of D. The *domination number* of a graph G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. The Minimum Domination problem is to find a dominating set of minimum cardinality of the input graph G. Given a positive integer k and a graph $G = (V, E)$, the Domination Decision problem is to decide whether G has a dominating set of cardinality at most k. The concept of domination and its variations are widely studied as can be seen in [10, 11].

For a set $S \subseteq V$ of the graph $G = (V, E)$, the subgraph of G *induced* by S is defined as $G[S] = (S, E_S)$, where $E_S = \{xy \in E | x, y \in S\}$. A set $D \subseteq V$ of a graph $G = (V, E)$ is called an *outer-connected dominating set* of G if D is a dominating set of G and $G[V \setminus D]$ is connected. The *outer-connected domination number* of a graph G, denoted by $\widetilde{\gamma}_c(G)$, is the cardinality of a minimum outer-connected dominating set of G. The concept of outer-connected domination number was introduced by Cyman [6] and further studied by others (see [1, 13, 12, 19]). This problem has possible applications in computer networks. Consider a client-server architecture based network in which any client must be able to communicate to one of the servers. Since overloading of severs is a bottleneck in such a network, every client must be able to communicate to another client directly (without interrupting any of the server). A smallest group of servers with these properties is a minimum outer-connected dominating set for the graph representing the computer network.

The Minimum Outer-connected Domination (MOCD) problem is to find an outer-connected dominating set of minimum cardinality of the input graph G. Given a positive integer k and a graph $G = (V, E)$, the Outer-connected Domination Decision (OCDD) problem is to decide whether G has an outer-connected dominating set of cardinality at most k. The Minimum Outer-connected Domination problem is studied for some subclasses of graphs (doubly chordal graphs, undirected path graphs, proper interval graphs and bipartite graphs) [6, 13].

In this paper, we study the algorithmic aspect of the Minimum Outer-connected Domination problem. The OCDD problem is known to be NP-complete for bipartite graphs. We strengthen the NP-completeness result of the OCDD problem by showing that this problem remains NP-complete for perfect elimination bipartite graphs. On the positive side, we propose a linear-time algorithm for computing a minimum outer-connected dominating set of a chain graph. We show that the OCDD problem can be solved in linear-time for graphs of bounded tree-width. Here, we also study the approximation aspect of the problem. We propose a $\Delta(G)$-approximation algorithm for the MOCD problem, where $\Delta(G)$ is the maximum degree of G. On the negative side, we derive some approximation hardness results.

The rest of the paper is organized as follows. In Section 2, some pertinent definitions and preliminary results are presented. In Section 3, the OCDD problem is shown to be NP-complete for perfect elimination bipartite graphs. In Section 4, the complexity difference of the Minimum Domination problem and the MOCD problem are highlighted. In Section 5, a linear-time algorithm for the MOCD problem in chain graphs, a subclass of perfect elimination bipartite graphs, is proposed. In Section 6, it is shown that the OCDD problem can be solved in linear-time for bounded tree-width graphs. In Section 7, an approximation algorithm for the MOCD problem is presented. We also prove that the MOCD problem cannot be approximated within a factor of $(1-\varepsilon)\ln|V|$ for any $\varepsilon > 0$, unless NP $\subseteq$ DTIME$(|V|^{O(\log\log|V|)})$. In Section 8, it is shown that the MOCD problem is APX-complete for graphs with bounded degree 4 and for bipartite graphs with bounded degree 7. Finally, Section 9 concludes the paper.

2 Preliminaries

For a graph $G = (V, E)$, the sets $N_G(v) = \{u \in V(G) | uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the *open neighborhood* and *closed neighborhood* of a vertex v, respectively. For a connected graph G, a vertex v is a *cut vertex* if $G \setminus \{v\}$ is disconnected. The *degree* of a vertex v is $|N_G(v)|$ and is denoted by $d_G(v)$. If $d_G(v) = 1$, then v is called a *pendant vertex*. For $S \subseteq V$, let $G[S]$ denote the subgraph induced by S on G. A graph $G = (V, E)$ is said to be *bipartite* if $V(G)$ can be partitioned into two disjoint sets X and Y such that every edge of G joins a vertex in X to a vertex in Y. Such a partition (X, Y) of V of a bipartite graph $G = (V, E)$ is called a *bipartition*. A bipartite graph with bipartition (X, Y) of V is denoted by $G = (X, Y, E)$. Let n and m denote the number of vertices and number of edges of G, respectively. A graph $H = (V', E')$ is a *spanning subgraph* of $G = (V, E)$ if $V' = V$ and $E' \subseteq E$. A connected acyclic spanning subgraph of G is a *spanning tree* of G. A tree with exactly one non-pendant vertex is a *star* and a tree with exactly two non-pendant vertices is called a *bi-star*.

Let G be a graph, T be a tree and ν be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair (T, ν) is called a *tree-decomposition* of G if it satisfies the following three conditions:

1. $V(G) = \bigcup_{t \in V(T)} V_t$,
2. for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t,
3. $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_1, t_2, t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3.

The *width* of (T, ν) is the number $\max\{|V_t| - 1 : t \in T\}$, and the *tree-width* $tw(G)$ of G is the least width of any tree-decomposition of G [7].

In the rest of the paper, by a graph we mean a connected graph with at least two vertices unless otherwise mentioned specifically. The following observations regarding outer-connected dominating set of a graph will be used throughout the paper.

Observation 1 *(a) Let v be a cut vertex of a connected graph $G = (V, E)$ and let $G[V \setminus \{v\}]$ have k components. If $v \in D$ for an outer-connected dominating set D of G, then D contains all the vertices of $k-1$ components of $G[V \setminus \{v\}]$.*

Proof: *Suppose that the above statement does not hold. Then there exists an outer-connected dominating set D containing v, and two vertices v_i and v_j belonging to two different components, say, G_i and G_j of $G[V \setminus \{v\}]$ such that v_i and v_j are not in D. Now there is no path from v_i to v_j in $G[V \setminus \{v\}]$, and hence there is no path from v_i to v_j in $G[V \setminus D]$. Hence $G[V \setminus D]$ is disconnected, which is a contradiction to the fact that D is an outer-connected dominating set. This proves the Observation 1(a).* □

(b) If v is a pendant vertex of $G = (V, E)$, then either $v \in D$ or $D = V \setminus \{v\}$ for every outer-connected dominating set D of G.

Proof: *Let D be any outer-connected dominating set of G and v be a pendant vertex of G. If $v \in D$, we are done. Suppose that $v \notin D$. Then the vertex adjacent to the pendant vertex v, say w, must belong to D. But w is cut vertex and one component of $G[V \setminus \{w\}]$ is the vertex v itself. By Observation 1(a), the vertices of all the components of $G[V \setminus \{w\}]$ other than one component must belong to D. Since $v \notin D$, $V \setminus \{v\} \subseteq D$, that is, $D = V \setminus \{v\}$. This proves the Observation 1(b).* □

(c) Let $G = (V, E)$ be a connected graph having at least three vertices. Then there is a minimum outer-connected dominating set of G containing all the pendant vertices of G.

Proof: *Let D_1^* be a minimum outer-connected dominating set of G. If D_1^* contains all the pendant vertices of G, then we are done. Assume that D_1^* does not contain a pendant vertex, say v, of G. Then by Observation 1(b), $D_1^* = V \setminus \{v\}$ and $\widetilde{\gamma}_c(G) = n - 1$. Since G is a connected graph having at least three vertices, G must contain a non-pendant vertex. Let w be the non-pendant vertex of G. Then the set $D_2^* = (D_1^* \setminus \{w\}) \cup \{v\}$ is also an outer-connected dominating set of G and $|D_2^*| = n-1 = \widetilde{\gamma}_c(G)$. Hence D_2^* is a minimum outer-connected dominating set containing all the pendant vertices of G. Hence the Observation 1(c) is proved.* □

(d) Every outer-connected dominating set D of cardinality at most $n - 2$ of a graph $G = (V, E)$ having n vertices contains all the pendant vertices of G.

Proof: *Proof follows from Observation 1(b).* □

(e) $\widetilde{\gamma}_c(G) = n - 1$ if and only if G is a star.

Proof: *Let G be a star having n vertices. If $n = 2$, then $\widetilde{\gamma}_c(G) = 1 = n-1$. If $n \geq 3$ then by Observation 1(c), $\widetilde{\gamma}_c(G) = n - 1$.*

Conversely suppose that $\widetilde{\gamma}_c(G) = n-1$. We need to prove that G is a star, that is, G contains at most one non-pendant vertex. On the contrary suppose that G contains two non-pendant vertices say x and y. If $xy \in E(G)$, then $V \setminus \{x, y\}$ is an outer-connected dominating set of cardinality $n-2$, which is a contradiction. If $xy \notin E(G)$, then at least one of the neighbors of x (same for y) must be a non-pendant vertex (otherwise G is not connected). Let z be the neighbor of x which is a non-pendant vertex. Then $V \setminus \{x, z\}$ is an outer-connected dominating set of cardinality $n-2$, again contradiction arises. Hence G must contain exactly one non-pendant vertex and hence is a star. □

3 NP-completeness proof for perfect elimination bipartite graphs

Let $G = (X, Y, E)$ be a bipartite graph. Then $uv \in E$ is a *bisimplicial edge* if $N_G(u) \cup N_G(v)$ induces a complete bipartite subgraph in G. Let $(e_1, e_2, \ldots, e_k)$ be an ordering of pairwise non-adjacent edges (no two edges have a common end vertex) of G (not necessarily all edges of E). Let S_i be the set of endpoints of edges $e_1, e_2, \ldots, e_i$ and let $S_0 = \emptyset$. Ordering $(e_1, e_2, \ldots, e_k)$ is a *perfect edge elimination ordering* for G if $G[(X \cup Y) \setminus S_k]$ has no edge and each edge e_i is bisimplicial in the remaining induced subgraph $G[(X \cup Y) \setminus S_{i-1}]$. G is a *perfect elimination bipartite graph* if G admits a perfect edge elimination ordering. The class of perfect elimination bipartite graphs was introduced by Golumbic and Goss [9].

To show the NP-completeness of the OCDD problem, we need to use a well known NP-complete problem, called VERTEX COVER DECISION problem [8]. A set $S \subseteq V$ of a graph $G = (V, E)$ is called a *vertex cover* of G if for every edge $uv \in E$, either $u \in S$ or $v \in S$.

VERTEX COVER DECISION problem
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a vertex cover of cardinality at most k?

We are now ready to prove the following theorem:

Theorem 2 *The OCDD problem is NP-complete for perfect elimination bipartite graphs.*

Proof: Given a perfect elimination bipartite graph $G = (V, E)$, a positive integer k and an arbitrary subset D of V, we can check in polynomial time whether $|D| \leq k$ and D is an outer-connected dominating set of G. Hence the OCDD problem for perfect elimination bipartite graphs is in NP. To show the hardness, we provide the polynomial time reduction from VERTEX COVER DECISION problem in general graphs to the OCDD problem in perfect elimination bipartite graphs.

Given a graph $G = (V, E)$, construct the graph $G' = (V', E')$ as follows:
If $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, define

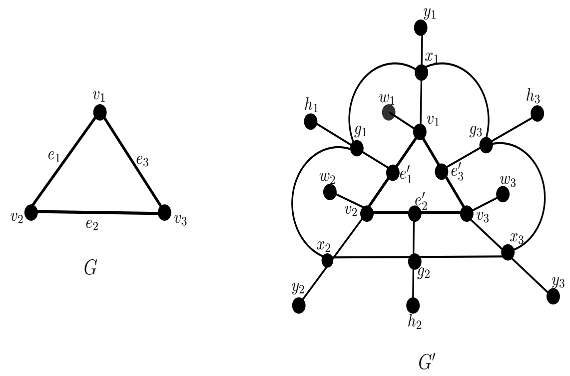

Figure 1: An illustration to the construction of G' from G

$V' = \{v_i, x_i, y_i, w_i \mid 1 \leq i \leq n\} \cup \{e'_i, g_i, h_i \mid 1 \leq i \leq m\}$ and
$E' = \{v_iw_i, v_ix_i, x_iy_i \mid 1 \leq i \leq n\} \cup \{e'_iv_j, e'_iv_k, g_ix_j, g_ix_k, e'_ig_i, g_ih_i \mid 1 \leq i \leq m,$ v_j and v_k are endpoints of edge $e_i\}$.

The graph $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_1\}$ and the associated graph G' are shown in Fig. 1 to illustrate the above construction.

Clearly G' is a perfect elimination bipartite graph since $(x_1y_1, x_2y_2, \ldots, x_ny_n, v_1w_1, v_2w_2, \ldots, v_nw_n, g_1h_1, g_2h_2, \ldots, g_mh_m)$ is perfect edge elimination ordering for G'.

Claim 3.1 *G has a vertex cover of size k if and only if G' has an outer-connected dominating set of size at most $2n + m + k$.*

Proof: Let us first assume that G has a vertex cover say V_c of size k. Then $V_c \cup \{w_i, y_i \mid 1 \leq i \leq n\} \cup \{h_i \mid 1 \leq i \leq m\}$ is an outer-connected dominating set of G' of size $2n + m + k$.

Conversely suppose that D is an outer-connected dominating set of G' of size $2n + m + k$. Define $S = \{h_i \mid 1 \leq i \leq m\} \cup \{w_i, y_i \mid 1 \leq i \leq n\}$ and $E' = \{e'_i \mid 1 \leq i \leq m\}$. By using Observation $1(d)$, all the pendant vertices must belong to D, hence $S \subseteq D$. But S does not dominate the vertices of E'. Define $S' = D \setminus S$. Hence all the vertices of E' are dominated by vertices in S'. Now to dominate e'_i, either $e'_i \in S'$ or $g_i \in S'$ or some $v_j \in S'$. If $e'_i \in S'$ or $g_i \in S'$, we remove it from S' and add v_j (i.e. adjacent to e'_i) in S'. Do this for all i between 1 to m. Define $V_t = V \cap S'$. Note that $|V_t| \leq k$. Since the vertices in V_t dominates all the vertices of E' in G', V_t is a vertex cover of G. This proves our claim. □

Hence our theorem is proved. □

4 Complexity difference in domination and outer-connected domination

Though outer-connected domination is a variation of domination, the problems differ in complexity; that is, there are graph classes in which one problem is polynomial time solvable while the other is NP-hard and vice versa. The Minimum Domination problem is polynomial time solvable for doubly chordal graphs [3], but the OCDD problem is NP-complete for this class of graphs [13]. On the other hand we construct a class of graphs for which the MOCD problem is trivially solvable, but the Domination Decision problem is NP-complete.

Definition 4.1 (GC graph) *A graph is said to be GC graph if it can be constructed from a general graph $G' = (V', E')$ where $|V'| = n > 1$ in the following way:*

(i) Take a complete graph on $2n$ vertices, say K_{2n}.

(ii) Take an arbitrary vertex u of G', an arbitrary vertex v of K_{2n}, join u and v by a path of length 2 by taking a new vertex w.

An example of GC graph is shown in Fig 3.

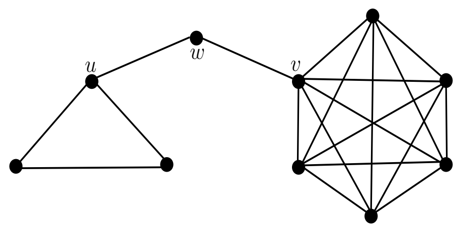

Figure 2: An example of GC graph

Theorem 3 *Let G be a GC graph constructed from a general graph $G' = (V', E')$ ($|V'| = n > 1$), by taking a path $P = uw, wv$, where u is an arbitrary vertex of G' and v is an arbitrary vertex of K_{2n}. Then $\widetilde{\gamma}_c(G) = n+1$ and $V' \cup \{x\}$ is an outer-connected dominating set of G, where x is any vertex of K_{2n} except v.*

Proof: It is easy to notice that $V' \cup \{x\}$ is an outer-connected dominating set of G. Suppose that D_o^* is a minimum cardinality outer-connected dominating set of G. Then $|D_o^*| \leq |V'| + 1$. To dominate the vertex w, at least one vertex from the set $\{u, w, v\}$ must belong to D_o^*.

If $v \in D_o^*$, then either $V' \cup \{w, v\} \subseteq D_o^*$ or $V(K_{2n}) \subseteq D_o^*$. In both the cases, we get a contradiction, since $|D_o^*| \leq n + 1$ and $n > 1$.

If $w \in D_o^*$, then either $V(K_{2n}) \cup \{w\} \subseteq D_o^*$ or $V' \cup \{w, y\} \subseteq D_o^*$ (where y is some vertex of K_{2n}). Again, in both the cases we get the condition, $|D_o^*| > n+1$, which is a contradiction.

If $u \in D_o^*$, then either $V(K_{2n}) \cup \{u, w\} \subseteq D_o^*$ or $V' \subseteq D_o^*$. If $V(K_{2n}) \cup \{u, w\} \subseteq D_o^*$, then $|D_o^*| > n + 1$, a contradiction. Thus the only possibility is $V' \subseteq D_o^*$. Now, to dominate all the vertices of clique K_{2n}, at least one vertex of K_{2n} should also belong to D_o^*. Hence $|D_o^*| \geq n + 1$, and this completes the proof of the theorem. □

Lemma 1 *Let G be a GC graph constructed from a general graph $G' = (V', E')$ ($|V'| = n > 1$), by taking a path P=$\{uw, wv\}$, where u is an arbitrary vertex of G' and v is an arbitrary vertex of K_{2n}. Then G' has a dominating set of cardinality k if and only if G has a dominating set of cardinality $k + 1$.*

Proof: Let D' be a dominating set of G' of cardinality k, then, clearly $D = D' \cup \{v\}$ is a dominating set of G of cardinality $k + 1$.

Conversely, suppose that D is a dominating set of G of cardinality $k + 1$. Then at least one vertex from the set $V(K_{2n})$ must be contained in D. Define $D' = D \setminus V(K_{2n})$. If $w \in D'$, then define $D' = (D' \setminus \{w\}) \cup \{u\}$. D' is a dominating set of G' of cardinality at most k. □

The following result for the Domination Decision problem is well known.

Theorem 4 [8] *The* Domination Decision *problem is NP-complete for general graphs.*

Theorem 5 *The* Domination Decision *problem is NP-complete for GC graphs.*

Proof: The proof directly follows from Lemma 1 and Theorem 4. □

5 Outer-connected domination in chain graphs

We have already seen that the OCDD problem is NP-complete even for perfect elimination bipartite graphs. In this section, we show that the problem of computing a minimum outer-connected dominating set of a chain graph can be solved in polynomial time.

A bipartite graph $G = (X, Y, E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a *chain*, that is, the vertices of X can be linearly ordered, say $x_1, x_2, \ldots, x_p$, such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_p)$. If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain [20]. An ordering $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq \cdots \supseteq N_G(y_q)$. It is well known that every chain graph admits a chain ordering [20, 14].

First we prove the following lemma, which will be helpful in proving the main result of this section.

Lemma 2 *Let $G = (X, Y, E)$ be a chain graph. If every vertex of G is either a pendant vertex or is adjacent to some pendant vertex, then G is either a star or bi-star.*

Proof: Suppose that to the contrary G is neither a star nor a bi-star. Then G contains at least three non-pendant vertices (that is, vertices of degree 2 or more), and at least two non-pendant vertices are present on same partite set. Let x_i and x_j be the non-pendant vertices belonging to the same partite set, say X. Since both x_i and x_j are not pendant vertices, they must be adjacent to some pendant vertices. By the definition of chain ordering, either $N_G(x_i) \subseteq N_G(x_j)$ or $N_G(x_j) \subseteq N_G(x_i)$. Without loss of generality we may assume that $N_G(x_i) \subseteq N_G(x_j)$. Then every vertex adjacent to x_i is also adjacent to x_j. Hence every vertex adjacent to x_i is of degree greater than or equal to 2. Thus x_i is neither a pendant vertex nor is adjacent to some pendant vertex, which is contrary to the assumptions of the theorem. This proves that G is either a star or bi-star. □

We are now ready to characterize the outer-connected domination number of a chain graph in terms of r, the number of pendant vertices it has. In fact, the $\widetilde{\gamma}_c(G)$ of a chain graph can take one of the four values $r-1, r, r+1$, and $r+2$. The following figure contains chain graphs with $\widetilde{\gamma}_c(G)$ taking these four distinct values.

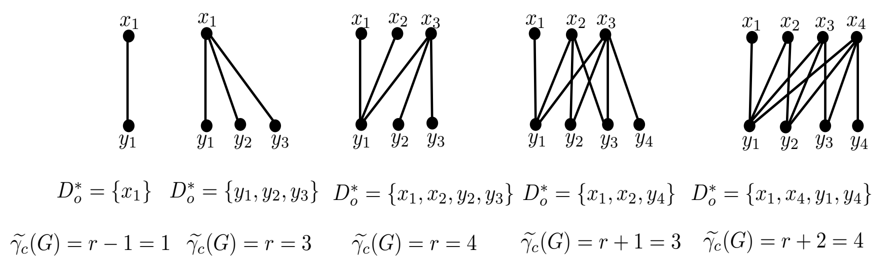

Figure 3: Chain graphs with their $\widetilde{\gamma}_c(G)$

Theorem 6 *Let $G = (X, Y, E)$ be a connected chain graph and $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ is chain ordering of $X \cup Y$. Then $r - 1 \leq \widetilde{\gamma}_c(G) \leq r + 2$, where r is the number of pendant vertices of G. Furthermore, the following are true.*

(a) $\widetilde{\gamma}_c(G) = r - 1$ if and only if $G = K_2$.

(b) $\widetilde{\gamma}_c(G) = r$ if and only if G is a star or bi-star of order greater than 2.

(c) Let P denotes the set of all pendant vertices of G and P_A denotes the set of vertices adjacent to the vertices of P. Then $\widetilde{\gamma}_c(G) = r + 1$ if and only if $G' = G[V \setminus (P \cup P_A)]$ is a star.

(d) If G is a graph other than the graphs described in the above statements then $\widetilde{\gamma}_c(G) = r + 2$.

Proof: Suppose that D is a minimum outer-connected dominating set of G. Then $|D| = \widetilde{\gamma}_c(G)$. Now by using Observation 1(b), either D contains all the pendant vertices of G or $D = V \setminus \{v\}$, where v is some pendant vertex. Thus either $\widetilde{\gamma}_c(G) \geq r$ or $\widetilde{\gamma}_c(G) = n - 1 \geq r - 1$. Hence $\widetilde{\gamma}_c(G) \geq r - 1$.

Let P denotes the set of pendant vertices of G. Now $D = P \cup \{x_p, y_1\}$ is an outer-connected dominating set of G. Hence $\widetilde{\gamma}_c(G) \leq r + 2$.

(a) If $G = K_2$. Then $r = 2$ and $\widetilde{\gamma}_c(G) = 1$ and hence $\widetilde{\gamma}_c(G) = r - 1$. Conversely suppose that $\widetilde{\gamma}_c(G) = r - 1$ and D be a minimum outer-connected dominating set of G. This implies that D does not contain at least one pendant vertex. Then by using Observation 1(b), D contains all the vertices of G other than one pendant vertex and hence $|D| = n - 1$. This implies that $r - 1 = n - 1$ and hence $r = n$. Thus all the vertices of G are pendant vertices. Since K_2 is the only such graph, $G = K_2$.

(b) If G is a star or a bi-star having at least 3 vertices, then clearly $\widetilde{\gamma}_c(G) = r$.

Conversely suppose that $\widetilde{\gamma}_c(G) = r$. If $r = 1$, then G contains at least one non-pendant vertex and hence $n \geq 3$. If $r \geq 2$, then since $\widetilde{\gamma}_c(G) \leq n - 1$, $n \geq 3$. Hence G has at least three vertices. Now let D be a minimum outer-connected dominating set of G and P be the set of all pendant vertices of G. Since $|D| = r$, either $D = P$ or $|D| = r = n - 1$. If $\widetilde{\gamma}_c(G) = n - 1$, then by Observation 1($e$), G is a star. If $D = P$, then every non-pendant vertex of G is adjacent to some pendant vertex, and hence by Lemma 2, G is either a star or bi-star. Hence in both the cases, G is a star or a bi-star.

(c) First suppose that $G' = G[V \setminus (P \cup P_A)]$ is a star. Note that P is not a dominating set. Since by Observation 1(c), P is properly contained in some minimum outer-connected dominating set of G, say D, $\widetilde{\gamma}_c(G) \geq r + 1$. Let u be the star center of G'. Then $D = P \cup \{u\}$ dominates all the vertices of G. Now the vertex adjacent to the pendant vertices in X, say v, is adjacent to all the vertices of X and the vertex adjacent to the pendant vertices in Y, say w, is adjacent to all the vertices of Y. Also v and w both are not taken in D. Hence $G[V \setminus D]$ is connected. So D is an outer-connected dominating set of G. Hence $\widetilde{\gamma}_c(G) = r + 1$.

Conversely suppose that $\widetilde{\gamma}_c(G) = r + 1$. By Observation 1($c$), there is a minimum outer-connected dominating set, say D, of G such that $P \subseteq D$. Now the vertices of $V \setminus (P \cup P_A)$ are dominated using only one vertex. This implies that $G[V \setminus (P \cup P_A)]$ is a star as it is a bipartite graph.

(d) Proof directly follows from above statements. □

A chain ordering of a chain graph $G = (X, Y, E)$ can be computed in linear-time [18]. The set P of pendant vertices of G can be computed in $O(n + m)$ time. If $|V(G)| = 2$, then take $D = \{v\}$, $v \in V(G)$. It can be checked in $O(n + m)$ whether G is a star or a bi-star. In that case, take $D = P$. If $G' = G[V \setminus (P \cup P_A)]$, where P_A be the set of vertices adjacent to a vertex in P, is a star with star-center v, then take $D = P \cup \{v\}$, otherwise take

$D = P \cup \{y_1, x_p\}$. By Theorem 6, D is a minimum outer-connected dominating set of G. Thus we have the following theorem.

Theorem 7 *A minimum outer-connected dominating set of a chain graph can be computed in $O(n + m)$ time.*

6 Outer-connected domination in bounded tree-width graphs

It is well known that every graph problem that can be described by counting monadic second-order logic (CMSOL) can be solved in linear-time in graphs of bounded tree-width, given a tree decomposition as input [5]. Graphs of tree-width at most k are exactly the partial k-trees [15].

In this section we show that the OCDD problem can be described by counting monadic second-order logic. Hence the OCDD problem can be solved in linear-time in graphs of bounded tree-width given a tree decomposition as input.

Definition 6.1 (Counting Monadic second-order logic) *A graph property $\mathcal{P}$ is expressible in counting monadic second-order logic, CMSOL for short, if $\mathcal{P}$ can be defined using:*

- *vertices, edges, sets of vertices and sets of edges of a graph G,*
- *the binary adjacency relation* adj *where $adj(u, v)$ holds if and only if, u, v are two adjacent vertices of G,*
- *binary incidence relation inc, where $inc(v, e)$ hold if and only if edge e is incident to vertex v in G,*
- *the unary cardinality operator card for sets of vertices of G,*
- *the logical operator OR ($\vee$), AND ($\wedge$), NOT ($\neg$),*
- *the membership relation $\in$, the equality operator $=$ for vertices and edges,*
- *the logical quantifiers $\exists$ and $\forall$ over vertices, edges, sets of vertices or sets of edges of G.*

The following result shows that many graph properties can be checked in linear-time for graphs of bounded tree-width.

Theorem 8 *[5] Let $\mathcal{P}$ be a graph property expressible in CMSOL and let c be a constant. Then, for any graph G of tree-width at most c, it can be checked in linear-time whether G has property $\mathcal{P}$.*

Let $OCD(G, k)$ denote the property that $\widetilde{\gamma}_c(G) \leq k$, given a graph G and a positive integer k.

Theorem 9 *Given a graph G and a positive integer k, $OCD(G, k)$ can be expressed in CMSOL.*

Proof: Given a graph $G = (V, E)$ and an integer k, the following CMSOL formula expresses the property that the graph G has a dominating set of size at most k.

$$\exists D, D \subseteq V, |D| \leq k, \forall x(x \in V \rightarrow (\exists y(y \in V \wedge y \in D \wedge adj(x, y)) \vee x \in D))$$

For a set $S \subseteq V$, the property that $G[S]$ is connected, can also be expressed in CMSOL. The graph $G[S]$ is disconnected if and only if the set S can be partitioned into two sets S_1 and S_2 such that there is no edge between a vertex in S_1 and a vertex in S_2. The following CMSOL logic formula expresses the property that $G[S]$ is connected.

$$\neg(\exists C, C \subseteq S, \neg(\exists e \in E, \exists u \in C, \exists v \in S \setminus C, (inc(u, e) \wedge inc(v, e))))$$

Now we can write the CMSOL logic formula which expresses the property $OCD(G, k)$ in the following way:
$\exists D, D \subseteq E, |D| \leq k, ((\forall x(x \in V \rightarrow (\exists y(y \in V \wedge y \in D \wedge adj(x, y)) \vee x \in D))) \wedge (\neg(\exists C, C \subseteq V \setminus D, \neg(\exists e \in E, \exists u \in C, \exists v(v \in V \wedge v \notin D \wedge v \notin C), (inc(u, e) \wedge inc(v, e))))))$.
Hence the theorem is proved. □

By Theorem 8 and Theorem 9, we have the following corollary.

Corollary 6.1 *The OCDD problem can be solved in linear-time for bounded tree-width graphs.*

Note that solving the OCDD problem is answering the question whether G has an outer-connected dominating set of cardinality at most k, for a given positive integer k. By asking this question at most n times, first for $k = 1$, then for $k = 2$ and so on, we can find the outer-connected domination number of a bounded tree-width graph G in at most $O(n^2)$ time.

As the tree-width of a tree is 1, the CMSOL approach gives an $O(n^2)$ algorithm for finding the outer-connected domination number of trees.

7 Approximation Algorithm and Hardness of Approximation

Let $G = (V, E)$ be any graph. Let D_o^* be any minimum outer-connected dominating set of G. Now $V = \cup_{v \in D_o^*} N_G[v]$. So,

$$\begin{aligned} n = |V| &= |\cup_{v \in D_o^*} N_G[v]| \leq \sum_{v \in D_o^*} |N_G[v]| \\ &\leq \sum_{v \in D_o^*} d_G(v) + 1 \leq \sum_{v \in D_o^*} (\Delta(G) + 1) \\ &\leq (\Delta(G) + 1) \cdot |D_o^*| \end{aligned}$$

Hence, $|D_o^*| \geq \lfloor \frac{n}{\Delta(G)+1} \rfloor$. Thus, we have the following result.

Lemma 3 *For any graph G of order n with maximum degree $\Delta(G)$,*

$$\widetilde{\gamma_c}(G) \geq \lfloor (\frac{n}{\Delta(G)+1}) \rfloor.$$

Hence for a graph $G = (V, E)$, $D_o = V(G)$ is an outer-connected dominating set such that $|D_o| \leq (\Delta(G)+1)\widetilde{\gamma_c}(G)$. Thus we have the following theorem.

Theorem 10 *The MOCD problem in any graph $G = (V, E)$ with maximum degree $\Delta(G)$ can be approximated with an approximation ratio of $\Delta(G) + 1$.*

The following approximation hardness result of the Minimum Domination problem will be used to establish an approximation hardness result of the MOCD problem.

Theorem 11 *[4] The* Minimum Domination *problem can not be approximated within a factor of $(1-\varepsilon)\ln |V|$ in polynomial time for any constant $\varepsilon > 0$ unless $NP \subseteq DTIME(\ |V|^{O(\log\log |V|)})$.*

Now we are ready to prove an approximation hardness result for the MOCD problem.

Theorem 12 *The MOCD problem for a graph $G = (V, E)$ can not be approximated within a factor of $(1-\varepsilon)\ln |V|$ in polynomial time for any constant $\varepsilon > 0$ unless $NP \subseteq$ DTIME$(|V|^{O(\log\log |V|)})$.*

Proof: We propose an approximation preserving reduction from the Minimum Domination problem to the MOCD problem. This together with the non-approximability bound of the Minimum Domination problem stated in Theorem 11 will provide the desired result.

Let us first describe the reduction from the Minimum Domination problem to the MOCD problem. Given a graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ construct a graph $G' = (V', E')$ as follows:

$V(G') = V(G) \cup \{w_1, w_2, \ldots, w_n\} \cup \{z\}$, and $E(G') = E(G) \cup \{v_i w_i | 1 \leq i \leq n\} \cup \{w_i w_j | 1 \leq i < j \leq n\} \cup \{z w_i | 1 \leq i \leq n\}$.

The graph $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1 v_2, v_2 v_3\}$ and the associated graph G' are shown in Fig. 4 to illustrate the above construction.

It is easy to see that if D^* is a minimum dominating set of G, then $D^* \cup \{z\}$ is a an outer-connected dominating set of G'.

Now assume that the minimum outer-connected dominating set can be approximated within a ratio of α, where $\alpha = (1-\varepsilon)\ln |V|$ for some (fixed) $\varepsilon > 0$, by using some algorithm, say algorithm A, that runs in polynomial time. Let l be a fixed positive integer. Consider the following algorithm:

Algorithm B
Input: A graph $G = (V, E)$
1. If a minimum dominating set D of cardinality $< l$ exists, construct it Else:
2. Construct G' as above.

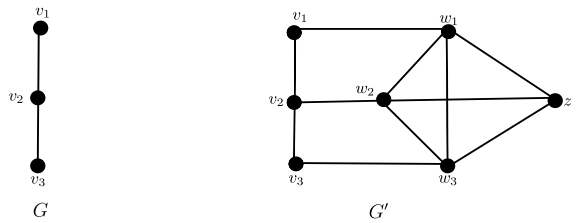

Figure 4: An illustration to the construction of G' from G

3. Compute outer-connected dominating set D_o in G' using algorithm A.
4. Compute D by following procedure
5. Define $D = D_o \bigcap V$
6. For each w_i, if $w_i \in D_o$ then $D = D \cup v_i$
6. Output D

This algorithm runs in polynomial time since algorithm A is a polynomial time algorithm and step 1 runs in polynomial time as l is a constant. Note that if D is a minimum dominating set of cardinality at most l, then it is optimal. In the following we will analyze the case where D is not a minimum dominating set of cardinality at most l.

Let D_o^* be a minimum outer-connected dominating set, then $|D_o^*| \geq l$. Given the graph $G = (V, E)$ algorithm B computes a dominating set D of cardinality $|D| \leq |D_o| \leq \alpha|D_o^*| \leq \alpha(1 + |D^*|) = \alpha(1 + 1/|D^*|)|D^*| \leq \alpha(1 + 1/l)|D^*|$

Hence Algorithm B approximates minimum dominating set within ratio $\alpha(1 + 1/l)$. Since $\alpha = (1 - \varepsilon)\ln|V|$ for some (fixed) $\varepsilon > 0$, for some positive integer l such that $1/l < \varepsilon/2$, algorithm B approximates minimum dominating set within ratio
$\alpha(1 + 1/l) < (1 - \varepsilon)(1 + \varepsilon/2)\ln(|V|) = (1 - \varepsilon')\ln(|V|)$ for $\varepsilon' = \varepsilon/2 + \varepsilon^2/2$.

By Theorem 11, if the Minimum Domination problem can be approximated within a ratio of $(1 - \varepsilon')\ln(|V|)$, then NP $\subseteq$ DTIME$(|V|^{O(\log\log|V|)})$. It follows that if the Minimum Outer-connected Domination problem can be approximated within a ratio of $(1 - \varepsilon)\ln(|V|)$ then NP $\subseteq$ DTIME$(|V|^{O(\log\log|V|)})$.

Since $\ln|V| \approx \ln(2|V|+1)$ for sufficiently large values of $|V|$, for a graph $G' = (V', E')$, where $|V'| = 2|V| + 1$, Minimum Outer-connected Domination problem cannot be approximated within a ratio of $(1 - \varepsilon)\ln|V'|$ unless NP $\subseteq$ DTIME$(|V'|^{O(\log\log|V'|)})$. □

8 APX-completeness

In this section, we show that the MOCD problem is APX-complete for graphs with maximum degree 4. We also show that the MOCD problem is APX-complete for bipartite graphs with maximum degree 7.

To this end, we need the concept of a very popular reduction, known as L-reduction. Let I_P denote the set of all instances of an optimization problem P and let $SOL_P(x)$ denote the set of solutions of an instance x of P. Let $m_P(x, z)$ denote the measure of the objective function value for $x \in I_P$ and $z \in SOL_P(x)$, and $opt_P(x)$ denotes the optimal value of the objective function for $x \in I_P$.

Definition 8.1 *Given two NP optimization problems F and G and a polynomial time transformation f from instances of F to instances of G, we say that f is an L-reduction if there are positive constants α and β such that for every instance x of F*

1. *$opt_G(f(x)) \leq \alpha \cdot opt_F(x)$.*
2. *for every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ we can in polynomial time find a solution y' of x with $m_F(x, y') = c_1$ such that $|opt_F(x) - c_1| \leq \beta |opt_G(f(x)) - c_2|$.*

To show the APX-completeness of a problem $\Pi \in$APX, it is enough to show that there is an L-reduction from some APX-complete problem to Π.

Since $\Delta(G) \leq k$ for some integer constant k, the following corollary follows from Theorem 10.

Corollary 8.1 *The MOCD problem for bounded degree graphs is in APX.*

Next we prove that the MOCD problem for bounded degree graphs is APX-hard.

8.1 APX-completeness for graphs with maximum degree 4

In this subsection we show that the MOCD problem is APX-complete for graphs with maximum degree 4.

Theorem 13 *The MOCD problem is APX-complete for graphs with maximum degree 4.*

Proof: By Corollary 8.1, the MOCD problem for bounded degree graphs is in APX. The Minimum Domination problem is known to be APX-hard for general graphs with maximum degree 3 [2]. We describe an L-reduction f from instances of the Minimum Domination Problem for graphs with maximum degree 3 to the instances of the MOCD problem. Given a graph $G = (V, E)$ of maximum degree 3, we construct a graph $G' = (V', E')$ as follows. Let $V = \{v_1, v_2, \ldots, v_n\}$. Let $V' = V \cup \{z_1, z_2, \ldots, z_n\} \cup \{y_1, y_2, \ldots, y_n\}$ and $E' = E \cup \{v_i y_i, y_i z_i | 1 \leq i \leq n\} \cup \{y_i y_{i+1} | 1 \leq i \leq n-1\}$. Note that the maximum degree of G' is 4. Now let us first prove the following claim:

Claim 8.1 *If D^* is a minimum cardinality dominating set of G, then the cardinality of minimum outer-connected dominating set, say D_o^*, in G' is $|D^*| + n$, where $n = |V|$.*

Proof: Suppose that D^* is a minimum cardinality dominating set of G, then $D^* \cup \{z_i \mid 1 \leq i \leq n\}$ is an outer-connected dominating set of cardinality $|D^*| + n$. Hence the cardinality of a minimum outer-connected dominating set, say D_o^* is less than or equal to $|D^*| + n$, that is, $|D_o^*| \leq |D^*| + n$.

Next suppose that D_o^* is a minimum cardinality outer- connected dominating set of G'. Define $D_o = V \cup \{z_i \mid 1 \leq i \leq n\}$. Then D_o is an outer-connected dominating set of cardinality $2n$. Hence $|D_o^*| \leq 2n$. So by Observation $1(d)$, all the pendant vertices of G' must belong to D_o^*. Hence z_i must belong to D_o^* for all $i, 1 \leq i \leq n$. Let $D' = D_o^* \setminus \{z_i \mid 1 \leq i \leq n\}$. Let $S = \{y_1, \ldots, y_n\} \cap D'$. Let $D'' = (D' \setminus S) \cup \{v_i | y_i \in S\}$. Then D'' is a dominating set of G and cardinality of D'' is less than or equal to $|D_o^*| - n$. Hence if D^* is minimum dominating set then $|D^*| \leq |D_o^*| - n$. Hence $|D_o^*| \geq |D^*| + n$.

This completes the proof of the claim. □

Let D^* and D_o^* be a minimum dominating set and a minimum outer-connected dominating set of G and G', respectively. Since G is of bounded degree 3, for any dominating set D of G, $|D| \geq \frac{n}{4}$. Thus $|D^*| \geq \frac{n}{4}$. Hence $|D_o^*| = |D^*| + n \leq |D^*| + 4|D^*|$ i.e. $|D_o^*| \leq 5|D^*|$. Now consider any outer-connected dominating set D_o of G', then we have the following two cases:

Case 1: z_i belong to D_o for all $i, 1 \leq i \leq n$.
Here y_i may or may not belong to D_o. Let $|D_o \cap \{y_1, y_2, \ldots, y_n\}| = r$ and $|D_o \cap V(G)| = k$. Then $|D_o| = n + r + k$. Now we try to find a dominating set D of G. First include those k vertices of V in D, which also belong to D_o. If $y_i \in D_o$ but $v_i \notin D_o$, then include v_i in D. Suppose that this happens for k' values of i, where $k' \leq r$. Then D is a dominating set of G and $|D| = k' + k$. Now $|D_o| - |D_o^*| = (n+r+k) - |D_o^*| = r + k - |D^*| \geq k' + k - |D^*| = |D| - |D^*|$ (as $|D_o^*| = |D^*| + n$). This implies $|D| - |D^*| \leq |D_o| - |D_o^*|$ in this case.

Case 2: At least one of the z_i does not belong to D_o for some i, where $1 \leq i \leq n$. In this case all the vertices except this particular z_i belong to D_o. Hence $|D_o| = 3n-1$. Now take $D = D_o \cap V = V$. Then D is a dominating set of G and $|D| = n$. Then $|D_o| - |D_o^*| = (3n-1) - (|D^*| + n) = (2n-1) - |D^*| \geq n - |D^*| = |D| - |D^*|$. This implies $|D| - |D^*| \leq |D_o| - |D_o^*|$ in this case.

Hence $|D| - |D^*| \leq |D_o| - |D_o^*|$ in both the cases and we have shown that f is an L-reduction with $\alpha = 5$ and $\beta = 1$.

Thus, the MOCD problem in graphs of bounded degree 4 is APX-complete. □

8.2 APX-completeness for bipartite graphs with maximum degree 7

In this subsection we prove the APX-completeness of the MOCD problem for bipartite graphs of bounded degree.

A set $S \subseteq V$ of a graph $G = (V, E)$ is a *total dominating set* if $N_G(v) \cap S \neq \emptyset$ for all $v \in V$. The Minimum Total Domination problem is to find a total dominating set of minimum cardinality of the input graph G. The Minimum

Total Domination problem is known to be APX-complete for bipartite graphs with maximum degree 3 [17].

Theorem 14 *The MOCD problem is APX-complete for bipartite graphs with maximum degree* 7*.*

Proof: By Corollary 8.1, the MOCD problem for bounded degree bipartite graphs is in APX. We describe an L-reduction f from instances of the Minimum Total Domination problem for bipartite graphs with maximum degree 3 to the instances of the MOCD problem for bipartite graphs of maximum degree 7. Given a bipartite graph $G = (V, E)$ of maximum degree 3 construct a graph $G' = (V', E')$ as follows. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $V' = V \cup \{w_1, w_2, \ldots, w_n\} \cup \{z_1, z_2, \ldots, z_n\} \cup \{y_1, y_2, \ldots, y_n\}$. Construct a spanning tree $T = (V, E_1)$ of G. Let $E_R = \{w_i w_j | v_i v_j \in E_1, 1 \leq i < j \leq n\}$. Let $E^i = \{w_i v_j | v_j \in N_G(v_i)\}$. Let $E' = E \cup E_R \cup \{w_i z_i, z_i y_i, 1 \leq i \leq n\} \cup (\cup_{i=1}^n E^i)$.

Clearly G' is a bipartite graph of maximum degree 7. The graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ and the associated graph G' are shown in Fig. 5 to illustrate the above construction.

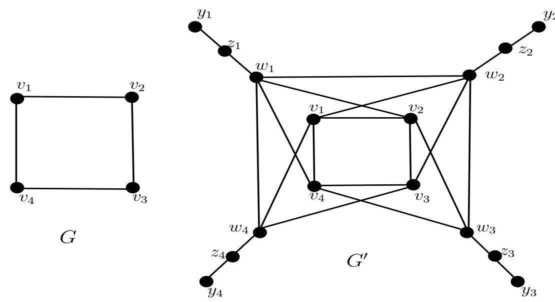

Figure 5: An illustration to the construction of G' from G

Let us first prove the following claim:

Claim 8.2 *If D_T^* is a minimum total dominating set of G and D_o^* is a minimum outer-connected dominating set of G', then $|D_o^*| = |D_T^*| + n$.*

Proof: Clearly $D_T^* \cup \{y_1, y_2, \ldots, y_n\}$ is an outer-connected dominating set. Hence $|D_o^*| \leq |D_T^*| + n$.

Now we construct a total dominating set of G of cardinality at most $|D_o^*| - n$ from the minimum outer-connected dominating set D_o^* of G' as follows.

The minimum outer-connected dominating set D_o^* of G' will necessarily contain all the $y_i, 1 \leq i \leq n$. Given D_o^*, we construct an outer-connected dominating set D_o^{**} such that $|D_o^*| = |D_o^{**}|$ and $D_o^{**} \cap \{w_1, w_2, \ldots, w_n\} = \emptyset$, as follows:

For each i, $1 \leq i \leq n$, if $w_i \in D_o^*$, then replace w_i with v_i.

Let us call the resultant set D_o^{**}. Define $D' = D_o^{**} \setminus \{y_1, y_2, \ldots, y_n\}$. Now to dominate w_i, either z_i belongs to D' or some neighbor v_j of w_i belongs to D'. If z_i belongs to D', then remove it from D' and add some neighbor v_j of w_i in D'. Then D' is a total dominating set of G and $|D'| \leq |D_o^{**}| - n = |D_o^*| - n$. Hence $|D_T^*| \leq |D_o^*| - n$. This proves our claim. □

Since the maximum degree of G is 3, for any total dominating set D_T of G, $|D_T| \geq n/3$. So $|D_T^*| \geq n/3$. Hence $|D_o^*| = |D_T^*| + n \leq |D_T^*| + 3|D_T^*|$. Thus $|D_o^*| \leq 4|D_T^*|$.

Now consider any outer-connected dominating set D_o, then we have following two cases:

Case 1: y_i belong to D_o for all $i, 1 \leq i \leq n$.

Define the sets $W = \{w_1, w_2, \ldots, w_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$. Now we construct an outer-connected dominating set D'_o from D_o, by replacing w_i with v_i, whenever $w_i \in D_o$, $1 \leq i \leq n$. Note that $\{y_1, y_2, \ldots, y_n\} \subseteq D'_o$ and $D'_o \cap W = \emptyset$.

Hence D'_o is an outer-connected dominating set of same or lesser cardinality than that of D_o. Now suppose that $|D'_o \cap Z| = r$ and $|D'_o \cap V| = k$, then $|D'_o| = n + r + k$.

Since for each i, $1 \leq i \leq n$, $N_{G'}(w_i) \cap V = N_G(v_i)$, $D'_o \cap V$ is a total dominating set of G whenever $(N_{G'}(w_i) \cap V) \cap D'_o = N_G(v_i) \cap D'_o \neq \emptyset$ for all i, $1 \leq i \leq n$. If not so, then suppose that there exist a set of vertices $S \subseteq V$ such that for every vertex $v_j \in S$, $N_G(v_j) \cap D'_o = \emptyset$, that is, $(N_{G'}(w_j) \cap V) \cap D'_o = \emptyset$. Now since $N_{G'}(w_j) \subseteq V \cup W \cup \{z_j\}$, z_j must belong to D'_o, as $N_G(v_j) \cap D'_o = \emptyset$. Now update D'_o as $D'_o = (D'_o \setminus \{z_j\}) \cup \{v_k\}$, where $v_k \in N_G(v_j)$. Do this for all the vertices in S. Now define $D_T = D'_o \cap V$. Then D_T is a total dominating set of G and $|D_T| = k + k_1$, where $k_1 \leq r$.

Now, $|D_T| - |D_T^*| = k + k_1 - |D_T^*| \leq n + r + k - (|D_T^*| + n) = |D'_o| - |D_o^*| \leq |D_o| - |D_o^*|$. This implies $|D_T| - |D_T^*| \leq |D_o| - |D_o^*|$ in this case.

Case 2: At least one y_i does not belong to D_o for some i, $1 \leq i \leq n$.

In this case all the vertices except this particular y_i belong to D_o. Hence $|D_o| = 4n - 1$. Now take $D_T = D_o \cap V = V$. Then D_T is a total dominating set of G and $|D_T| = n$. Then $|D_o| - |D_o^*| = (4n - 1) - (|D_T^*| + n) = (3n - 1) - |D_T^*| \geq n - |D_T^*| = |D_T| - |D_T^*|$. This implies $|D_T| - |D_T^*| \leq |D_o| - |D_o^*|$ in this case.

Hence $|D_T| - |D_T^*| \leq |D_o| - |D_o^*|$ in both the cases and we have shown that f is an L-reduction with $\alpha = 4$ and $\beta = 1$. □

9 Conclusion

In this paper, we studied the algorithmic and complexity aspects of the MOCD problem. The OCDD problem is known to be NP-complete for bipartite graphs. In this paper, we proved that the OCDD problem remains NP-complete for perfect elimination bipartite graphs. On the positive side, we proposed a linear-time algorithm for computing a minimum outer-connected dominating set of a chain graph, a subclass of bipartite graphs. It remains interesting to study the problem for further subclasses of bipartite graphs. We also derived a $\Delta(G)$-approximation algorithm for the MOCD problem, where $\Delta(G)$ is the maximum degree of G. On the negative side, we proved that the MOCD problem can not be approximated within a factor of $(1 - \varepsilon) \ln |V|$ for any $\varepsilon > 0$, unless NP $\subseteq$ DTIME($|V|^{O(\log \log |V|)}$). It would also be interesting to try to close the gap between positive and negative approximability results. One may also observe that the MOCD problem is trivially solvable for graphs with bounded degree 2. However, the MOCD problem becomes APX-complete for graphs with bounded degree 4 as we have proved in this paper. The complexity status of the problem is still open for graphs with bounded degree 3.

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