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On Universal Point Sets for Planar Graphs

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Abstract

A set P of points in $\mathbb{R}^2$ is n-universal if every planar graph on n vertices admits a plane straight-line embedding on P. Answering a question by Kobourov, we show that there is no n-universal point set of size n, for any $n \geq 15$. Conversely, we use a computer program to show that there exist universal point sets for all $n \leq 10$ and to enumerate all corresponding order types. Finally, we describe a collection $\mathcal{G}$ of $7'393$ planar graphs on 35 vertices that do not admit a simultaneous geometric embedding without mapping, that is, no set of 35 points in the plane supports a plane straight-line embedding of all graphs in $\mathcal{G}$.

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1 Introduction

The Fáry-Wagner theorem [30, 20, 29] states that every planar graph admits a *plane straight-line embedding*: a drawing in $\mathbb{R}^2$ where vertices are represented by pairwise distinct points, every edge is represented by a line segment connecting its endpoints, and no two edges intersect except at a common endpoint. De Fraysseix, Pach, and Pollack [16] showed in 1988 that every planar graph admits a plane straight-line embedding that places the vertices onto a triangular subset of the rectangular grid $(2n-3) \times (n-1)$. This set is universal in the sense that it does not depend on any particular planar graph: it works for all of them. Formally, we say that a set of points in the plane is called *n-universal* for planar graphs if it admits a *plane straight-line embedding* of all planar graphs on n vertices.

A long-standing open problem is to give tight bounds on the minimum number of points in an n-universal point set. The currently known asymptotic bounds are apart by a linear factor. On the one hand, the n-universal set given by De Fraysseix, Pach, and Pollack [16] has $n^2 - O(n)$ points. This upper bound was improved by Schnyder [28] to $n^2/2 - O(n)$, then by Brandenburg [8] to $4n^2/9 + O(n)$, and finally by Bannister et al. [5] to $n^2/4 - \Theta(n)$. On the other hand, Kurowski [23] showed that at least $1.235n$ points are necessary [23], improving earlier bounds of $1.206n$ by Chrobak and Karloff [14] and $n + \sqrt{n}$ by De Fraysseix, Pach, and Pollack [16].

The following related question was asked around 2002 by Kobourov [17]: what is the largest value of n for which a universal point set of size n exists? We prove the following in Section 3.

Theorem 1 *There is no n-universal point set of size n for any $n \geq 15$.*

Our proof of Theorem 1 combines a labeled counting scheme for *planar 3-trees* (also known as *stacked triangulations*) with known lower bounds on the rectilinear crossing number [1, 25]. The labeled counting scheme is very similar to the one Kurowski [23] used in his asymptotic lower bound argument.

To complement Theorem 1, we use a computer program to show that there exist n-universal point sets of size n for all $n \leq 10$. We give the total number of n-universal *order types* of size n sets for each n in Section 5. Point set order types [22] are a combinatorial abstraction of planar point sets that encode the orientation of all point triples, which in particular determines whether or not any two line segments cross. For $n \leq 11$, there is a database with realizations of every (realizable) order type [2]. As a side remark: it is not clear that the property "there exists an n-universal point set of size n" is monotone in n.

Simultaneous embeddings. For a collection $\mathcal{G} = \{G_1, \ldots, G_k\}$ of planar graphs on n vertices, a *simultaneous geometric embedding without mapping* for $\mathcal{G}$ is a collection of plane straight-line embeddings $\phi_i : G_i \to P$ onto the same set $P \subset \mathbb{R}^2$ of n points [9]. If there exists an n-universal point set of size n, then such a simultaneous embedding is always possible. In Section 4, we consider the following problem: what is the largest natural number σ such that every collection of σ planar graphs on the same number of vertices admits a simultaneous geometric embedding without mapping? From the Fáry-Wagner Theorem we know that $\sigma \geq 1$. We prove the following upper bound:

Theorem 2 *There is a collection of* $7'393$ *planar graphs on* 35 *vertices that do not admit a simultaneous plane straight-line embedding without mapping, hence* $\sigma < 7'393$.

To our knowledge these are the best bounds currently known. It is a very interesting and probably challenging open problem to determine the exact value of σ: the question whether $\sigma = 2$ was already posed in the paper that introduced the problem [9].

Related work. There has been much interest in universal point sets in recent years. In one variation on the theme, we do not insist on straight-line edges but instead allow more freedom in drawing the edges. For instance, we could draw every edge with at most k *bends*, i.e., as a polygonal curve with at most $k + 1$ segments. Everett et al. [19] showed that there is an n-universal point set of size n for $k = 1$. If one insists that the bend-points are also embedded on the universal set, then Dujmović et al. [18] showed that $O(n^2/\log n)$ points suffice for $k = 1$, $O(n \log n)$ points suffice for $k = 2$, and $O(n)$ points suffice for $k = 3$. Very recently, Löffler and Tóth [24] showed that $6n - 10$ points suffice for $k = 1$. Angelini et al. [4] showed that n points suffice when we draw the edges as circular arcs.

Alternatively, we could insist on straight-line edges, but restrict ourselves to smaller graph classes. Fulek and Tóth [21] showed that there exist n-universal point sets of size $O(n^{3/2} \log n)$ for planar 3-trees. Most other results in this direction focus on variations of outerplanarity. Bose [7] showed that every set of n points in general position is n-universal for outerplanar graphs. A graph embedding is k-*outerplane* if deleting the vertices on its outer face yields a $(k - 1)$-outerplane embedding, where 1-outerplane is defined simply as outerplane. A k-*outerplanar* graph is any graph that admits a k-outerplane embedding. Very recently, Bruckdorfer et al. [12] described an n-universal point set of size $O(n \log n)$ for 2-outerplanar graphs. Angelini et al. [3] showed that $O(n(\log n/ \log\log n)^2)$ points suffice for simply-nested planar graphs, a subclass of k-outerplanar graphs.

2 Preliminaries

We follow the convention of explicitly distinguishing the terms *planar* (a property of a graph) and *plane* (a property of an embedding). A plane straight-line embedding of a graph $G = (V, E)$ is completely determined by the embedding of the vertex set. Hence, we frequently represent a plane straight-line embedding by an injection $\phi : V \to \mathbb{R}^2$. Alternatively, if P is a set of $|V|$ points, we represent such an embedding by a bijection $\pi : V \to P$. We define $\pi(G)$ to be the plane straight-line embedding defined by $\pi(V)$. An (unlabeled) *planar 3-tree* is a maximal planar graph defined recursively as follows:

- A triangle is a planar 3-tree.
- If $G = (V, E)$ is a planar 3-tree and $\langle u, v, w\rangle$ is any face of G, then the graph obtained by adding a new vertex to V and connecting it to u, v and w is again a planar 3-tree.

In the literature, planar 3-trees are sometimes considered to have an associated embedding or a distinguished outer face. In this paper, we call planar 3-trees with a distinguished outer face *plane 3-trees*. Since a planar 3-tree is a maximal planar graph, it has n vertices and $2n-4$ triangular faces and its combinatorial embedding is fixed up to the choice of the outer face. Hence, the set of faces of a planar 3-tree is uniquely defined, even if we do not consider a concrete embedding. A *triangle* of a planar 3-tree is any (not necessarily facial) 3-cycle.

3 Large universal point sets

For every integer $n \geq 4$, we define a family $\mathcal{T}_n$ of *labeled* planar 3-trees on the set of vertices $V_n := \{v_1, \ldots, v_n\}$ as follows:

- $\mathcal{T}_4$ contains only the complete graph K_4 on vertex set $\{v_1, \ldots, v_4\}$,
- If T is a graph in $\mathcal{T}_{n-1}$ and $\langle v_i, v_j, v_k\rangle$ is a face of T, then the graph obtained by adding v_n to T and connecting v_n to v_i, v_j, and v_k is in $\mathcal{T}_n$.

We insist on the fact that $\mathcal{T}_n$ is a set of *labeled* abstract graphs, many of which can in fact be isomorphic if considered as abstract (unlabeled) graphs. We also point out that for $n > 4$, the class $\mathcal{T}_n$ does not contain *all* labeled planar 3-trees on n vertices. For instance, the four graphs in $\mathcal{T}_5$ are shown in Figure 1, and there is no graph for which both v_1 and v_2 have degree three.

Lemma 1 $|\mathcal{T}_n| = 2^{n-4} \cdot (n-3)!$ *for all* $n \geq 4$.

Proof: By definition, $|\mathcal{T}_4| = 1$. Every graph in $\mathcal{T}_n$ is constructed by splitting one of the $2(n-1)-4 = 2n-6$ faces of a graph in $\mathcal{T}_{n-1}$. Each choice of a face results in a different graph. We therefore have

$$|\mathcal{T}_n| = |\mathcal{T}_{n-1}| \cdot (2n-6) = \prod_{i=5}^{n}(2i-6) = 2^{n-4}\prod_{i=5}^{n}(i-3) = 2^{n-4} \cdot (n-3)!$$

as required. □

Lemma 2 *Given a set* $P_n = \{p_1, \ldots, p_n\}$ *of* $n \geq 4$ *labeled points in the plane and a bijection* $\pi : V_n \to P_n$*, there is* at most one $T \in \mathcal{T}_n$ *such that* π *is a plane straight-line embedding of* T.

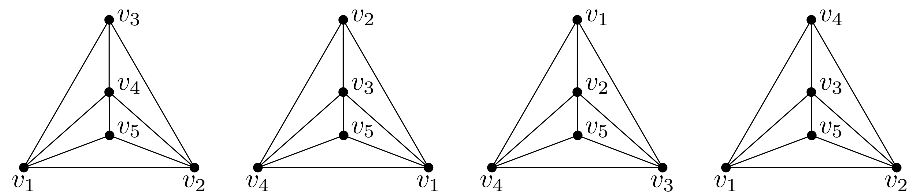


Figure 1: The four planar 3-trees in $\mathcal{T}_5$, with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$.

Proof: Consider any such labeled point set P_n and assume without loss of generality that $\pi(v_i) = p_i$ for all i. Let π_k be the restriction of π to $\{v_1, \ldots, v_k\}$. Let S_k be the set of graphs $T_k \in \mathcal{T}_k$ for which $\pi_k(T_k)$ is plane. The statement of the lemma is equivalent to $|S_n| \leq 1$.

We prove that $|S_n| \leq 1$ by induction on n. For $n = 4$ we have $|S_n| \leq |\mathcal{T}_n| = 1$, as required. Suppose that $n \geq 5$. If $S_n = \emptyset$ then we are done, so suppose that there is a graph $T_n \in S_n$. If we delete v_n from $\pi_n(T_n)$, we get a plane straight-line embedding $\pi_{n-1}(T_{n-1})$ of a graph T_{n-1}. By definition of $\mathcal{T}$ we have $T_{n-1} \in \mathcal{T}_{n-1}$ and since $\pi_{n-1}(T_{n-1})$ is plane we have $T_{n-1} \in S_{n-1}$. By the induction hypothesis, $|S_{n-1}| \leq 1$, and hence $S_{n-1} = \{T_{n-1}\}$. T_n is one of the $2n - 6$ possible extensions of T_{n-1}. In $\pi_{n-1}(T_{n-1})$ the point p_n is contained in a triangular region $\langle p_i, p_j, p_k \rangle$ corresponding to some face $\langle v_i, v_j, v_k \rangle$ of T_{n-1}. If T_n was constructed by inserting v_n into a face different from $\langle v_i, v_j, v_k \rangle$, then v_n is adjacent to some vertex $v_\ell \notin \{v_i, v_j, v_k\}$. But then the line segment $p_n p_\ell$ in $\pi_n(T_n)$ crosses the triangle $\langle p_i, p_j, p_k \rangle$: a contradiction to the fact that $T_n \in S_n$. Hence, T_n must be the unique graph obtained by taking T_{n-1} and inserting v_n into the face $\langle v_i, v_j, v_k \rangle$. Since T_n was chosen arbitrarily in S_n, we conclude that $S_n = \{T_n\}$. Hence, $|S_n| \leq 1$. $\square$

Lemma 2 states that π is a plane straight-line embedding for *at most one* $T \in \mathcal{T}_n$. The reason for this is that some bijections $\pi : V_n \to P_n$ induce a crossing for *every* labeled planar 3-tree in T_n. For example, recall that $\{v_1, v_2, v_3, v_4\}$ form a K_4 for every graph in $\mathcal{T}_n$. If π maps these vertices to set of four points in convex position, then there will necessarily be a crossing, regardless of which $T \in \mathcal{T}_n$ we consider. See Figure 2.

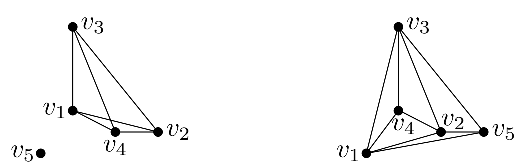


Figure 2: If π maps v_1, v_2, v_3, v_4 to four points in convex position (on the left), then $\pi(T)$ has a crossing for all $T \in \mathcal{T}_n$. Otherwise, $\pi(T)$ is plane for at most one $T \in \mathcal{T}_n$ (on the right).

Lemma 3 *Given a point set $P \subset \mathbb{R}^2$ of n points in general position, more than a $\frac{3}{8} \cdot \frac{n-4}{n}$-fraction of all four-element subsets of P is in convex position.*

Proof: Ábrego and Fernández-Merchant [1] proved that every plane straight-line embedding of the complete graph K_n has at least

$$c_n := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

crossings. Note that for $n \leq 4$ at least one of the floor expressions is zero, whereas for $n = 5$ the theorem states that every straight-line embedding of K_5 has at least one crossing. Every pair of crossing edges corresponds to a four-tuple of points in convex position. Hence, c_n is a lower bound on the number of convex four-gons in P. We proceed to derive a floor-free lower bound on c_n. For odd n we have

$$c_n = \frac{1}{4} \left(\frac{n-1}{2} \right) \left(\frac{n-1}{2} \right) \left(\frac{n-3}{2} \right) \left(\frac{n-3}{2} \right)$$

and for even n we have

$$c_n = \frac{1}{4} \left(\frac{n-0}{2} \right) \left(\frac{n-2}{2} \right) \left(\frac{n-2}{2} \right) \left(\frac{n-4}{2} \right)$$

and so

$$\begin{aligned} c_n &> \frac{1}{4} \left(\frac{n-1}{2} \right) \left(\frac{n-2}{2} \right) \left(\frac{n-3}{2} \right) \left(\frac{n-4}{2} \right) \\ &= \frac{3}{8} \cdot \frac{n-4}{n} \cdot \frac{n(n-1)(n-2)(n-3)}{4 \cdot 3 \cdot 2} \\ &= \frac{3}{8} \cdot \frac{n-4}{n} \cdot \binom{n}{4} \end{aligned}$$

for all n. $\square$

We use Lemma 3 to bound the number of graphs from $\mathcal{T}_n$ that admit a plane straight-line embedding on a given point set in Lemma 4 below. As opposed to Lemma 2, the point set P in the statement below is *not* a labeled point set. That is, when we say that a labeled planar 3-tree $G \in \mathcal{T}_n$ admits a plane straight-line embedding on P, it means simply that there is a bijection φ between P and G such that φ is a plane straight-line embedding of G.

Lemma 4 *On any set $P \subset \mathbb{R}^2$ of $n \geq 4$ points, fewer than $\frac{1}{8}(5n+12)(n-1)!$ graphs from $\mathcal{T}_n$ admit a plane straight-line embedding.*

Proof: Let $P \subset \mathbb{R}^2$ be a set of n points and denote by $\mathcal{F}_n \subseteq \mathcal{T}_n$ the set of labeled planar 3-trees from $\mathcal{T}_n$ that admit a plane straight-line embedding onto P. We represent a straight-line embedding of a graph $T \in \mathcal{T}_n$ onto P by a permutation π of the points of P, where each vertex v_i is mapped to point $\pi(i)$.

Let S_n be the set of all permutations of P. We define a map $\psi : \mathcal{F}_n \to S_n$ by assigning to each $T \in \mathcal{F}_n$ some $\psi(T) \in S_n$ such that $\psi(T)$ is a plane straight-line embedding of T (such an embedding exists by definition of $\mathcal{F}_n$).

By Lemma 2, every permutation $\pi \in S_n$ is a plane straight-line embedding of *at most one* $T \in \mathcal{F}_n$. It follows that ψ is a injection, and hence $\psi : \mathcal{F}_n \to \Pi$, with $\Pi = \mathrm{Im}(\psi)$, is a bijection and so $|\mathcal{F}_n| = |\Pi| \le |S_n| = n!$.

Next we quantify the difference between Π and S_n using Lemma 3. Note that the general position assumption is not a restriction, since in case of collinearities, a slight perturbation of the point set yields a new point set that still admits all plane straight-line embeddings of the original point set. Consider a permutation $\pi = \langle p_1, \ldots, p_n \rangle$ such that $\langle p_1, p_2, p_3, p_4 \rangle$ form a convex quadrilateral. As discussed after Lemma 2, π is not a plane straight-line embedding for any $T \in \mathcal{F}_n$. It follows that $\pi \in S_n \setminus \Pi$. We know from Lemma 3 that more than a fraction of $(3/8) \cdot (n-4)/n$ of the 4-tuples of P are in convex position. Hence, a $(3/8) \cdot (n-4)/n$-fraction of all permutations in S_n start with four points in convex position: these permutations are not in Π. So we can bound the number of possible labeled plane straight-line embeddings by

$$|\Pi| < \left(1 - \frac{3}{8} \cdot \frac{n-4}{n}\right) n! = \left(\frac{8n - 3n + 12}{8n}\right) n! = \frac{1}{8}(5n+12)(n-1)!$$

as required. $\square$

We are now ready to prove Theorem 1.

Theorem 1 *There is no n-universal point set of size n for any $n \ge 15$.*

Proof: Consider an n-universal point set $P \subset \mathbb{R}^2$ with $|P| = n$. Being universal, in particular P has to accommodate all graphs from $\mathcal{T}_n$. By Lemma 1, there are exactly $2^{n-4} \cdot (n-3)!$ graphs in $\mathcal{T}_n$, whereas by Lemma 4 no more than $\frac{1}{8}(5n+12)(n-1)!$ graphs from $\mathcal{T}_n$ admit a plane straight-line drawing on P. Combining both bounds we obtain

$$2^{n-1} \le (5n+12)(n-1)(n-2).$$

Setting $n = 15$ yields $2^{14} = 16'384 \le 87 \cdot 14 \cdot 13 = 15'834$, which is a contradiction and so there is no 15-universal set of 15 points. For $n = 14$ the inequality reads $2^{13} = 8'192 \le 82 \cdot 13 \cdot 12 = 12'792$ and so there is no indication that there cannot be a 14-universal set of 14 points. To prove the claim for any $n > 15$, consider the two functions $f(n) = 2^{n-1}$ and $g(n) = (5n+12)(n-1)(n-2)$ that constitute the inequality. Since f is exponential in n and g is just a cubic polynomial, f certainly dominates g, for sufficiently large n. Moreover, we know that $f(15) > g(15)$. Noting that $f(n)/f(n-1) = 2$ and $g(n) > 0$, for $n > 2$, it suffices to show that $g(n)/g(n-1) < 2$ for all $n \ge 16$. We can bound

$$\begin{aligned}\frac{g(n)}{g(n-1)} &= \frac{(5n+12)(n-1)(n-2)}{(5(n-1)+12)(n-2)(n-3)} = \frac{(5n+12)(n-1)}{(5n+7)(n-3)} \\ &< \frac{(5n+15)n}{5n(n-3)} = \frac{5n(n+3)}{5n(n-3)} = \frac{n+3}{n-3},\end{aligned}$$

which is easily seen to be upper bounded by two, for $n \ge 9$. $\square$

4 Simultaneous Geometric Embeddings

The number of non-isomorphic planar 3-trees on n vertices was computed by Beineke and Pippert [6], and appears as sequence A027610 on Sloane's Encyclopedia of Integer Sequences. For $n = 15$, this number is $321'776$. Hence we can also phrase our result in the language of simultaneous embeddings [9].

Corollary 1 *There is a collection of* $321'776$ *planar graphs that do not admit a simultaneous geometric embedding without mapping.*

In the following we will give an explicit construction for a much smaller family of graphs that not admit a simultaneous embedding without mapping. In order to do so, we first study how planar 3-trees on n vertices can be embedded on a fixed set P of n points. In particular, we want to show that after selecting the outer face of a planar 3-tree G and embedding it on three points of P, there is at most one way to complete the embedding to a plane straight-line embedding of G on P. We prove this with the following two lemmas. We do not claim originality for the proofs: ideas similar to Lemma 5 appeared in [26] and a proof for Lemma 6 in a slightly different setting appeared in [27].

Lemma 5 *Let G be a labeled planar 3-tree on the vertex set $V_n = \{v_1, \ldots, v_n\}$, for $n \geq 3$, and let C denote any triangle in G. Then G can be constructed starting from C by iteratively adding a new vertex and connecting it to the three vertices of some facial triangle of the partial graph constructed so far.*

Proof: Note that C does not have to be a facial triangle. We prove the statement by induction on n. For $n = 3$ we have $G = C$. For $n > 3$, we can iteratively construct G from *some* triangle in the way described by definition of planar 3-trees. Without loss of generality suppose that adding vertices in the order $\langle v_1, v_2, \ldots, v_n \rangle$ yields such a construction sequence. Denote by G_i the graph that is constructed by the sequence $\langle v_1, \ldots, v_i \rangle$, for $1 \leq i \leq n$.

Let $C = \langle v_i, v_j, v_k \rangle$ such that $i < j < k$. Consider the graph G_k: In the last construction step, v_k is added as a new vertex into some facial triangle T of G_{k-1}. As v_k is a neighbor of both v_i and v_j in G, both v_i and v_j are vertices of T; denote the third vertex of T by v_x. Note that all of $\langle v_i, v_j, v_k \rangle$ and $\langle v_i, v_k, v_x \rangle$ and $\langle v_j, v_x, v_k \rangle$ are facial triangles in G_k.

If $k = 4$, then $\langle v_i, v_j, v_k, v_x, v_5, \ldots, v_n \rangle$ is a construction sequence for G, starting with C, as required. If $k > 4$, then $\langle v_i, v_j, v_x \rangle$ is a separating triangle in G_k. By the induction hypothesis we obtain a construction sequence S for G_{k-1} starting with the triangle $\langle v_i, v_j, v_x \rangle$. The desired sequence for G is obtained as $\langle v_i, v_j, v_k, v_x, S^-, v_{k+1}, \ldots, v_n \rangle$, where S^- is the suffix of S that excludes the starting triangle $\langle v_i, v_j, v_x \rangle$. □

And now we can prove the desired property that the mapping for the selected outer face completely determines the mapping for the remaining vertices.

Lemma 6 *Given a labeled planar 3-tree G on vertex set $V_n = \{v_1, \ldots, v_n\}$, a triangle $c = \langle c_1, c_2, c_3 \rangle$ in G, and a set $P \subset \mathbb{R}^2$ of n points with convex hull $\langle p_1, p_2, p_3 \rangle$, there is at most one way to complete the partial embedding $\{c_1 \mapsto p_1, c_2 \mapsto p_2, c_3 \mapsto p_3\}$ to a plane straight-line embedding of G on P.*

Proof: We use Lemma 5 to relabel the vertices in such a way that $\langle c_1, c_2, c_3 \rangle$ becomes $\langle v_1, v_2, v_3 \rangle$ and the order $\langle v_1, \ldots, v_n \rangle$ is a construction sequence for G. Embed vertices $\langle v_1, v_2, v_3 \rangle$ onto $\langle p_1, p_2, p_3 \rangle$. We iteratively embed the remaining vertices as follows. When we construct G by following the construction sequence $\langle v_1, \ldots, v_n \rangle$, vertex v_i is inserted into some face $\langle v_j, v_k, v_\ell \rangle$. Note that v_j, v_k, v_ℓ have already been embedded on some points p_j, p_k, p_ℓ. The vertices contained in the triangle $\langle v_j, v_k, v_\ell \rangle$ (except v_i) are partitioned into three sets by the cycles $\langle v_i, v_j, v_k \rangle$ (n_1 vertices) and $\langle v_i, v_k, v_\ell \rangle$ (n_2 vertices) and $\langle v_i, v_\ell, v_j \rangle$ (n_3 vertices). We claim that we must embed v_i on a point p_i such that $\langle p_i, p_j, p_k \rangle$ contains exactly n_1 points, $\langle p_i, p_k, p_\ell \rangle$ contains exactly n_2 points and $\langle p_i, p_\ell, p_j \rangle$ contains exactly n_3 points. Indeed, if some triangle has too few points, then it will not be possible to embed the subgraph of G enclosed by the corresponding cycle there, and the resulting straight-line embedding will have a crossing. It remains to show that there is always at most one choice for p_i. Suppose that there are two candidates for p_i, say p'_i and p''_i. Then p''_i must be contained in $\langle p'_i, p_j, p_k \rangle$ or $\langle p'_i, p_k, p_\ell \rangle$ or $\langle p'_i, p_\ell, p_j \rangle$ (or vice versa). Without loss of generality, let it be contained in $\langle p'_i, p_j, p_k \rangle$: now $\langle p''_i, p_j, p_k \rangle$ contains fewer points than $\langle p'_i, p_j, p_k \rangle$, which is a contradiction. The lemma follows by induction. □

In light of this lemma, it is not surprising that we can easily find three graphs that do not admit a simultaneous geometric embedding without mapping if the mapping for the outer face is specified for each of them.

Lemma 7 *There is no set $P \subset \mathbb{R}^2$ of five points with convex hull p_a, p_b, p_c such that every graph shown in Figure 3 has a plane straight-line embedding on P where the vertices a, b and c are mapped to the points p_a, p_b and p_c, respectively.*

Proof: The point p for the central vertex that is connected to all of a, b, c must be chosen so that (i) it is not in convex position with p_a, p_b and p_c and (ii) the number of points in the three resulting triangles is one in one triangle and zero in the other two. That requires three distinct choices for p, but there are only two points available. □

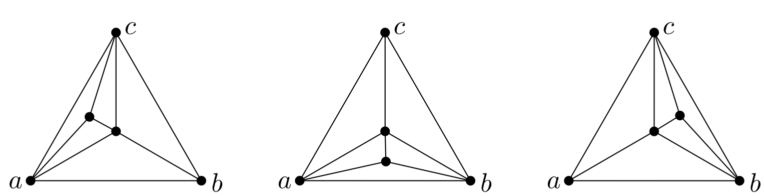
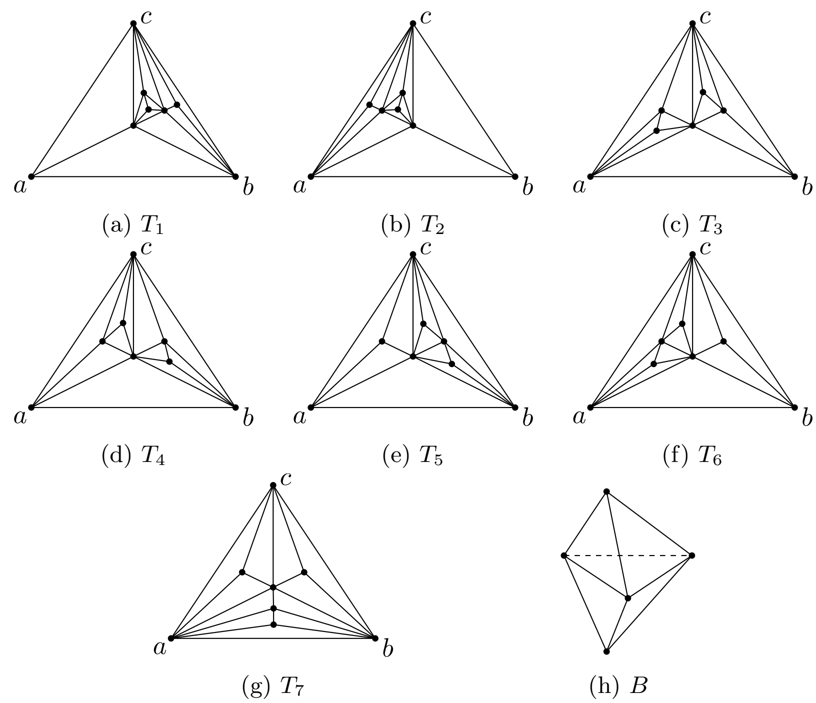


Figure 3: Three planar graphs that do not admit a simultaneous geometric embedding with a fixed mapping for the outer face.



(a) T_1 (b) T_2 (c) T_3 (d) T_4 (e) T_5 (f) T_6 (g) T_7 (h) B

Figure 4: (a)–(g): Seven planar graphs, no three of which admit a simultaneous geometric embedding with a fixed mapping for the outer face; (h): the skeleton B of a triangular bipyramid.

In fact, there are many such triples of graphs. Denote by $\mathcal{T} = \{T_1, \ldots, T_7\}$ the family of seven graphs on eight vertices depicted in Figure 4. We consider these graphs as abstract but *rooted* graphs, that is, one face is designated as the outer face and the counterclockwise order of the vertices along the outer face (the *orientation* of the face) is $\langle a, b, c\rangle$ in each case. Observe that all graphs in $\mathcal{T}$ are planar 3-trees.

Lemma 8 *Let $P \subset \mathbb{R}^2$ be a set of eight points with convex hull $\langle p_a, p_b, p_c\rangle$. Then the partial embedding $\{a \to p_a, b \to p_b, c \to p_c\}$ can be extended to a complete plane straight-line embedding for at most two of graphs in $\mathcal{T}$.*

Informally, no three of the graphs in Figure 4(a-g) admit a simultaneous geometric embedding with a fixed mapping for the outer face. The lemma can be verified with help of a computer program that exhaustively checks all order types. Refer to Appendix A for some details on the implementation. The graphs in $\mathcal{T}$ were discovered using a (different) computer program.

Using $\mathcal{T}$ we construct a family $\mathcal{G}$ of graphs as follows. Start from the skeleton B of a *triangular bipyramid*, that is, a triangle and two additional vertices, each of which is connected to all vertices of the triangle. See Figure 4h. The graph B has five vertices and six faces and it is a planar 3-tree.

We obtain $\mathcal{G}$ from B by planting one of the graphs from $\mathcal{T}$ onto each of the six faces of B. Each face of B is a (combinatorial) triangle where one vertex has degree three (one of the pyramid tips) and the other two vertices have degree four (the vertices of the starting triangle). On each face f of B a selected graph T from $\mathcal{T}$ is planted by identifying the three vertices bounding f with the three vertices bounding the outer face of T in such a way that vertex c (which appears at the top in Figure 4) is mapped to the vertex of degree three (in B) of f. In the next paragraph, we will explain why we do not need to specify how a and b are matched to f. The same graph from $\mathcal{T}$ may be used for more than one face. We define $t(f') := T$ for each new face f' that is created by planting T onto f. The family $\mathcal{G}$ consists of all graphs on $5+6\cdot 5 = 35$ vertices that can be obtained in this way. By construction all these graphs are planar 3-trees. Therefore by Lemma 6 on any given set of 35 points, the plane straight-line embedding is unique (if it exists), once an outer face has been selected and its mapping onto the point set has been fixed.

Observe that $\mathcal{T}$ is *flip-symmetric* with respect to horizontal reflection. In other (more combinatorial) words, for every $T \in \mathcal{T}$ we can exchange the role of the bottom two vertices a and b of the outer face (and thereby also its orientation) to obtain a graph that is also in $\mathcal{T}$. The graphs form symmetric pairs of siblings (T_1, T_2), (T_3, T_4), (T_5, T_6), and T_7 flips to itself. Therefore, regardless of the orientation in which we plant a graph from $\mathcal{T}$ onto a face of B, we obtain a graph in $\mathcal{G}$, and so $\mathcal{G}$ is well-defined.

Next, we give a lower bound on the number of nonisomorphic graphs in $\mathcal{G}$.

Lemma 9 *The family $\mathcal{G}$ contains at least $9'805$ pairwise nonisomorphic graphs.*

Proof: Consider the bipyramid B as a face-labeled object. There are 7^6 different ways to assign a graph from $\mathcal{T}$ to each of the six now distinguishable faces. Denote this class of face-labeled graphs by $\mathcal{F}$. For many of these assignments the corresponding graphs are isomorphic if considered as abstract (unlabeled) graphs. However, the following argument shows that every isomorphism between two such graphs maps the vertex set of B to itself.

The two tips of B have degree three and are incident to three faces. Onto each of the faces one graph from $\mathcal{T}$ is planted, which increases the degree by four (for $T_1, \ldots, T_6$) or three (for T_7) to a total of at least twelve. The three triangle vertices start with degree four and are incident to four faces. Every graph from $\mathcal{T}$ planted there adds at least one more edge, to a total degree of at least eight. But the highest degree among the interior vertices of the graphs in $\mathcal{T}$ is seven, which proves the claim.

Hence we have to look for isomorphisms only among the symmetries of the bipyramid B. The tips are distinguishable from the triangle vertices, because the former are incident to three high degree vertices, whereas the latter are incident to four high degree vertices. There are thus 12 ways to map one bipyramid to another (2 ways to map the tips and $3! = 6$ ways to map the triangle). Hence, every graph in $\mathcal{F}$ is isomorphic to at most 12 graphs from $\mathcal{F}$. It follows that there are at least $7^6/12 > 9'804$ pairwise nonisomorphic graphs in $\mathcal{G}$. $\square$

We now give an upper bound on the number of graphs of $\mathcal{G}$ that can be simultaneously embedded on a common point set.

Lemma 10 *At most* $7'392$ *pairwise nonisomorphic graphs of* $\mathcal{G}$ *admit a simultaneous geometric embedding without mapping.*

Proof: Consider a subset $\mathcal{G}' \subseteq \mathcal{G}$ of pairwise nonisomorphic graphs and a point set P that admits a simultaneous embedding of $\mathcal{G}'$. Since $\mathcal{G}'$ is a class of maximal planar graphs, the convex hull of P must be a triangle. For each $G \in \mathcal{G}'$ we select an outer face $f(G)$ and a mapping $\pi(G)$ for the vertices bounding $f(G)$ to the convex hull of P so that the resulting straight-line embedding, which by Lemma 6 is completely determined by $f(G)$ and $\pi(G)$, is plane.

Recall that we associated with each face f' of G the graph $t(f') \in \{T_1, \ldots, T_7\}$ from which it originates. Let us group the graphs from $\mathcal{G}'$ into bins, according to the maps f and π. For f, there are $7 \cdot 11$ possible choices: one of the eleven faces of one of the seven graphs in $\mathcal{T}$. For π there are three choices: one of the three possible rotations to map the face chosen by f to the convex hull of P. Note that regarding π there is no additional factor of two for the orientation of the face, because by flip-symmetry such a change corresponds to a different graph (for $T_1, \ldots, T_6$) or a different face of the graph (for T_7), that is, a different choice for f. Altogether this yields a partition of $\mathcal{G}'$ into $3 \cdot 77 = 231$ bins.

Now consider one of the bins X and let $\{G_1, \ldots, G_k\} = X$. For all $1 \leq i, j \leq k$ we have $t(f(G_i)) = t(f(G_j))$ by the way in which we divided the graphs into bins. Let G'_i be the subgraph of G_i that contains only $t(f(G_i))$ and B. Note that $G'_i = G'_j$: the graphs G_i and G_j can differ only in what was embedded in the remaining five faces of B. Furthermore, the subgraphs that were embedded in each of these five remaining faces of B contain exactly five extra vertices (not counting the vertices belonging to B), because each of $T_1, \ldots, T_7$ has the same size. Hence, following the proof of Lemma 6, we see that the plane embedding of G'_i onto P with the mapping π of $f(G'_i)$ onto the convex hull not only exists (by choice of f and π) but is in fact unique, provided we insist that each of the remaining five faces of B contains the correct (five) number of points of P in the embedding.

It follows that for all graphs in the same bin the graphs from $\mathcal{T}$ planted onto the faces of B are mapped to the same point sets. Any two (nonisomorphic) graphs from $\mathcal{G}'$ differ in at least one of those faces – and by definition not in the one in which the outer face was selected by f. In order for the graphs in a bin to be simultaneously embeddable on P, by Lemma 8 there are at most two different graphs from $\mathcal{T}$ mapped to any of the remaining five faces of B. Therefore there cannot be more than $2^5 = 32$ graphs from $\mathcal{G}'$ in any bin. Hence $|\mathcal{G}'| \leq 231 \cdot 32 = 7'392$, as claimed. $\square$

Since there are strictly more nonisomorphic graphs in $\mathcal{G}$ than can possibly be simultaneously embedded, not all graphs of $\mathcal{G}$ admit a simultaneous embedding. In particular, any subset of $7'392 + 1$ nonisomorphic graphs in $\mathcal{G}$ is a collection that does not have a simultaneous embedding. This proves our Theorem 2.

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	≥ 15
# universal point sets:	1	1	1	1	1	5	45	364	5′955	2′072	?	?	?	?	0

Table 1: The number of (non-equivalent) n-universal point sets of size n.

5 Small n-universal point sets

As we have seen in Section 3, there are no n-universal point sets of size n for $n \geq 15$. In this section, we consider the case $n < 15$. Specifically, we used a computer program to show the following:

Theorem 3 *There exist n-universal point sets of size n for all* $1 \leq n \leq 10$.

We use a straightforward brute-force approach. The two main ingredients are the aforementioned order type database [2] with point sets of size $n \leq 10$ and the *plantri* program for generating maximal planar graphs [10, 11]. Refer to Appendix A for some details on the implementation. Work on the case $n = 11$ is still in progress at the time of writing. For $n > 11$ the approach unfortunately becomes infeasible; it is unknown whether or not there exist n-universal point sets of size n for $11 \leq n \leq 14$. Table 1 gives the number of n-universal point sets of size n and Figure 5 shows one universal point set for each $n = 5, \ldots, 10$.

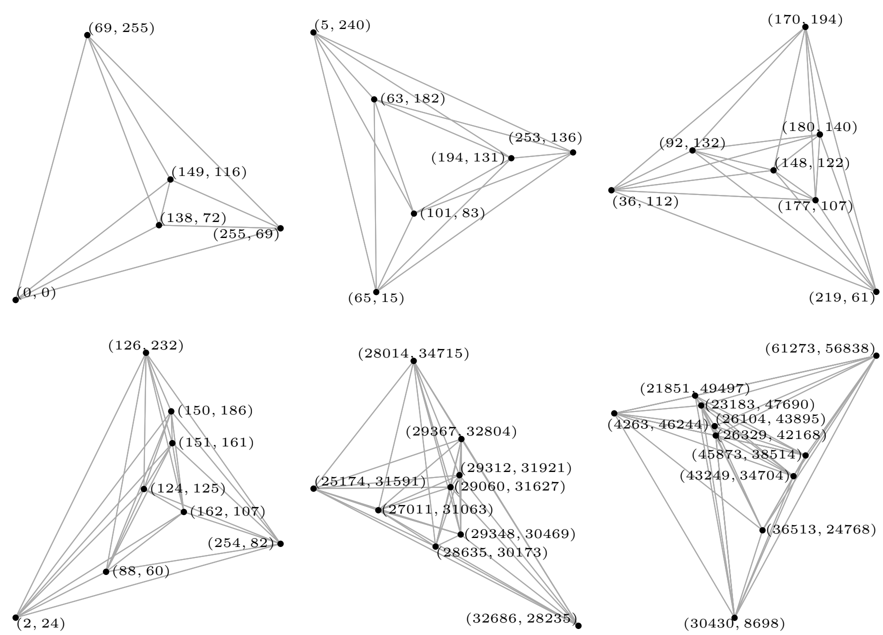


Figure 5: One universal point set for each $n = 5, \ldots, 10$. Each pair of points is connected with a line segment.

6 Conclusions

We proved that there exists an n-universal point set of size n for every $n \leq 10$ and that no such point set exists for $n \geq 15$. The cases $11 \leq n \leq 14$ are still open. The main open problem remains to close the gap between the linear lower bound and the quadratic upper bound on the size of an n-universal point set. On the topic of simultaneous embeddings we proved that $\sigma < 7393$ by describing 7393 graphs that do not admit a simultaneous geometric embedding without mapping. The tantalizing question of whether $\sigma = 2$, as originally asked by Brass et al. [9], remains open.

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A Notes on implementation

We provide the code to prove Lemma 8 and Theorem 3 at [13]. This package contains the source code of a program *universal* to generate all universal point sets of size $n \leq 10$ and the source code of a program *verifygraphs* to verify Lemma 8. The programs are similar and rely on two dependencies: the order type database [2] with point sets of size $n \leq 10$ and the *plantri* program for generating maximal planar graphs [10, 11]. The file *README* in the package contains detailed instructions on how to download the dependencies, compile the programs, and run the programs. In the remainder we give some notes on the implementation of these programs.

A.1 Common code

The programs *universal* and *verifygraphs* share some basic code. They both use an implementation of the Graham-Andrew algorithm to compute the convex hull of a point set, following the pseudocode given in [15, pages 6–7]. They also share code to verify whether a straight-line embedding of a graph on a point set P is plane. One naive approach is to test for every pair of edges whether the corresponding line segments intersect. Instead, the programs use the fact that the *plantri* program outputs not only the maximal planar graphs but also their unique combinatorial embedding in the form of a rotation system. The *rotation system* of an embedding describes for every vertex v the circular order of the edges incident to v. To determine whether a straight-line embedding $\mathcal{E}$ of G onto P is plane, it suffices to check whether the rotation system output by *plantri* coincides (up to global reflection) with the rotation system of $\mathcal{E}$. All common code has been tested against an independent implementation.

A.2 Generating universal point sets

The program *universal* enumerates all point sets in the order type database. To determine if a point set P of size n is n-universal, it tests if for all maximal planar graphs $G = (V, E)$ on n vertices, there exists a bijection $\varphi : V \to P$ such that straight-line drawing of G induced by φ is plane. If such a bijection exists for all G, then P is universal. Otherwise, there is a graph G that has no plane straight-line embedding on P. Note that it is sufficient to consider maximal planar graphs since adding edges only makes the embedding problem more difficult.

A.3 Verifying Lemma 8

The program *verifygraphs* iterates over all point sets P of size $n = 8$ in the order type database. Let $P = \{p_0, \ldots, p_7\}$ be such that $\{p_0, p_1, p_2\}$ forms the convex hull. Then for every possible bijection $\phi : \{a, b, c\} \to \{p_0, p_1, p_2\}$, the program counts the number of graphs from Figure 4(a-g) for which ϕ can be extended to a plane straight-line embedding. This number is always at most two: this proves Lemma 8.

A.4 Heuristics

The program *universal* has two compile-time options that enable heuristics to improve the running time of the program. These heuristics are disabled by default, since they increase the complexity of the code and may thus make it more difficult to check its correctness. The code outputs the same universal point sets, regardless of which heuristics are enabled. Enabling both heuristics decreases the time required to generate all universal point sets of size $n \leq 9$ on a 2011 notebook computer from approximately 90 minutes to 20 minutes.

The *reorder heuristic* reorders the list of graphs whenever a graph is found that could not be embedded on the point set under consideration. This graph is moved to the front of the list. The idea is that some graphs may admit an embedding on fewer point sets and hence computation time may decrease if such graphs are checked first.

The *permutation heuristic* exploits an observation we made earlier in the paper. Let us label the vertices of G by $\langle v_1, \ldots, v_n \rangle$ and treat a straight-line embedding of G as a permutation $\langle p_1, \ldots, p_n \rangle$ of P so that each v_i is mapped to p_i. If the partial embedding of $\langle v_1, \ldots, v_k \rangle$ onto $\langle p_1, \ldots, p_k \rangle$ already contains a crossing, then none of the permutations of P that start with $\langle p_1, \ldots, p_k \rangle$ will result in a plane straight-line embedding of G. The permutation heuristic proceeds as follows. It first checks whether $\langle p_1, \ldots, p_n \rangle$ is a plane straight-line embedding of G. If not, it performs a binary search to find the k such that there is no crossing for the partial embedding onto $\langle p_1, \ldots, p_{k-1} \rangle$, but there is a crossing for the partial embedding onto $\langle p_1, \ldots, p_k \rangle$. It then skips all permutations that start with $\langle p_1, \ldots, p_k \rangle$.