Intersection-Link Representations of Graphs

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Abstract

We consider drawings of graphs that contain dense subgraphs. We introduce *intersection-link representations* for such graphs, in which each vertex u is represented by a geometric object $R(u)$ and each edge (u, v) is represented by the intersection between $R(u)$ and $R(v)$, if it belongs to a dense subgraph, or by a curve connecting the boundaries of $R(u)$ and $R(v)$, otherwise. We study a notion of planarity, called CLIQUE PLANARITY, for intersection-link representations of graphs in which the dense subgraphs are cliques.

1   Introduction

In several applications there is the need to represent graphs that are globally sparse but contain dense subgraphs. As an example, a social network is often composed of communities, whose members are closely interlinked, connected by a network of relationships that is much less dense. The visualization of such networks poses challenges that are attracting the study of several researchers (see, e.g., [8, 14]). One frequent approach is to rely on clustering techniques to collapse dense subgraphs and then represent only the links between clusters. However, this has the drawback of hiding part of the graph structure. Another approach that has been explored is the use of hybrid drawing standards, where different conventions are used to represent the dense and the sparse portions of the graph: In the drawing standard introduced in [6, 15] each dense part is represented by an adjacency matrix while two adjacent dense parts are connected by a curve. The complexity of constructing such representations without crossings has been recently studied [12].

In this paper we study *intersection-link representations*, which are hybrid representations where in the dense parts of the graph the edges are represented by the intersection of geometric objects (*intersection* representation) and in the sparse parts the edges are represented by curves (*link* representation).

More formally, we introduce the following problem. Suppose that a pair (G, S) is given where G is a graph and S is a set of cliques that partition the vertex set of G. In an *intersection-link* representation, vertices are represented by geometric objects that are translates of the same rectangle. Consider an edge (u, v) and let $R(u)$ and $R(v)$ be the rectangles representing u and v, respectively. If (u, v) is part of a clique (*intersection-edge*) we represent it by drawing $R(u)$ and $R(v)$ so that they intersect, otherwise (*link-edge*) we represent it by a curve connecting $R(u)$ and $R(v)$. An example is provided in Fig. 1.
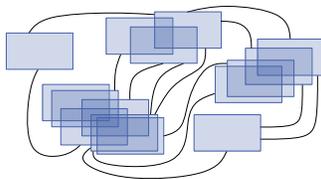
Figure 1: Intersection-link representation of a graph with five cliques.

We introduce and study the Clique Planarity problem, which asks whether a pair (G, S) has an intersection-link representation such that link-edges do not cross each other and do not intersect the interior of any rectangle. The main challenge of the problem lies in the interplay between the geometric constraints imposed by the rectangle arrangements and the topological constraints imposed by the link edges.

Several problems are related to Clique Planarity; here we mention two notable ones. The problem of recognizing intersection graphs of translates of the
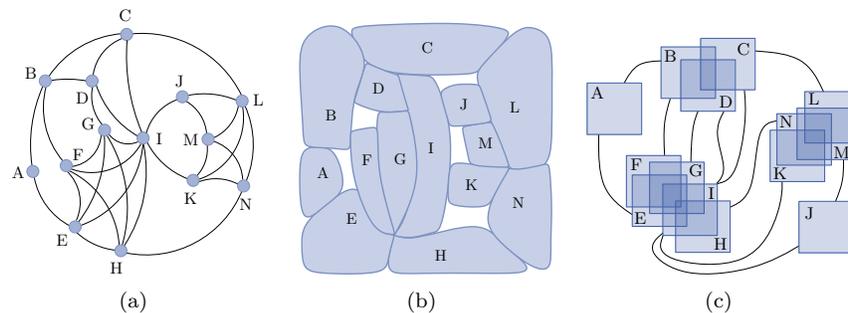
Figure 2: A graph (a) that is both a map graph (b) and a clique-planar graph (c).

same rectangle is $\mathcal{NP}$-complete [9]. Note that this does not imply $\mathcal{NP}$-hardness for our problem, since cliques always have such a representation. *Maps* consist of internally-disjoint connected regions of the plane, representing the vertices of a graph whose edges correspond to contacts between adjacent regions. Since each contact can even be a single point, maps allow to represent graphs containing large cliques in a readable way. The class of graphs that admit such representations are the *map graphs*; see Fig. 2 for an example of a graph that is both clique planar and a map graph. The recognition of map graphs has been studied in [10, 18, 19].

We now describe our contribution. Our study includes several interesting and, at a first glance, seemingly unrelated theoretical problems.

- In Section 3 we establish that the clique-planar graphs and the map graphs are different graph classes. Namely, we show that there are graphs that admit a clique-planar representation, while not admitting any representation as a map, and vice versa.

- In Section 4 we show that Clique Planarity is $\mathcal{NP}$-complete even if S contains just one clique with more than one vertex. This result is established by observing a relationship between Clique Planarity and a natural constrained version of the Clustered Planarity problem, in which we ask whether a path (rather than a tree as in the usual Clustered Planarity problem) can be added to each cluster to make it connected while preserving clustered planarity; we prove this problem to be $\mathcal{NP}$-complete, a result that might be interesting in its own right.

- In Section 5, we show how to decide Clique Planarity in polynomial time if each clique has a prescribed geometric representation, via a reduction to testing planarity for a graph with a given partial representation [5].

- In Section 6, we concentrate on instances of Clique Planarity composed of two cliques. While we are unable to settle the complexity of this case, we show that the problem becomes equivalent to an interesting variant of the 2-Page Book Embedding problem, in which the graph is bipartite and the vertex ordering in the book embedding has to respect the vertex partition of the graph. This problem is in our opinion worthy

of future research efforts. For now, we use this equivalence to establish a polynomial-time algorithm for the case in which the link-edges are assigned to the pages of the book embedding.

- In Section 7, we study a Sugiyama-style problem where the cliques are arranged on levels according to a hierarchy. In this practical setting we show that CLIQUE PLANARITY is solvable in polynomial time. This is achieved via a reduction to the $\mathcal{T}$-LEVEL PLANARITY problem [3].

Conclusions and open problems are presented in Section 8.

2   The intersection-link model

Let G be a graph and S be a set of cliques inducing a partition of the vertex set of G. In an *intersection-link representation* of (G, S):

- each vertex u is a geometric object $R(u)$, which is a translate of an axis-aligned rectangle $\mathcal{R}$;

- two rectangles $R(u)$ and $R(v)$ intersect if and only if edge (u, v) is an intersection-edge, that is, if and only if (u, v) belongs to a clique in S; and

- if (u, v) is a link-edge, then it is represented by a curve connecting the boundaries of $R(u)$ and $R(v)$.

To avoid degenerate intersections we assume that no two rectangles have their sides on the same horizontal or vertical line. The CLIQUE PLANARITY problem asks whether an intersection-link representation of a pair (G, S) exists such that: (i) no two curves intersect; and (ii) no curve intersects the interior of a rectangle. Such a representation is called *clique-planar*. A pair (G, S) is *clique-planar* if it admits a clique-planar representation.

We now present two simple, yet important, combinatorial properties of intersection-link representations. Let Γ be an intersection-link representation of $(K_n, \{K_n\})$ and let B be the outer boundary of Γ. We have the following.

Lemma 1 *Traversing B clockwise, the sequence of encountered rectangles is not of the form $..., R(u), ..., R(v), ..., R(u), ..., R(v)$, for any $u, v \in G$.*

Proof: The statement follows from the fact that the outer boundary of the union of $R(u)$ and $R(v)$ consists of two maximal portions, one belonging to $R(u)$ and one to $R(v)$. □

Lemma 2 *Traversing B clockwise, the sequence of encountered rectangles is a subsequence of $R(u_1), R(u_2), ..., R(u_n), R(u_{n-1}), ..., R(u_2)$, for some permutation $u_1, ..., u_n$ of the vertices of K_n.*

Proof: We prove a sequence of claims.

(Claim A): Every maximal portion of B belonging to a single rectangle $R(u)$ contains (at least) one corner of $R(u)$. Namely, if part of a side of $R(u)$ belongs to B, while its corners do not, then two distinct rectangles $R(v)$ and $R(z)$ enclose
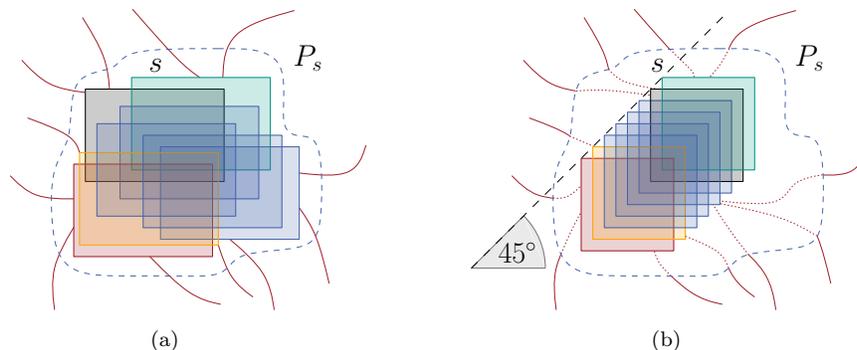
Figure 3: Illustration for the proof of Lemma 3. (a) Clique-planar representation of a clique s. (b) Representation of s by axis-aligned unit squares with their upper-left corners along a common line with slope 1.

those corners. However, this implies that $R(v)$ and $R(z)$ do not intersect, a contradiction to the fact that Γ is a representation of $(K_n, \{K_n\})$.

(Claim B): If two adjacent corners of the same rectangle $R(u)$ both belong to B, then the entire side of $R(u)$ between them belongs to B. Namely, if a rectangle $R(v) \neq R(u)$ intersects a side of $R(u)$, then at least one of the two corners of that side lies in the interior of $R(v)$, given that $R(u)$ and $R(v)$ are translates of the same rectangle; hence that corner does not belong to B.

(Claim C): No rectangle defines three distinct maximal portions of B. Suppose, for a contradiction, that a rectangle $R(u)$ defines three distinct maximal portions of B. By Claim A, each maximal portion of B belonging to $R(u)$ contains a corner of $R(u)$. This implies the existence of two adjacent corners belonging to two distinct maximal portions of B. However, by Claim B the side of $R(u)$ between those corners belongs to B, hence those corners belong to the same maximal portion of B, a contradiction.

Claim C and Lemma 1 imply the statement of the lemma. $\square$

The following lemma allows us to focus, without loss of generality, on special clique-planar representations, which we call *canonical*.

Lemma 3 *Let (G, S) admit a clique-planar representation Γ. There exists a clique-planar representation Γ' of (G, S) such that:*

- *each vertex is represented by an axis-aligned unit square; and*

- *for each clique $s \in S$, all the squares representing vertices in s have their upper-left corner along a common line with slope 1.*

Proof: Initialize $\Gamma' = \Gamma$. Rescale Γ' in such a way that the unit distance is very small with respect to the size of the rectangles representing vertices in Γ.
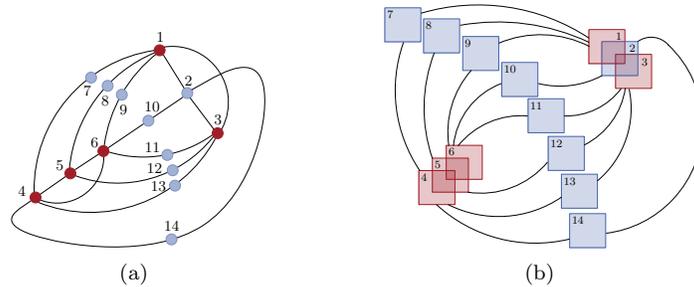
Figure 4: (a) A graph that does not admit a representation as a map, but admits a clique-planar representation (b).

For each clique $s \in S$, consider a closed polyline P_s "very close" to the representation of s, so that it contains all and only the rectangles representing the vertices of s and it crosses at most once each curve representing a link-edge of G; refer to Fig. 3. Traverse P_s clockwise. By Lemma 2 and by the clique planarity of Γ, the circular sequence of encountered curves representing link-edges and crossing P_s contains edges incident to a subsequence of $R(u_1), R(u_2), ..., R(u_{|s|}), R(u_{|s|-1}), ..., R(u_2)$, for some permutation $u_1, ..., u_{|s|}$ of the vertices of s. Remove the interior of P_s. Put in the interior of P_s unit squares $Q(u_1), Q(u_2), ..., Q(u_{|s|})$ representing $u_1, u_2, ..., u_{|s|}$ as required by the lemma and such that they all share a common point of the plane. Reroute the curves representing link-edges from the border of P_s to the suitable ending squares. This can be done without introducing any crossings, because the circular sequence of the squares encountered when traversing the boundary of the square arrangement clockwise is $Q(u_1), Q(u_2), ..., Q(u_{|s|}), Q(u_{|s|-1}), ..., Q(u_2)$.   □

3   Relationship with Map Graphs

As shown in Fig. 2 there are graphs that are both map graphs and clique-planar graphs. However, in this section we show that neither of the classes is contained into the other.

Lemma 4 *There exists a clique-planar graph that is not a map graph.*

Proof: Consider the graph G of Fig. 4(a). As shown in Fig. 4(b) graph G is clique-planar. Suppose, for a contradiction, that G admits a representation as a map Γ. Let R_1, R_2, and R_3 be the pairwise-touching regions representing the vertices of triangle $\triangle 1, 2, 3$ in Γ. Without loss of generality, we can assume that the region $R_1 \cap R_2 \cap R_3$ is empty; indeed, since no vertex of G is adjacent to all the vertices of the triangle $\triangle 1, 2, 3$, a local deformation can get rid of any point of $R_1 \cap R_2 \cap R_3$ while preserving the pairwise adjacencies between the regions. The same argument holds for triangle $\triangle 4, 5, 6$. Also, $G \setminus \{1, 2, 3, 4, 5, 6\}$ does not contain any clique of size larger than 2. Thus, one could obtain a planar drawing

of G by placing each vertex u in the interior of the region R_u representing it in Γ and by drawing each edge (u, v) as a curve entirely contained in the interior of $R_u \cup R_v$. This is a contradiction, since graph G is not planar. In fact, vertices $1, 3, 4, 5$, and 6 (filled red in Fig. 4(a)) form a K_5 subdivision. This concludes the proof of the lemma. □

Lemma 5 *There exists a map graph that is not a clique-planar graph.*

Proof: Consider a graph $G_h = (V, E)$ composed by three sets V_1, V_2, and V_3 of h vertices each, where the graph induced by $V_1 \cup V_2$ is a clique and the graph induced by $V_2 \cup V_3$ is a clique; see Fig. 5(a). We have that G_h is a map graph; see Fig. 5(b). Observe that in any partition S of V into vertex-disjoint cliques there are at least $h/2$ vertices in V_2 that do not fall into the same clique with the vertices of V_1 or V_3. The link edges among such vertices induce a $K_{\frac{h}{2}, h}$. Therefore, for $h \geq 6$ the clique planarity of G_h would imply the planarity of $K_{3,3}$. □

4 Hardness Results on Clique Planarity

In this section we prove that the CLIQUE PLANARITY problem is not solvable in polynomial time, unless $\mathcal{P} = \mathcal{NP}$. In fact, we have the following.

Theorem 1 *It is $\mathcal{NP}$-complete to decide whether a pair (G, S) is clique-planar, even if S contains just one clique with more than one vertex.*

We prove Theorem 1 by showing a polynomial-time reduction from a constrained clustered planarity problem, which we prove to be $\mathcal{NP}$-complete, to the CLIQUE PLANARITY problem.

A *clustered graph* (G, T) is a pair such that G is a graph and T is a rooted tree whose leaves are the vertices of G. The internal nodes of T different from the root are the *clusters* of G. Each cluster $\mu \in T$ is associated with a set containing all and only the vertices of G that are the leaves of the subtree of T rooted at μ. We call *cluster* also this set. A clustered graph is *flat* if every cluster is a
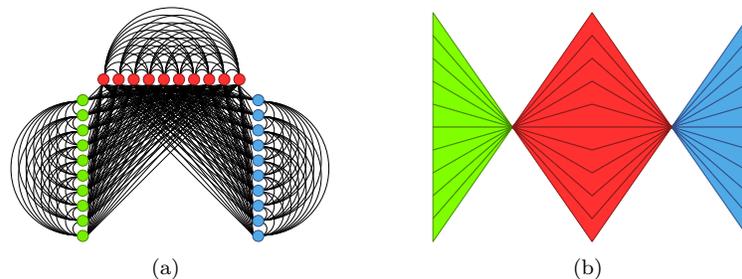
Figure 5: (a) Graph G_{10} and (b) a corresponding map.

child of the root. The CLUSTERED PLANARITY problem asks whether a given clustered graph (G, T) admits a *c-planar drawing*, i.e., a planar drawing of G, together with a representation of each cluster μ in T as a simple region R_μ of the plane such that: (i) every region R_μ contains all and only the vertices in μ; (ii) every two regions R_μ and R_ν are either disjoint or one contains the other; and (iii) every edge intersects the boundary of each region R_μ at most once.

Polynomial-time algorithms for testing the existence of a c-planar drawing of a clustered graph are known only in special cases, most notably, if it is *c-connected*, i.e., each cluster induces a connected subgraph [11, 13]. It has long been known [13] that a clustered graph (G, T) is c-planar if and only if a set of edges can be added to G so that the resulting graph is c-planar and c-connected. Any such set of edges is called a *saturator*, and the subset of a saturator composed of those edges between vertices of the same cluster μ defines a *saturator for μ*. A saturator is *linear* if the saturator for each cluster is a path.

The CLUSTERED PLANARITY WITH LINEAR SATURATORS (CPLS) problem takes as input a flat clustered graph (G, T) such that each cluster in T induces an independent set of vertices, and asks whether (G, T) admits a linear saturator.

Lemma 6 *Let (G, T) be an instance of* CPLS *with $G = (V, E)$ and let $E^\star \subseteq \binom{V}{2} \setminus E$ be such that in $G^\star = (V, E \cup E^\star)$ every cluster induces a path. Then $E^\star$ is a linear saturator for (G, T) if and only if $G^\star$ is planar.*

Proof: Clearly, if $E^\star$ is a linear saturator, then $(G^\star, T)$ is c-planar and thus G^* is planar. Conversely, assume that $G^\star$ is planar and let $\Gamma^\star$ be a planar drawing of it. Since the vertices of each cluster are isolated in G, the region R_μ for each cluster μ can be represented by a sufficiently narrow region around the corresponding path in $G^\star$ yielding a c-planar drawing of $G^\star$. It follows that $E^\star$ is a linear saturator. □

The following lemma connects the problem CLIQUE PLANARITY with the problem CLUSTERED PLANARITY WITH LINEAR SATURATORS.

Lemma 7 *Given an instance (G, T) of the* CPLS *problem, an equivalent instance (G', S) of the* CLIQUE PLANARITY *problem can be constructed in quadratic time.*

Proof: Instance (G', S) is defined as follows. Initialize $G' = G$. For each cluster $\mu \in T$, add edges to G' such that μ forms a clique and add this clique to S. Clearly, instance (G', S) can be constructed in quadratic time. We prove that (G, T) admits a linear saturator if and only if (G', S) is clique-planar.

Suppose that (G, T) admits a linear saturator. This implies that there exists a c-planar drawing $\Gamma^\star$ of $(G^\star, T)$, where $G^\star$ is obtained by adding the saturator to G. We construct a clique-planar representation Γ of (G', S) starting from $\Gamma^\star$ as follows.

Consider any cluster μ of T represented by region R_μ, let B_μ be the boundary of R_μ, and let $u_1, ..., u_k$ be the vertices of μ ordered as they appear along the saturator for μ. For each edge (u, v) of $G^\star$ crossing B_μ, subdivide (u, v) with

a dummy vertex at this crossing point. Note that the order of the vertices of μ, corresponding to the order in which their incident edges cross B_μ, is a subsequence of $u_1, ..., u_{k-1}, u_k, u_{k-1}, ..., u_2$.

Remove from $\Gamma^\star$ all the vertices and (part of the) edges contained in the interior of R_μ. Represent $u_1, ..., u_k$ by pairwise-intersecting rectangles $R(u_1), ..., R(u_k)$ that are translates of each other and whose upper-left corners touch a common line in this order. Scale $\Gamma^\star$ such that the arrangement can be placed in the interior of R_μ. Then connect the subdivision vertices on B_μ with the suitable rectangles. This is possible without introducing crossings since the order of the subdivision vertices on B_μ defines an order of their end-vertices in μ which is a subsequence of $u_1, ..., u_{k-1}, u_k, u_{k-1}, ..., u_2$, while the circular order in which the rectangles occur along the boundary of their arrangement is $R(u_1), ..., R(u_k), R(u_{k-1}), ..., R(u_2)$. By treating every other cluster of T analogously, we get a clique-planar representation of (G', S).

Conversely, suppose that (G', S) has a clique-planar representation Γ, which we can assume to be canonical by Lemma 3. We define a set $E^\star$ as follows. For each clique $s \in S$, let $R(u_1), ..., R(u_k)$ be the order in which the rectangles corresponding to s touch the line with slope 1 through their upper-left corners in Γ; add to $E^\star$ all the edges (u_i, u_{i+1}), for $i = 1, ..., k-1$. We claim that $E^\star$ is a linear saturator for (G, T). Indeed, by Lemma 6, it suffices to show that $G + E^\star$ admits a planar drawing.
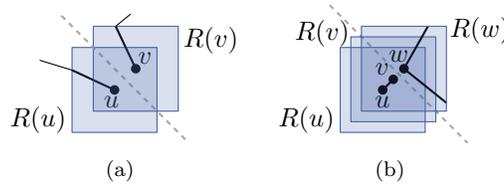
Figure 6: Construction of a linear saturator from a clique-planar representation.

Initialize $\Gamma^\star = \Gamma$. We place each vertex v at the center of the square $R(v)$ and remove $R(v)$ from $\Gamma^\star$. We extend each edge (u, v) with two straight-line segments from the boundaries of $R(u)$ and $R(v)$ to u and v, respectively. This does not produce crossings; in fact, only the segments of two vertices u and v such that $R(u)$ and $R(v)$ intersect might cross. However, such segments are separated by the line through the intersection points of the boundaries of $R(u)$ and $R(v)$; see Fig. 6(a). We now draw the edges in $E^\star$ as straight-line segments. As before, this may not introduce a crossing with any other segment or edge. In fact consider an edge (u, v) in $E^\star$ and any segment e_w incident to a vertex $w \neq u, v$ in the same clique. Assume u, v, w are in this order along the line with slope 1 through them. Then (u, v) is separated from e_w by the line through the two intersection points of the boundaries of $R(v)$ and $R(w)$; see Fig. 6(b). This concludes the proof. □

Next, we prove that the CPLS problem is $\mathcal{NP}$-complete.

Theorem 2 *The* CPLS *problem is* $\mathcal{NP}$*-complete, even if the underlying graph is a subdivision of a triangulated planar graph and there is just one cluster containing more than one vertex.*

Proof: The problem clearly lies in $\mathcal{NP}$. We give a polynomial-time reduction from the HAMILTONIAN PATH problem in biconnected planar graphs [17].

(a)   (b)
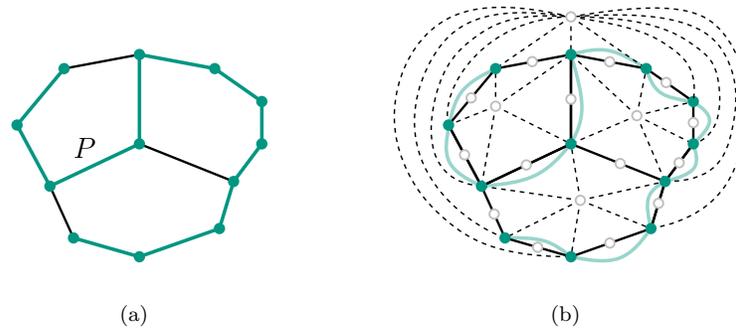
Figure 7: (a) A biconnected planar graph G with a Hamiltonian path P. (b) The clustered graph (G', T) obtained from G and the linear saturator for (G', T) corresponding to P.

Given a biconnected planar graph G we construct an instance (G', T) of CPLS that admits a linear saturator if and only if G has a Hamiltonian path. Initialize $G' = G$. Let $\mathcal{E}$ be a planar embedding of G', as in Fig. 7(a). For each face f, add a vertex v_f inside f and connect it to all the vertices incident to f. Since G is biconnected, each face f of $\mathcal{E}$ is bounded by a simple cycle, hence G' is a triangulated planar graph. Subdivide with a dummy vertex each edge of G' that is not incident to a vertex v_f, for any face f of $\mathcal{E}$, as in Fig. 7(b). Finally, add a cluster μ to T containing all the vertices of G and, for each of the remaining vertices, add to T a cluster containing only that vertex. Note that each cluster of (G', T) induces an independent set.

Suppose that G admits a Hamiltonian path $P = v_1, ..., v_n$ and let $E^\star = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$. Since P is Hamiltonian, $E^\star$ is a path connecting μ. Let $G^\star = G' + E^\star$. Since every cluster different from μ contains only one vertex, all the clusters of $(G^\star, T)$ induce paths. A planar drawing of $G^\star$ can be obtained from a planar drawing Γ of G' as follows. Note that, for each edge $(v_i, v_{i+1}) \in E^\star$, vertices v_i and v_{i+1} share two faces in Γ since the dummy vertex added to subdivide edge (v_i, v_{i+1}) has degree 2. Hence, each saturator edge (v_i, v_{i+1}) can be routed inside one of these faces arbitrarily close to the length-2 path between v_i and v_{i+1} neither crossing an edge of G' nor another saturator edge. Thus $G^\star$ is planar, and by Lemma 6 the set $E^\star$ is a linear saturator for (G', T).

Conversely, suppose that (G', T) admits a linear saturator $E^\star$. We claim that $E^\star$ is a Hamiltonian path of G. By construction, the vertices of μ are

exactly the vertices of G; also, each edge of $E^\star$ corresponds to an edge of G, due to the fact that two vertices of μ are incident to a common face in any planar drawing of G' if and only if they are adjacent in G. Hence, the path of G corresponding to $E^\star$ is Hamiltonian. This concludes the proof. □

The $\mathcal{NP}$-completeness of the CPLS problem proved in Theorem 2, together with the reduction from the CPLS problem to the CLIQUE PLANARITY problem proved in Lemma 7, implies Theorem 1.

5 Clique Planarity with Given Vertex Representations

In this section we show how to test CLIQUE PLANARITY in polynomial time for instances (G, S) with given vertex representations. That is, a clique-planar representation Γ' of (G', S) is given, where G' is obtained from G by removing its link-edges, and the goal is to test whether the link-edges of (G, S) can be drawn in Γ' to obtain a clique-planar representation Γ of (G, S).

Let n be the number of vertices of G. First, in a preprocessing step, we compute the boundary of the arrangement representing each clique $s \in S$ in Γ'. This can be performed in total $O(n \log n)$ time, since this boundary can be easily computed once the rectangles have been sorted by the x-coordinates of their left sides and by the y-coordinates of their bottom sides.

(a)　　　　　(b)　　　　　(c)
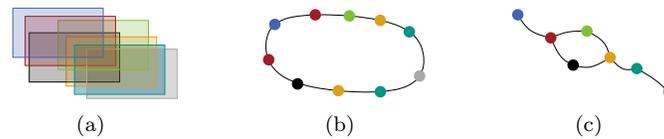
Figure 8: (a) An intersection-link representation Γ of $(K_7, \{s = K_7\})$. (b) A simple cycle with a vertex for each maximal portion of the boundary of Γ belonging to a single rectangle. (c) Planar drawing $\mathcal{H}'_s$ of graph H'_s corresponding to Γ.

We then check whether all the rectangles representing the vertices of each clique are pairwise intersecting. This can be done in total $O(n)$ time by computing the maximum x- and y-coordinates x_M and y_M among all bottom-left corners, the minimum x- and y-coordinates x_m and y_m among all top-right corners, and by checking whether $x_M < x_m$ and $y_M < y_m$.

We next test in total $O(n)$ time whether every vertex of G incident to a link-edge is represented in Γ' by a rectangle incident to the outer boundary of the clique it belongs to. If the test fails, the instance is negative. Otherwise, we proceed as follows.

We give a linear-time reduction to the PARTIAL EMBEDDING PLANARITY problem [5], which asks whether a planar drawing of a graph H exists extending a given drawing $\mathcal{H}'$ of a subgraph H' of H.

First, we define a connected component H'_s of H' corresponding to a clique $s \in S$ and its drawing $\mathcal{H}'_s$. We remark that H'_s is a *cactus graph*, that is a connected graph that admits a planar embedding in which all the edges are incident to the outer face. Denote by B the boundary of the representation of s in Γ' (see Fig. 8(a)). If s has one or two vertices, then H'_s is a vertex or an edge, respectively (and $\mathcal{H}'_s$ is any drawing of H'_s). Otherwise, initialize H'_s to a simple cycle containing a vertex for each maximal portion of B belonging to a single rectangle (see Fig. 8(b)). Let $\mathcal{H}'_s$ be any planar drawing of H'_s with a suitable orientation. Each rectangle in Γ' may correspond to two vertices of H'_s, but no more than two by Lemma 2. Insert an edge in H'_s between every two vertices representing the same rectangle and draw it in the interior of $\mathcal{H}'_s$. By Lemma 1, these edges do not alter the planarity of $\mathcal{H}'_s$. Contract the inserted edges in H'_s and $\mathcal{H}'_s$ (see Fig. 8(c)). This completes the construction of H'_s, together with its planar drawing $\mathcal{H}'_s$.

Graph H' is the union of graphs H'_s, over all the cliques $s \in S$; the drawings $\mathcal{H}'_s$ of H'_s are in the outer face of each other in $\mathcal{H}'$. Note that, because of the initial test, the end-vertices of each link-edge of G are ensured to be vertices of H'; then we define H as the graph obtained from H' by adding, for each link-edge (u, v) of G, an edge between the vertices of H' corresponding to u and v. We have the following.

Lemma 8 *There exists a planar drawing of H extending $\mathcal{H}'$ if and only if there exists a clique-planar representation of (G, S) coinciding with Γ' when restricted to (G', S).*

Proof: Let $\mathcal{H}$ be a planar drawing of H extending $\mathcal{H}'$. We construct a clique-planar representation Γ of (G, S) as follows. Initialize $\Gamma = \mathcal{H}$. For each clique $s \in S$, consider a closed polyline P_s close to $\mathcal{H}'_s$ so that it contains all and only the vertices and edges of H'_s in its interior and it crosses at most once every other edge of H. Scale Γ up so that, for every clique $s \in S$, a rectangle which is the bounding box of the representation of s in Γ' fits in the interior of P_s. Remove the interior of P_s and put in its place a copy of the representation of s in Γ'. Reroute the curves representing link-edges from the border of P_s to the suitable ending rectangles. This can be done without introducing any crossings, because the vertices of H'_s appear along the walk delimiting the outer face of $\mathcal{H}'_s$ in the same order as the corresponding rectangles appear along the boundary B of the representation of s in Γ', by construction. Finally, a homeomorphism of the plane can be exploited to translate the representation of each clique to the position it has in Γ', while maintaining the clique planarity of the representation.

Let Γ be a clique-planar representation of (G, S). We construct a planar drawing $\mathcal{H}$ of H extending $\mathcal{H}'$ as follows. Initialize $\mathcal{H} = \Gamma$. For each clique $s \in S$, consider a closed polyline P_s close to the representation of s in $\mathcal{H}$ so that it contains all and only the rectangles representing vertices of s and it crosses at most once each curve representing a link-edge of G. Remove the interior of P_s and put in its place a scaled copy of $\mathcal{H}'_s$. Reroute the curves representing link-edges from the border of P_s to the suitable end-vertices. As

in the previous direction, this can be done without introducing any crossings. Finally, a homeomorphism of the plane can be exploited to transform $\mathcal{H}$ into a planar drawing that coincides with $\mathcal{H}'$ when restricted to H.

This concludes the proof of the lemma.   □

We get the following main theorem of this section.

Theorem 3 CLIQUE PLANARITY *can be decided in $O(n \log n)$ time for a pair (G, S) if the rectangle representing each vertex of G is given as part of the input, where n is the number of vertices of G.*

Proof: The preprocessing step can be performed in $O(n \log n)$ time. The initial tests can be performed in $O(n)$ time. The described reduction to PARTIAL EMBEDDING PLANARITY can be performed in $O(n)$ time. Indeed, each graph H_s is initialized to a simple cycle by traversing the boundary B of each clique s, which takes $O(|s|)$ time since B has $O(n)$ complexity, by Lemma 2. Then edges are added to H_s and contracted afterwards; each contraction requires merging the adjacency lists of the end-vertices of the contracted edge, which can be done in constant time since these vertices have constant degree, again by Lemma 2. Finally, the PARTIAL EMBEDDING PLANARITY problem can be solved in $O(n)$ time [5].   □

6   Clique Planarity for Graphs with Two Cliques

In this section we study the CLIQUE PLANARITY problem for pairs (G, S) such that $|S| = 2$. Observe that, if $|S| = 1$, then the CLIQUE PLANARITY problem is trivial, since in this case G is a clique with no link-edge and a clique-planar representation of (G, S) can be easily constructed. The case in which $|S| = 2$ is already surprisingly non-trivial. Indeed, we could not determine the computational complexity of CLIQUE PLANARITY in this case. However, we establish the equivalence between our problem and a book embedding problem whose study might be interesting on its own; by means of this equivalence we show a polynomial-time algorithm for a special version of the CLIQUE PLANARITY problem. This book embedding problem is defined as follows.

A *2-page book embedding* is a plane drawing of a graph where the vertices are cyclically arranged along a closed curve ℓ, called the *spine*, and each edge is entirely drawn in one of the two regions of the plane delimited by ℓ. The 2-PAGE BOOK EMBEDDING problem asks whether a 2-page book embedding exists for a given graph. This problem is $\mathcal{NP}$-complete [20].

Now consider a bipartite graph $G(V_1 \cup V_2, E)$. We define a *bipartite 2-page book embedding* of G as a 2-page book embedding such that all the vertices in V_1 occur consecutively along the spine (and all the vertices in V_2 occur consecutively, as well). We call the corresponding embedding problem BIPARTITE 2-PAGE BOOK EMBEDDING.

Finally, we define a *bipartite 2-page book embedding with spine crossings* (B2PBESC), as a bipartite 2-page book embedding in which edges are not restricted to lie in one of the two regions delimited by ℓ, but each of them might

cross ℓ once. These crossings are only allowed to happen in the two portions of ℓ delimited by a vertex of V_1 and a vertex of V_2. We call the corresponding embedding problem BIPARTITE 2-PAGE BOOK EMBEDDING WITH SPINE CROSSINGS.

We now prove that the B2PBESC problem is equivalent to CLIQUE PLANARITY for instances (G, S) such that $|S| = 2$. Consider any instance $(G', \{s_1, s_2\})$ of the CLIQUE PLANARITY problem. An instance $G(V_1 \cup V_2, E)$ of the B2PBESC problem can be defined in which V_1 is the vertex set of s_1 and V_2 is the vertex set of s_2; also, E consists of all the link-edges of G'. Conversely, given an instance $G(V_1 \cup V_2, E)$ of the B2PBESC problem, an instance $(G', \{s_1, s_2\})$ of CLIQUE PLANARITY can be constructed in which s_1 is a clique on V_1 and s_2 is a clique on V_2; the set of link-edges of G' coincides with E. Observe that, since link-edges only connect vertices of different cliques and since edges of E only connect vertices of V_1 to vertices of V_2, each mapping generates a valid instance for the other problem. Also, these mappings define a bijection, hence the following lemma establishes the equivalence between the two problems.

Lemma 9 $(G', \{s_1, s_2\})$ *is clique-planar if and only if* $G(V_1 \cup V_2, E)$ *admits a* B2PBESC.

Proof: Suppose that there exists a B2PBESC $\mathcal{B}$ of $G(V_1 \cup V_2, E)$. We construct a clique-planar representation Γ of (G', S) as follows. Initialize $\Gamma = \mathcal{B}$. Relabel the vertices in V_1 (resp. in V_2) as $u_1, ..., u_k$ (resp. $v_1, ..., v_h$) according to the order in which they appear along ℓ. Draw a closed curve λ_1 (λ_2) enclosing a portion of the spine ℓ containing all and only the vertices $u_1, ..., u_k$ (resp. $v_1, ..., v_h$). Scale Γ up so that λ_1 and λ_2 are large enough to contain a square of size $(1+\epsilon) \times (1+\epsilon)$ in their interiors, with $\epsilon > 0$. Remove the interior of λ_1 and λ_2. Draw pairwise-intersecting unit squares $Q(u_1), ..., Q(u_k)$ (resp. $Q(v_1), ..., Q(v_h)$) all in the interior of λ_1 (resp. of λ_2) with their upper-left corners in this order along a common line l_1 (resp. l_2). Reroute the curves representing portions of link-edges from the border of λ_1 and λ_2 to the suitable ending squares inside them. This can be done without introducing any crossings, because the vertices of V_1 (V_2) appear along ℓ in the same order as the corresponding squares touch l_1 (resp. l_2); also, the portion of a link-edge connecting a point on λ_1 with a point on λ_2 is contained in the original drawing of the edge of G, hence no two such portions cross each other. Thus, Γ is a clique-planar representation of (G', S).

Suppose that there exists a clique-planar representation Γ of $(G', \{s_1, s_2\})$. We construct a B2PBESC $\mathcal{B}$ of $G(V_1 \cup V_2, E)$ as follows. Refer to Fig. 9. Initialize $\mathcal{B} = \Gamma$. By Lemma 2, there exists a labeling $u_1, ..., u_k$ of the vertices in s_1 such that the order in which the rectangles representing these vertices are encountered when traversing the boundary B_1 of their arrangement clockwise is a subsequence of $R(u_1), ..., R(u_k), R(u_{k-1}), ..., R(u_2)$. Hence, for any two points p_1^a on $R(u_1) \cap B_1$ and p_1^b on $R(u_k) \cap B_1$, there exists a curve ℓ_1 between p_1^a and p_1^b entering $R(u_1), ..., R(u_k)$ in this order. Place vertex u_i of V_1 at the point where ℓ_1 enters $R(u_i)$. Define ℓ_2, p_2^a and p_2^b, and draw the vertices of
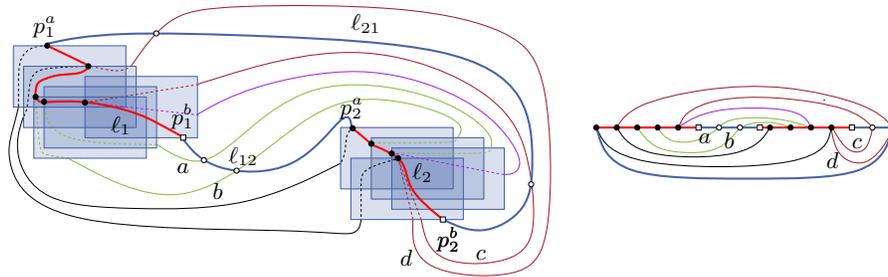
Figure 9: Constructing a B2PBESC from a clique-planar representation.

V_2 analogously. Further, add to $\mathcal{B}$ two curves ℓ_{12} and ℓ_{21}, not intersecting each other, not intersecting the same link-edge, each intersecting a link edge at most once, and connecting p_1^a to p_2^b, and p_2^a to p_1^b, respectively. Reroute the curves representing portions of the edges in E from B_1 and B_2 to the suitable ending vertices inside them. This can be done without introducing any crossings, because the vertices of V_1 (V_2) appear along ℓ_1 (along ℓ_2) in the same order as the corresponding rectangles appear along B_1 (along B_2) in Γ, by construction. Finally, consider the curve ℓ composed of ℓ_1, ℓ_{12}, ℓ_2, and ℓ_{21}. We have that all the vertices of V_1 (of V_2) appear consecutively along ℓ, since they all lie on ℓ_1 (on ℓ_2); also, each edge $e \in E$ crosses ℓ at most once, either on ℓ_{12} or on ℓ_{21}. Hence, $\mathcal{B}$ is a B2PBESC of $G(V_1 \cup V_2, E)$. This concludes the proof of the lemma. □

We now consider a variant of the CLIQUE PLANARITY problem for two cliques in which the link-edges incident to a clique are partitioned into two sets, and the goal is to construct a clique-planar representation in which the link-edges in different sets of the partition exit the clique on "different sides". This constraint finds a correspondence with the variant of the (non-bipartite) 2-PAGE BOOK EMBEDDING problem, called the PARTITIONED 2-PAGE BOOK EMBEDDING problem, in which vertices are allowed to be arbitrarily permuted along the spine, while the edges are pre-assigned to the pages of the book [4, 16].
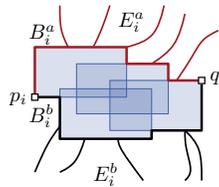
Figure 10: An intersection-link representation Γ_i of s_i. The top side of Γ_i and the link-edges in E_i^a are red, while the bottom side of Γ_i and the link-edges in E_i^b are black.

More formally, let $(G, S = \{s_1, s_2\})$ be an instance of CLIQUE PLANARITY

and let $\{E_i^a, E_i^b\}$ be a partition of the link-edges incident to s_i, with $i \in \{1, 2\}$. Consider an intersection-link representation Γ_i of s_i with outer boundary B_i, let p_i be the bottom-left corner of the leftmost rectangle in Γ_i, and let q_i be the upper-right corner of the rightmost rectangle in Γ_i; see Fig. 10. Let B_i^a be the part of B_i from p_i to q_i in clockwise direction (this is the *top side* of Γ_i) and let B_i^b be the part of B_i from q_i to p_i in clockwise direction (this is the *bottom side* of Γ_i). We aim to construct a clique-planar representation of (G, S) in which all the link-edges in E_i^a (resp. in E_i^b) intersect the arrangement Γ_i of rectangles representing s_i on the top side (resp. on the bottom side) of Γ_i. We call the problem of determining whether such a representation exists 2-PARTITIONED CLIQUE PLANARITY. We prove that 2-PARTITIONED CLIQUE PLANARITY can be solved in quadratic time.

The algorithm is based on a reduction to equivalent special instances of the SIMULTANEOUS EMBEDDING WITH FIXED EDGES (SEFE) problem that can be decided in quadratic time. Given two graphs G_1 and G_2 on the same vertex set V, the SEFE problem asks to find planar drawings of G_1 and G_2 that coincide on V and on the common edges of G_1 and G_2. We have the following.

Lemma 10 *Let* $(G, \{s_1, s_2\})$ *and* $\{E_1^a, E_1^b, E_2^a, E_2^b\}$ *be an instance of the* 2-PARTITIONED CLIQUE PLANARITY *problem. An equivalent instance* $\langle G_1, G_2 \rangle$ *of* SEFE *such that* $G_1 = (V, E_1)$ *and* $G_2 = (V, E_2)$ *are 2-connected and such that the common graph* $G_\cap = (V, E_1 \cap E_2)$ *is connected can be constructed in linear time.*

Proof: By Lemma 9, we can describe $(G, \{s_1, s_2\})$ by its equivalent instance $G(V_1 \cup V_2, E)$ of the B2PBESC problem, where V_1 is the vertex set of s_1, V_2 is the vertex set of s_2, and E is the set of link-edges of $(G, \{s_1, s_2\})$. The partition $\{E_i^a, E_i^b\}$ of the edges incident to s_i translates to constraints on the side of the spine ℓ of the B2PBESC each of these edges has to be incident to. Namely, for each vertex $u \in V_1$, all the edges in E_1^a (in E_1^b) incident to u have to exit u from the internal (resp. external) side of ℓ; and analogously for the edges of E_2^a and E_2^b. This implies that edges in $E_1^a \cap E_2^a$ entirely lie inside ℓ, edges in $E_1^b \cap E_2^b$ entirely lie outside ℓ, while the other edges have to cross ℓ.

We now describe how to construct $\langle G_1, G_2 \rangle$. Refer to Fig. 11.

The common graph $G_\cap$ contains a cycle $\mathcal{C} = (t_1, r_1, t_2, t_3, r_2, t_4, q_2, q_1)$. Also, $G_\cap$ contains two stars Q_1 and Q_2 centered at q_1 and q_2, respectively, where Q_1 has a leaf $w(u)$ for each vertex $u \in V_1$ and a leaf $w(e)$ for each edge $e \in E_1^a \cap E_2^b$, and where Q_2 has a leaf $w'(v)$ for each vertex $v \in V_2$ and a leaf $w'(e)$ for each edge $e \in E_1^b \cap E_2^a$. Further, $G_\cap$ contains trees T_i rooted at t_i, for $i = 1, ..., 4$, defined as follows.

Tree T_1 contains a leaf $z(e)$ adjacent to t_1 for each edge $e \in E_1^a \cap E_2^b$; also, it contains a vertex w_1 adjacent to t_1; finally, it contains a leaf $z(u)$ adjacent to w_1 for each vertex $u \in V_1$ that is incident to at least one edge in E_1^b.

Tree T_2 contains a leaf $z'(e)$ adjacent to t_2 for each edge $e \in E_1^a \cap E_2^b$; also, it contains a vertex w_2 adjacent to t_2; finally, it contains a leaf $z'(u)$ adjacent to w_2 for each vertex $u \in V_1$ that is incident to at least one edge in E_1^a.
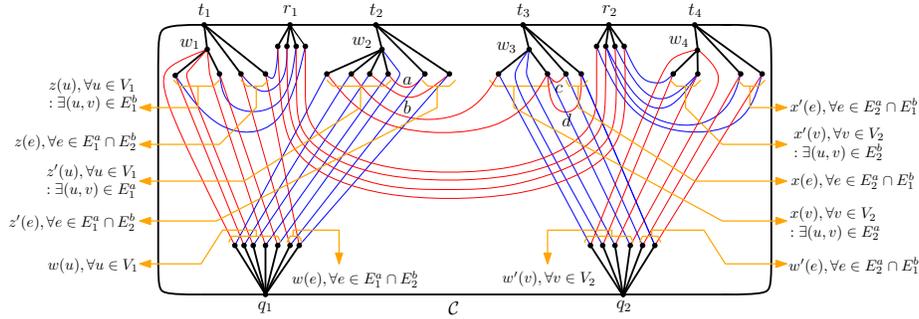
Figure 11: The SEFE $\langle \Gamma_1, \Gamma_2 \rangle$ of $\langle G_1, G_2 \rangle$ corresponding to Γ from Fig. 9.

Tree T_3 contains a leaf $x(e)$ adjacent to t_3 for each edge $e \in E_2^a \cap E_1^b$; also, it contains a vertex w_3 adjacent to t_3; finally, it contains a leaf $x(v)$ adjacent to w_3 for each vertex $v \in V_2$ that is incident to at least one edge in E_2^a.

Tree T_4 contains a leaf $x'(e)$ adjacent to t_4 for each edge $e \in E_2^a \cap E_1^b$; also, it contains a vertex w_4 adjacent to t_4; finally, it contains a leaf $x'(v)$ adjacent to w_4 for each vertex $v \in V_2$ that is incident to at least one edge in E_2^b.

Finally, $G_\cap$ contains two stars R_1 and R_2 centered at r_1 and r_2, respectively, with the same number of leaves as T_1. Namely, R_1 (R_2) contains a leaf $r_1(e)$ (a leaf $r_2(e)$) adjacent to r_1 (to r_2, resp.) for each edge $e \in E_1^a \cap E_2^b$; also, it contains a leaf $r_1(u)$ (a leaf $r_2(u)$, resp.) for each vertex $u \in V_1$ that is incident to at least one edge in E_1^b.

Graph G_1 contains $G_\cap$ plus the following edges. Consider each edge $e = (u, v)$ with $u \in V_1$ and $v \in V_2$. If $e \in E_1^a \cap E_2^b$, graph G_1 has an edge $(w(e), z(e))$ and an edge $(z'(u), z'(e))$; if $e \in E_1^b \cap E_2^a$, graph G_1 has an edge $(w'(e), x'(e))$ and an edge $(x(v), x(e))$; if $e \in E_1^a \cap E_2^a$, graph G_1 has an edge $(z'(u), x(v))$. For each vertex $u \in V_1$, if u is incident to at least one edge in E_1^b, then G_1 contains edge $(w(u), z(u))$ and edge $(r_1(u), r_2(u))$, otherwise it contains edge $(w(u), w_1)$. For each vertex $v \in V_2$, if v is incident to at least one edge in E_2^b, then G_1 contains edge $(w'(v), x'(v))$, otherwise it contains edge $(w'(v), w_4)$.

Graph G_2 contains $G_\cap$ plus the following edges. Consider each edge $e = (u, v)$ with $u \in V_1$ and $v \in V_2$. If $e \in E_1^a \cap E_2^b$, graph G_2 has an edge $(w(e), z'(e))$, an edge $(z(e), r_1(e))$, and an edge $(r_2(e), x'(v))$. If $e \in E_2^a \cap E_1^b$, graph G_2 has an edge $(r_2(u), x'(e))$ and an edge $(w'(e), x(e))$. If $e \in E_1^b \cap E_2^b$, graph G_1 has an edge $(r_2(u), x'(v))$. For each vertex $u \in V_1$, if u is incident to at least one edge in E_1^a, then G_2 contains edge $(w(u), z'(u))$, otherwise it contains edge $(w(u), w_2)$. Also, if u is incident to at least one edge in E_1^b, then G_2 contains edge $(w(u), r_1(u))$. For each vertex $v \in V_2$, if v is incident to at least one edge in E_2^a, then G_2 contains edge $(w'(v), x(v))$, otherwise it contains edge $(w'(v), w_3)$.

The order of the leaves of Q_1 represents the order in which the vertices of s_1 and the crossings between edges of $E_1^a \cap E_2^b$ appear along ℓ; analogously, the order of the leaves of Q_2 represents the order in which the vertices of s_2 and

the crossings between edges of $E_2^a \cap E_1^b$ appear along ℓ. Hence, concatenating these two orders yields a total order of the vertices and of the crossings along ℓ. In the following we prove that this correspondence determines a B2PBESC, when starting from a SEFE, and vice versa.

Suppose that $\langle G_1, G_2 \rangle$ admits a SEFE. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the orders in which the leaves of Q_1 and of Q_2 appear around q_1 and q_2, respectively. We obtain a B2PBESC $\mathcal{B}$ by concatenating $\mathcal{O}_1$ and $\mathcal{O}_2$, where each vertex $u \in V_1$ corresponds to leaf $w(u)$ of Q_1, each vertex $v \in V_2$ corresponds to leaf $w'(v)$ of Q_2, and the crossing between an edge e and the spine ℓ of $\mathcal{B}$ corresponds to either leaf $w(e)$ of Q_1 or leaf $w'(e)$ of Q_2, depending on whether $e \in E_1^a \cap E_2^b$ or $e \in E_2^a \cap E_1^b$.

Since the leaves of Q_1 corresponding to vertices of V_1 are all connected to vertex w_1 by paths belonging to the same graph G_1, they appear consecutively around q_1, and hence the corresponding vertices of V_1 appear consecutively along ℓ. Analogously, all the vertices of V_2 appear consecutively along ℓ. These two facts imply that all the edges cross the spine in a point lying between a vertex of V_1 and a vertex of V_2. Hence, the order of the vertices along the spine ℓ of $\mathcal{B}$ is consistent with a valid B2PBESC. We now show that this order also allows us to draw the edges without crossings.

The routing of each edge $e = (u, v)$ is performed as follows. If $e \in E_1^a \cap E_2^a$, then e is drawn as a curve on the internal side of ℓ. If $e \in E_1^b \cap E_2^b$, then e is drawn as a curve on the external side of ℓ. If $e \in E_1^a \cap E_2^b$, then e is drawn as a curve whose portion between u and the crossing point y_e is on the internal side of ℓ, and whose portion between y_e and v is on the external side of ℓ. If $e \in E_1^b \cap E_2^a$, then e is drawn as a curve whose portion between u and y_e is on the external side of ℓ, and whose portion between y_e and v is on the internal side of ℓ.

First observe that, because of the edges connecting Q_1 with T_1 and T_2, the clockwise order of the leaves of Q_1 coincides with the counterclockwise order of the corresponding leaves of T_1 and T_2. The same holds for Q_2 with respect to T_3 and T_4. Also, because of the edges connecting R_1 with T_1 and R_2, the clockwise order of the leaves of R_2 is the same as the one of the corresponding leaves of T_1.

We now prove that no two edges in the constructed book embedding cross each other. Observe that each edge either entirely lies in one of the two sides of ℓ, or it crosses ℓ once, hence it is composed of two portions in different sides of ℓ. Clearly, it suffices to prove that no two edges (or portions) on the same side of ℓ cross each other.

Consider the portions of the edges of $G(V_1 \cup V_2, E)$ that lie on the same side of ℓ, say on the internal side of ℓ: these are the edges $e = (u, v)$ in $E_1^a \cap E_2^a$, the portions of the edges $e = (u, v)$ in $E_1^a \cap E_2^b$ between u and y_e, and the portions of the edges $e = (u, v)$ in $E_1^b \cap E_2^a$ between y_e and v. Every edge of the first type corresponds to an edge $(z'(u), x(v))$ of G_1; every portion of an edge of the second type corresponds to an edge $(z'(u), z'(e))$ of G_1; and every portion of an edge of the third type corresponds to an edge $(x(e), x(v))$ of G_1. Since all these edges belong to G_1, they do not cross in the given SEFE. By construction,

for each vertex of G incident to at least one of the considered edges, there is a corresponding leaf of either T_2 or T_3 in $G_\cap$; also, for each of these edges that crosses ℓ, there is a corresponding leaf of either T_2 or T_3 in $G_\cap$. Further, these leaves appear, in the left-to-right order of the leaves of T_2 and T_3, in the same order as they appear along ℓ, by the previous observation that the order of the leaves of T_2 (of T_3) coincides with the one of Q_1 (of Q_2, resp.). This implies that the considered edges of $G(V_1 \cup V_2, E)$ do not cross in the internal side of ℓ. Analogous arguments can be used to prove that the portions of the edges of $G(V_1 \cup V_2, E)$ lying in the external side of ℓ do not cross each other. In this case, the edges of the SEFE that have to be considered are $(r_2(u), x'(v))$, $(r_2(u), x'(e))$, and $(r_2(e), x'(v))$, which all belong to G_2.

We now prove the opposite direction. Suppose that $G(V_1 \cup V_2, E)$ admits a B2PBESC $\mathcal{B}$. We construct a SEFE of $\langle G_1, G_2 \rangle$ as follows.

We define a linear ordering σ_1 of the vertices of V_1 and the crossings between ℓ and the edges in $E_1^a \cap E_2^b$; σ_1 is the clockwise order in which such vertices and crossings appear along ℓ starting at any vertex of V_2. Observe that, since $\mathcal{B}$ is a B2PBESC, all the vertices in V_1 appear consecutively in σ_1. Analogously, we define a linear ordering σ_2 of the vertices of V_2 and the crossings between ℓ and the edges in $E_2^a \cap E_1^b$; σ_2 is the clockwise order in which such vertices and crossings appear along ℓ starting at any vertex of V_1. Observe that, since $\mathcal{B}$ is a B2PBESC, all the vertices in V_2 appear consecutively in σ_2.

Recall that each leaf of Q_1 either corresponds to a vertex in V_1 or to an edge in $E_1^a \cap E_2^b$. Thus, we can define a clockwise linear ordering of the leaves of Q_1 around q_1 as the corresponding vertices and crossings appear in σ_1; this linear ordering starts after edge (q_1, t_1). A clockwise linear ordering of the leaves of Q_2 around q_2 is defined analogously from σ_2 starting from edge (q_2, q_1).

The order of the leaves of trees T_i, with $i = 1, 2, 3, 4$, and of stars R_1 and R_2 is also decided based on σ_1 and σ_2. This allows us to draw all the edges of G_1 and G_2 that are incident to stars Q_1, Q_2, R_1, and R_2 without crossings; in particular, the paths connecting q_1 to w_1 and w_2, and those connecting q_2 to w_3 and w_4 can be drawn without crossings since the vertices of V_1 (and the vertices of V_2) are consecutive along ℓ in $\mathcal{B}$. The fact that the edges connecting the leaves of T_2 and T_3, and the edges connecting the leaves of R_2 and T_4 can be drawn without crossings is again due to the fact that, by construction, these edges correspond to portions of edges of G lying on the same side of ℓ.

Graph $G_\cap$ is connected, by construction. The fact that G_1 and G_2 are 2-connected can be proved as follows. Graph G_1 is composed of the outer cycle $\mathcal{C}$, plus a set of 2-connected components connecting pairs of vertices of $\mathcal{C}$. One component connects q_1 to t_1; one component connects r_1 to r_2; one connects q_2 to t_4; and another one connects t_2 to t_3. In particular, in order for this latter component to actually exist, at least one edge in $E_1^a \cap E_2^a$ must exist. However, this fact can be assumed without loss of generality, as otherwise two dummy vertices and an edge in $E_1^a \cap E_2^a$ between them could be added to the instance without altering the possibility of finding a B2PBESC. As for G_2, it is also composed of the outer cycle $\mathcal{C}$, plus a set of 2-connected components connecting pairs of vertices of $\mathcal{C}$. One component connects q_1 to t_2; one connects t_1 to r_1;

one connects r_2 to t_4; and another one connects q_2 to t_3.

As $\langle G_1, G_2 \rangle$ can be easily constructed in linear time, the lemma follows. □

Theorem 4 *Problem* 2-PARTITIONED CLIQUE PLANARITY *can be solved in quadratic time for instances* (G, S) *in which* $|S| = 2$.

Proof: Apply Lemma 10 to construct in linear time an instance $\langle G_1, G_2 \rangle$ of SEFE that is equivalent to (G, S) such that G_1 and G_2 are biconnected and their intersection graph $G_\cap$ is connected. The statement follows since instances of the SEFE problem with this property can be solved in quadratic time [7]. □

7   Hierarchical Clique Planarity

In this section we study a version of the CLIQUE PLANARITY problem in which the cliques are given together with a hierarchical relationship among them. Namely, let (G, S) be an instance of CLIQUE PLANARITY and let $\psi : S \to \{1, ..., k\}$, with $k \leq |S|$, be an assignment of the cliques in S to k levels such that, for each link-edge (u, v) of G connecting a vertex u of a clique s' to a vertex v of a clique s'', we have $\psi(s') \neq \psi(s'')$; an instance is *proper* if $\psi(s') = \psi(s'') \pm 1$ for each link-edge.

We aim to construct canonical clique-planar representations of (G, S) such that:

Property 1 For each clique $s \in S$, the top side of the bounding box of the representation of s lies on the line $y = 2\psi(s)$, while the bottom side lies above the line $y = 2\psi(s) - 2$; and

Property 2 Each link-edge (u, v), with $u \in s'$, $v \in s''$, $\psi(s') < \psi(s'')$, is drawn as a y-monotone curve from the top side of $R(u)$ to the bottom side of $R(v)$.

We call the problem of testing whether such a representation exists LEVEL CLIQUE PLANARITY.

We show how to test level clique planarity in quadratic time for proper instances via a linear-time reduction to equivalent proper instances of $\mathcal{T}$-LEVEL PLANARITY [3].

A $\mathcal{T}$-*level graph* $(V, E, \gamma, \mathcal{T})$ consists of:
 (i) a graph $G = (V, E)$;
 (ii) a function $\gamma : V \to \{1, ..., k\}$ such that $\gamma(u) \neq \gamma(v)$ for each $(u, v) \in E$, where the set $V_i = \{v \mid \gamma(v) = i\}$ is the i-th *level* of G; and
 (iii) a set $\mathcal{T} = \{T_1, ..., T_k\}$ of rooted trees such that the leaves of T_i are the vertices in V_i.

A $\mathcal{T}$-*level planar drawing* of $(V, E, \gamma, \mathcal{T})$ is a planar drawing of G where the edges are y-monotone curves and the vertices in V_i are placed along the line $y = i$, denoted by ℓ_i, according to an order *compatible* with T_i; that is, for each internal node μ of T_i, the leaves of the subtree of T_i rooted at μ are consecutive along ℓ_i. A $\mathcal{T}$-level graph is $\mathcal{T}$-*level planar* if it admits a $\mathcal{T}$-level planar drawing.

The $\mathcal{T}$-LEVEL PLANARITY problem asks to test whether a $\mathcal{T}$-level graph is $\mathcal{T}$-level planar. We have the following.

Lemma 11 *Given a proper instance of* LEVEL CLIQUE PLANARITY*, an equivalent proper instance of* $\mathcal{T}$-LEVEL PLANARITY *can be constructed in linear time.*

Proof: Given $(G(V,E), S, \psi)$, an instance $(V, E', \gamma, \mathcal{T})$ of $\mathcal{T}$-LEVEL PLANARITY can be constructed as follows. The vertex sets of the graphs coincide and E' coincides with the set of link-edges in E. For each vertex v in a clique $s \in S$ we have $\gamma(v) = \psi(s)$. Finally, for $i = 1, ..., k$, where k is the number of levels in (G, S, ψ), tree $T_i \in \mathcal{T}$ has root r_i, a child w_s of r_i for each $s \in S$, and the vertices of s as children of w_s.

(a)

(b)
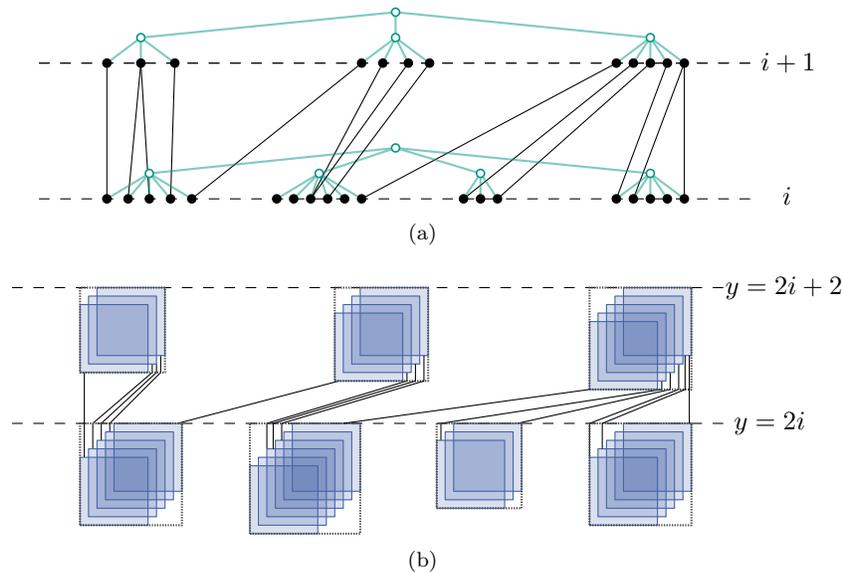
Figure 12: Construction of a clique-planar representation of $(G(V,E), S, \psi)$ from a $\mathcal{T}$-level planar drawing Γ of $(V, E', \gamma, \mathcal{T})$. (a) The part of Γ between two levels i and $i+1$. The edges and the internal nodes of trees T_i and T_{i+1} are green, while the vertices in V and the edges in E' are black. (b) The corresponding clique-planar representation between levels i and $i+1$. The bounding box of the representation of each clique is dotted.

Suppose that $(V, E', \gamma, \mathcal{T})$ admits a $\mathcal{T}$-level planar drawing Γ, as in Fig. 12(a). We construct a clique-planar representation with the desired properties as follows; refer to Fig. 12(b). For each clique $s \in S$ with $\psi(s) = i$, we construct a canonical representation of s in a bounding box of size $(1 + \varepsilon) \times (1 + \varepsilon)$, with $0 < \varepsilon < 1$, and plug it between lines $y = 2\psi(s)$ and $y = 2\psi(s) - 2$ with the top side of the bounding box lying on line $y = 2\psi(s)$. Note that the bottom side of the bounding box is above the line $y = 2\psi(s) - 2$. Cliques on the same

level i are placed side-by-side, so that they do not touch each other, in the same order as the corresponding children of r_i appear around r_i in Γ. Finally, for each two consecutive levels V_i and V_{i+1}, consider the edges in E' connecting a vertex $u \in V_i$ with a vertex $v \in V_{i+1}$ as they appear in Γ from left to right; we draw the corresponding link-edge $(u, v) \in E$ as a polyline, lying to the right of any previously drawn edge between V_i and V_{i+1}, composed of three segments: the first is a vertical segment connecting a point on the top side of $R(u)$ with a point p_u on the top side of the bounding box of s', the third is a vertical segment connecting a point on the bottom side of $R(v)$ with a point p_v on the bottom side of the bounding box of s'', the middle one is a straight-line segment connecting p_u with p_v.

The obtained representation satisfies Properties 1 and 2 by construction. That the link-edges do not cross each other descends from the following three facts. First, the order of the cliques along the same level i is the same as the order of the corresponding children of the root of T_i. Second, the node w_s in $T_{\psi(s)}$ enforces vertices of the same clique s to be consecutive along $\ell_{\psi(s)}$ in Γ. Third, for each clique $s \in S$, the squares representing the vertices of s are in the same order as the corresponding vertices of s along $\ell_{\psi(s)}$. This implies that, for any two cliques $s' \in V_i$ and $s'' \in V_{i+1}$, the left-to-right order in which the link-edges between s' and s'' intersect the line $y = 2\psi(s')$ is the same as the one in which they intersect the line $y = 2(\psi(s'')) - (1 + \varepsilon)$, hence no two such link-edges cross.

Suppose that $(G(V, E), S, \psi)$ admits a clique-planar representation satisfying Properties 1 and 2. We construct a $\mathcal{T}$-level planar drawing Γ of $(V, E', \gamma, \mathcal{T})$ as follows. For $i = 1, ..., k$, consider a line ℓ_i defined as $y = 2i - 1$. Place each vertex $v \in V_i$ at the intersection between ℓ_i and the left side of $R(v)$. Note that such an intersection exists since the clique-planar representation of $(G(V, E), S, \psi)$ satisfies Property 1. Draw each edge (u, v) with $u \in V_i$ and $v \in V_{i+1}$ as a curve composed of three parts: The middle part coincides with the drawing of the corresponding link-edge, which connects a point p_u on the top side of $R(u)$ with a point p_v on the bottom side of $R(v)$; the first part is a curve connecting u with p_u entirely contained inside $R(u)$ not crossing any other edge (this can be done by routing the curve first following the left side of $R(u)$ and then following the top side of $R(u)$); and the last part is a curve connecting v with p_v entirely contained inside $R(v)$ not crossing any other edge (this can be done by routing the curve first following the left side of $R(v)$ and then following the bottom side of $R(v)$).

We show that Γ is a $\mathcal{T}$-level planar drawing of $(V, E', \gamma, \mathcal{T})$. No two edges cross in Γ since the middle parts of the edges in E' have the same drawing as the link-edges in E, which do not cross by hypothesis, while the first and the last parts do not cross by construction. Finally, the fact that the ordering of the vertices of V_i along ℓ_i is compatible with T_i descends from the fact that ℓ_i intersects all the rectangles of each clique s with $\psi(s) = i$ and that no two rectangles representing vertices belonging to different cliques overlap. Hence, vertices belonging to the same clique, and hence children of the same internal node of T_i, are consecutive along ℓ_i. The construction can be performed in

linear time, thus proving the lemma. $\square$

We thus get the main result of this section.

Theorem 5 LEVEL CLIQUE PLANARITY *is solvable in quadratic time for proper instances and in quartic time for general instances.*

Proof: Any instance (G, S, ψ) of LEVEL CLIQUE PLANARITY can be made proper by introducing dummy cliques composed of single vertices to split link-edges spanning more than one level. This does not alter the level clique planarity of the instance and might introduce a quadratic number of vertices. Lemma 11 constructs in linear time an equivalent proper instance of $\mathcal{T}$-LEVEL PLANARITY. The statement follows since $\mathcal{T}$-LEVEL PLANARITY can be solved in quadratic time [3] for proper instances. $\square$

8   Conclusions and Open Problems

We initiated the study of hybrid representations of graphs in which vertices are geometric objects and edges are either represented by intersections (if part of dense subgraphs) or by curves (otherwise). Several intriguing questions arise from our research.

1. How about considering families of dense graphs richer than cliques? Other natural families of dense graphs could be considered, say interval graphs, complete bipartite graphs, or triangle-free graphs.

2. How about using different geometric objects for representing vertices? Even simple objects like equilateral triangles or unit circles seem to pose great challenges, as they give rise to arrangements with a complex combinatorial structure. For example, we have no counterpart of Lemma 2 in those cases.

3. What is the complexity of the BIPARTITE 2-PAGE BOOK EMBEDDING problem? We remark that, in the version in which spine crossings are allowed, this problem is equivalent to the clique planarity problem for instances with two cliques.

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