Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 21, no. 6, pp. 983–1002 (2017)
DOI: 10.7155/jgaa.00446

Column planarity and partially-simultaneous geometric embedding

Luis Barba[1] *William Evans*[2] *Michael Hoffmann*[1] *Vincent Kusters*[1] *Maria Saumell*[3] *Bettina Speckmann*[4]

[1]Dep. of Computer Science, ETH Zürich, Switzerland
[2]Dep. of Computer Science, University of British Columbia, Canada
[3]Inst. of Computer Science, The Czech Academy of Sciences, Czech Republic
[4]Dep. of Mathematics and Computer Science, TU Eindhoven, Netherlands

Abstract

We introduce the notion of *column planarity* of a subset R of the vertices of a graph G. Informally, we say that R is column planar in G if we can assign x-coordinates to the vertices in R such that any assignment of y-coordinates to them produces a partial embedding that can be completed to a plane straight-line drawing of G. Column planarity is both a relaxation and a strengthening of unlabeled level planarity. We prove near tight bounds for the maximum size of column planar subsets of trees: every tree on n vertices contains a column planar set of size at least $14n/17$ and for any $\epsilon > 0$ and any sufficiently large n, there exists an n-vertex tree in which every column planar subset has size at most $(5/6+\epsilon)n$. In addition, we show that every outerplanar graph has a column planar set of size at least $n/2$.

We also consider a relaxation of simultaneous geometric embedding (SGE), which we call partially-simultaneous geometric embedding (PSGE). A PSGE of two graphs G_1 and G_2 allows some of their vertices to map to two different points in the plane. We show how to use column planar subsets to construct k-*PSGEs*, which are PSGEs in which at least k vertices are mapped to the same point for both graphs. In particular, we show that every two trees on n vertices admit an $11n/17$-PSGE and every two outerplanar graphs admit an $n/4$-PSGE.

Submitted:	Reviewed:	Revised:	Reviewed:	Revised:
December 2015	May 2016	September 2016	November 2016	May 2017
	Accepted:	Final:	Published:	
	July 2017	September 2017	October 2017	
	Communicated by:		Article type:	
	S.-H. Hong		Regular Paper	

L. Barba is supported by the ETH Postdoctoral Fellowship. M. Hoffmann and V. Kusters are partially supported by the ESF EUROCORES programme EuroGIGA, CRP GraDR and the Swiss National Science Foundation, SNF Project 20GG21-134306. M. Saumell was supported by project LO1506 of the Czech Ministry of Education, Youth and Sports, by project NEXLIZ CZ.1.07/2.3.00/30.0038, co-financed by the European Social Fund and the state budget of the Czech Republic, and by institutional support RVO:67985807. W. Evans is partially supported by NSERC Canada. Preliminary versions of this paper appeared in [2, 16].

E-mail addresses: luis.barba@inf.ethz.ch (Luis Barba) will@cs.ubc.ca (William Evans) hoffmann@inf.ethz.ch (Michael Hoffmann) vincent.kusters@inf.ethz.ch (Vincent Kusters) saumell@cs.cas.cz (Maria Saumell) b.speckmann@tue.nl (Bettina Speckmann)

1 Introduction

A graph $G = (V, E)$ on n vertices is *unlabeled level planar* (ULP) if for all injections $\gamma : V \to \mathbb{R}$, there exists an injection $\varrho : V \to \mathbb{R}$, so that embedding each $v \in V$ at $(\varrho(v), \gamma(v))$ results in a plane straight-line embedding of G. In other words, for any assignment of y-coordinates to the vertices of G, there exists an assignment of x-coordinates that results in a plane straight-line embedding of G. Estrella-Balderrama, Fowler and Kobourov [14] originally introduced ULP graphs and characterized ULP trees in terms of forbidden subgraphs. Fowler and Kobourov [17] extended this characterization to general ULP graphs. ULP graphs are exactly the graphs that admit a simultaneous geometric embedding with a monotone path: this was the original motivation for studying them.

In this paper we introduce the notion of *column planarity* of a subset R of the vertices V of a graph $G = (V, E)$. Informally, we say that R is column planar in G if we can assign x-coordinates to the vertices in R such that any assignment of y-coordinates to them produces a partial embedding that can be completed to a plane straight-line drawing of G. Column planarity is both a relaxation and a strengthening of unlabeled level planarity. It is a relaxation since it applies only to a subset R of the vertices and a strengthening since the requirements on R are more strict than in the case of unlabeled level planarity.

More formally, for $R \subseteq V$, we say that R is *column planar in* $G = (V, E)$ if there exists an injection $\varrho : R \to \mathbb{R}$ such that for all ϱ*-compatible* injections $\gamma : R \to \mathbb{R}$, there exists a plane straight-line embedding of G where each $v \in R$ is embedded at $(\varrho(v), \gamma(v))$. Injection γ is ϱ-compatible if the combination of ϱ and γ does not embed three vertices on a line. Clearly, if R is column planar in G then any subset of R is also column planar in G. We say that R is ϱ*-column planar* when we need to emphasize the injection ϱ (see Figure 1 for an example). If $R = V$ is column planar in G then G is ULP since column planarity implies the existence of one assignment of x-coordinates to vertices that will produce a planar embedding for all assignments of y-coordinates, while to be a ULP graph the x-coordinate assignment may depend on the y-coordinate assignment. In this sense, column planarity of V is strictly more restrictive than unlabeled level

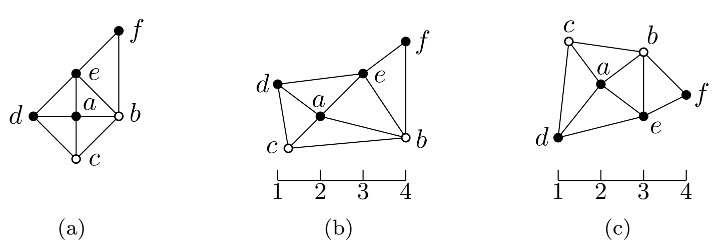

Figure 1: (a) A graph $G = (V, E)$ with $R = \{a, d, e, f\}$ which is ϱ-column planar for $\varrho = \{d \mapsto 1, a \mapsto 2, e \mapsto 3, f \mapsto 4\}$. (b-c) Two assignments of y-coordinates to the vertices R and corresponding plane straight-line completions of G.

planarity of G. Di Giacomo et al. [10] studied column planarity under a different name. Specifically, they defined EAP graphs as the graphs $G = (V, E)$ where V is column planar in G. They considered a family of graphs called *fat caterpillars* and proved that these are exactly the EAP graphs.

As mentioned above, the study of ULP graphs was originally motivated by simultaneous geometric embedding, a concept introduced by Brass et al. [5]. Formally, given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of n vertices, they defined a *simultaneous geometric embedding* (SGE) of G_1 and G_2 as an injection $\varphi : V \to \mathbb{R}^2$ such that the straight-line drawings of G_1 and G_2 induced by φ are both plane. With slight abuse of notation, we refer to these drawings as $\varphi(G_1)$ and $\varphi(G_2)$. Figure 2c depicts an SGE of the graphs in Figure 2a and Figure 2b. Bläsius et al. [3] gave an excellent survey of the subsequent papers on SGE with a comprehensive list of results. On the positive side, Brass et al. [5] proved that two paths, cycles or caterpillars always admit an SGE. Cabello et al. [6] proved that a matching and a tree or outerpath (a type of outerplanar graph) always admit an SGE. On the negative side, Brass et al. [5] proved that three paths sometimes do not admit an SGE. Erten and Kobourov [13] proved that a planar graph and a path may not admit an SGE. Frati, Kaufmann and Kobourov [18] strengthened this result to the case where the planar graph and the path do not share any edges. Geyer, Kaufmann and Kobourov [19] described two trees that do not admit an SGE. Angelini et al. [1] closed a long-standing open question by describing a tree and a path that admit no SGE. Finally, Estrella-Balderrama et al. [15] showed that the decision problem for SGE is NP-hard.

In light of the restrictiveness of simultaneous geometric embedding, several other variations on the abstract problem have been studied. Cappos et al. [8] considered a version of SGE where edges are embedded as circular arcs or with bends. Di Giacomo et al. [11] considered *matched drawings*: a version of SGE where the location of a vertex in the drawing of G_1 needs to have only the same y-coordinate as its location in the drawing of G_2.

In this paper we study a variant on SGE which we call *partially-simultaneous geometric embedding* (PSGE). We do not require *every* vertex to map to a single point in the plane. Instead, some vertices can have a "split personality" and map to two different locations, one associated with G_1 and one associated with

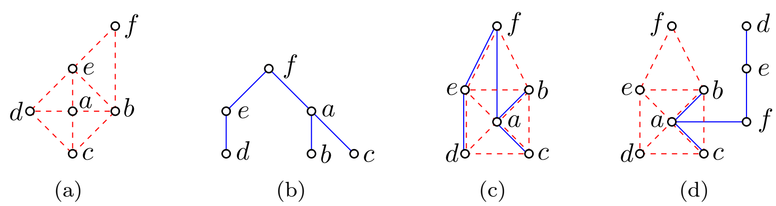

Figure 2: (a–b) Two graphs on the same vertex set. (c) An SGE of these graphs. (d) A 3-PSGE of these graphs.

G_2. Specifically, given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of n vertices, a *k-partially-simultaneous geometric embedding* (k-PSGE) of G_1 and G_2 is a pair of injections $\varphi_1 : V \to \mathbb{R}^2$ and $\varphi_2 : V \to \mathbb{R}^2$ such that (i) the straight-line drawings $\varphi_1(G_1)$ and $\varphi_2(G_2)$ are both plane; (ii) if $\varphi_1(v_1) = \varphi_2(v_2)$ then $v_1 = v_2$ and; (iii) $\varphi_1(v) = \varphi_2(v)$ for at least k vertices $v \in V$. An n-PSGE is simply an SGE. Figure 2(d) depicts a 3-PSGE of the graphs in Figure 2(a) and Figure 2(b).

PSGE is related to the notion of *planar untangling*: Given a straight-line drawing of a planar graph, change the embedding of as few vertices as possible in order to obtain a plane drawing. Goaoc et al. [20] described an improvement of a result by Bose et al. [4] to show that $\sqrt[4]{(n+1)/2}$ vertices can always be kept in their original positions. Since we can simply take any plane embedding of G_1, use the same embedding for G_2 and then untangle G_2, it immediately follows that every two planar graphs on n vertices admit a $\sqrt[4]{(n+1)/2}$-PSGE. In the other direction, Cano et al. [7] showed the existence of a graph and a straight-line drawing of this graph such that in any planar untangling, the embedding of all but $O(n^{.4965})$ vertices has to change. However, an upper bound for untangling does not have direct implications for PSGE because untangling is more restrictive compared to PSGE: In the untangling problem the positions of all vertices, in particular, of those that remain fixed, are given, whereas in PSGE one may select suitable positions for those vertices.

Results and organization. We prove that n-vertex trees have column planar sets of size at least $14n/17$ and we show that there exist n-vertex trees where every column planar subset has size at most $5n/6$. For n-vertex outerplanar graphs, we can always find a column planar subset of size $n/2$. These results complement the recent developments of Da Lozzo et al. [9] who showed that n-vertex triconnected cubic planar graphs have column planar sets of size at least $\lceil \frac{n}{4} \rceil$, and that n-vertex planar graph of treewidth at most three have a column planar sets of size at least $\lceil \frac{n-3}{8} \rceil$. Moreover, they proved that n-vertex planar graphs of treewidth k have column planar sets of size $\Omega(k^2)$.

Next, we establish the relation between column planarity and PSGE. We show that every two trees admit an $11n/17$-PSGE, that every tree and ULP graph admit a $14n/17$-PSGE, and that every two outerplanar graphs admit an $n/4$-PSGE.

2 Column planar sets in trees

In this section, we show how to find large column planar sets in trees. Let $p(v)$ be the parent of vertex v in a rooted tree T, and let $r(T)$ be the root of T. Given a subset R of the vertices of T, let $C_R(v)$ be the non-leaf children of v in R and let $C_R^+(v)$ be those vertices in $C_R(v)$ with at least one child in R. We first prove that subsets of T satisfying certain conditions are always column planar and next that every tree contains a large such subset.

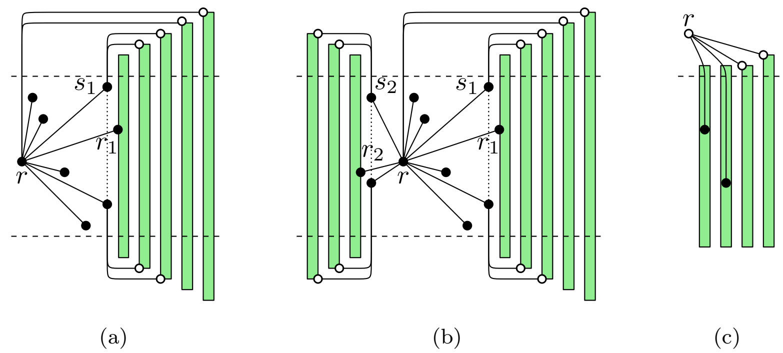

Figure 3: Embedding a tree with a column planar set. The column planar vertices are black. Figure (a) refers to the case $r \in R$ and $p(r) \in R$, (b) refers to the case $r \in R$ and $p(r) \notin R$, and (c) refers to the case $r \notin R$.

Lemma 1 *For a rooted tree $T = (V, E)$, a set $R \subseteq V$ is column planar in T if for all $v \in R$, either*

1. *$p(v) \in R$, the number of non-leaf children of v in R is at most two, and at most one of these children has a child in R (i.e. $|C_R(v)| \leq 2$ and $|C_R^+(v)| \leq 1$); or*
2. *$p(v) \notin R$, the number of non-leaf children of v in R is at most four, and at most two of these children have a child in R (i.e. $|C_R(v)| \leq 4$ and $|C_R^+(v)| \leq 2$).*

Proof: We will embed T recursively. The x-coordinates of V will be fixed in such a way that any assignment $\gamma : R \to \mathbb{R}$ of y-coordinates to R can be accommodated by embedding the vertices of $V \setminus R$ with y-coordinates much larger than $\max \gamma$ or much smaller than $\min \gamma$. Thus, the edges between $V \setminus R$ and R are embedded as near-vertical line segments. In the figures that accompany this proof, such edges will be drawn as curves.

For a subtree T' of T, let $p(T')$ be the parent of $r(T')$. If $r(T')$ is the root of T then $p(T')$, though it does not exist, is viewed as not in R. Our embedding will have the following properties for each subtree T':

(i) if $r(T') \notin R$ or $\{r(T'), p(T')\} \subseteq R$, then $r(T')$ has either the smallest or largest x-coordinate among all vertices in T';

(ii) if $r(T') \notin R$, then the y-coordinate of $r(T')$ can be chosen arbitrarily, as long as it is sufficiently small or sufficiently large; and

(iii) no almost-vertical ray from $r(T')$ intersects any edge from T'.

Let T be the rooted tree we want to embed. Let $r = r(T)$. First we discuss the case $r \in R$. There are two subcases, corresponding to the two cases of the statement of this lemma.

Suppose first that we are in Case 1 of the lemma, that is, $p(T) \in R$ and $|C_R(r)| \leq 2$ while $|C_R^+(r)| \leq 1$. Embed r at $x = 1$ and its ℓ leaf children at $x = 2, \ldots, \ell + 1$. (Their y-coordinates are determined by γ.) Let r_1 and s_1 be vertices of T such that $C_R(r) \subseteq \{r_1, s_1\}$ and $C_R^+(r) \subseteq \{r_1\}$. Embed r_1 and its subtree recursively and scale its x-coordinates to lie in $[\ell + 3, \ell + 4]$. By Property (i), and possibly after mirroring the embedding of the subtree rooted at r_1 horizontally, the edge $\{r, r_1\}$ does not cross edges in the subtree rooted at r_1. See Figure 3(a).

Embed s_1 at $x = \ell + 2$. Let $\langle T_1, \ldots, T_k\rangle$ be the child subtrees of s_1. Embed T_i recursively and scale its x-coordinates to lie in $[\ell + 3 + 2i, \ell + 4 + 2i]$ for all $1 \leq i \leq k$. Vertex s_1 will be above $\{r, r_1\}$ for some γ and below $\{r, r_1\}$ for other γ. If it is above, let $r(T_1), \ldots, r(T_k)$ have progressively larger y-coordinates (by Property (ii)). If it is below, let them have progressively smaller y-coordinates. Then none of the edges $\{s_1, r(T_i)\}$ cross $\{r, r_1\}$ and the edge $\{s_1, r(T_i)\}$ does not cross any edges in T_i by (i) and (ii).

Recursively embed the remaining subtrees $\langle T_1', \ldots, T_t'\rangle$ of r (none of their roots are in R) with x-coordinates in $[\ell + 3 + 2k + 2i, \ell + 4 + 2k + 2i]$ for all $1 \leq i \leq t$ such that $\langle r(T_1'), \ldots, r(T_t')\rangle$ have progressively larger y-coordinates. The edge $\{r, r(T_i')\}$ does not cross any edges in T_i' by (ii). Note that r has the lowest x-coordinate, and thus (i) is satisfied. Properties (ii) and (iii) are trivially satisfied.

Suppose now that we are in Case 2 of the lemma, that is, $p(T) \notin R$ and $|C_R(v)| \leq 4$ while $|C_R^+(v)| \leq 2$. Let r_1, r_2, s_1 and s_2 be vertices of T such that $C_R(r) \subseteq \{r_1, r_2, s_1, s_2\}$ and $C_R^+(r) \subseteq \{r_1, r_2\}$. In order to embed r and its child subtrees different from the ones rooted at r_2 and s_2, proceed as in the previous case. Mirror the recursive embedding of the subtree rooted at r_2 horizontally and scale it to have x-coordinates in $[-3, -2]$. Embed the subtree rooted at s_1 as in the previous case. In order to embed the subtree rooted at s_2, proceed similarly to s_1, but embed s_2 and its subtree to the left of r. See Figure 3(b). Properties (i)-(iii) are trivially satisfied.

Finally, suppose that $r = r(T) \notin R$. Embed its child subtrees $T_1, \ldots, T_t$ to have x-coordinates in $[2i, 2i + 1]$ for all $1 \leq i \leq t$. Embed r sufficiently high on the line $x = 1$. For subtrees T_i with $r(T_i) \in R$, note that the edge $\{r, r(T_i)\}$ does not cross any edges of T_i due to Property (iii). For the other ones, $\{r, r(T_i)\}$ does not cross edges of T_i due to Properties (i) and (ii). See Figure 3(c). Properties (i)-(iii) are satisfied. □

It remains to show that every tree contains a large subset that satisfies the conditions imposed by Lemma 1. We show that every tree on n vertices contains such a subset of size at least $14n/17$ and that there are trees with no column planar subset of size larger than $5n/6$. Note that $14/17 \approx 5/6 - 0.01$, and thus our results are almost tight.

Lemma 2 *Let T be a tree on n vertices rooted at any vertex $r(T)$. Let c_i be the number of vertices with exactly i children. Then* $c_0 = (n+1+\sum_{i=1}^{n-1}(i-2)c_i)/2$.

Proof: The number of edges in T is $n-1$ and also equals the degree sum divided by two. Thus, $\sum_{i=0}^{n-1} c_i(i+1) = 2(n-1)+1 = 2n-1$. Since $\sum_{i=0}^{n-1} c_i = n$, we get $\sum_{i=0}^{n-1} c_i(i-2) + 3n = 2n-1$ and thus $-2c_0 = -n-1-\sum_{i=1}^{n-1} c_i(i-2)$. The lemma follows. □

Theorem 1 *A tree T on n vertices contains a column planar set of size at least $14n/17$.*

Proof: Let R denote the column planar set to be constructed. Note that the set formed by all leaves in T and all vertices with exactly one child in T is column planar by Lemma 1 (for each vertex v in this set, it holds that $|C_R^+(v)| \leq 1$ and $|C_R(v)| \leq 1$). Intuitively, if this set is large enough, then we are done. Otherwise, we construct a column planar set greedily as follows. Root T at an arbitrary non-leaf vertex $r(T)$. Orient every edge towards the root and topologically sort T to obtain an order $\langle v_1, \ldots, v_n \rangle$ where $v_n = r(T)$. We will greedily add vertices to R in this order. More precisely, let $R_0 = \emptyset$ and let $R_i := R_{i-1} \cup \{v_i\}$ if $R_{i-1} \cup \{v_i\}$ satisfies Lemma 1 and let $R_i := R_{i-1}$ otherwise. Let $R = R_n$ be our final subset of T.

We say that a vertex is *marked* if it is in R. Consider a vertex $v = v_i \notin R$. The reason that v is not in R is that $R_{i-1} \cup \{v\}$ does not satisfy the condition in Lemma 1 for v or for a child u of v (or both). More precisely, v is contained in exactly one of the following sets:

$$\begin{aligned}
X_a &= \{v \in T \setminus R : && |C_R^+(v)| > 2\},\\
X_b &= \{v \in T \setminus R \setminus X_a : && |C_R(v)| > 4\},\\
X_c &= \{v \in T \setminus R \setminus X_a \setminus X_b : && |C_R^+(u)| > 1 \wedge v = p(u)\},\\
X_d &= \{v \in T \setminus R \setminus X_a \setminus X_b \setminus X_c : && |C_R(u)| > 2 \wedge v = p(u)\}.
\end{aligned}$$

We associate with each such v a witness tree $W(v)$ as follows (see Figure 4). If $v \in X_a$, then let $W(v)$ consist of v, three vertices of $C_R^+(v)$, and a marked child of each of them (which must exist by definition of $C_R^+(v)$). If $v \in X_b$, then let $W(v)$ consist of v and five marked children of v. If $v \in X_c$, then let $W(v)$ consist of v and u, two vertices of $C_R^+(u)$, and a marked child of each of them. If $v \in X_d$, let $W(v)$ consist of v and u, and three marked children of u. Note

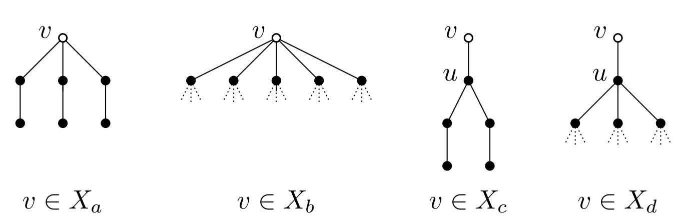

Figure 4: The witness tree $W(v)$ when v is in X_a, X_b, X_c or X_d. The marked vertices are black. Dotted line segments indicate that a vertex has at least one child.

that $W(v)$ and $W(v')$ are disjoint for $v, v' \in T \setminus R$ with $v \neq v'$. We have the following.

$$|X_a| + |X_b| + |X_c| + |X_d| + |R| = n. \tag{1}$$

Let L_t and I_t be the set of marked vertices of $\bigcup_{v \in X_t} W(v)$ that are leaves and internal vertices in T, respectively, for $t = a, b, c, d$. We have

$$|I_a| + |L_a| = 6|X_a| \qquad |L_a| = 3|X_a| \tag{2}$$
$$|I_b| + |L_b| = 5|X_b| \qquad |L_b| = 0 \tag{3}$$
$$|I_c| + |L_c| = 5|X_c| \qquad |L_c| = 2|X_c| \tag{4}$$
$$|I_d| + |L_d| = 4|X_d| \qquad |L_d| = 0 \tag{5}$$

Since R always contains all leaves of T, we have

$$|R| \geq c_0 + |I_a| + |I_b| + |I_c| + |I_d|, \tag{6}$$

where c_i is the number of vertices with exactly i children in T. Note that $W(v)$ contains a vertex with at least three children if $v \in X_a \cup X_b \cup X_d$. Hence, by Lemma 2,

$$c_0 > \frac{n - c_1 + \sum_{i=3}^{n-1} c_i}{2} \geq \frac{n - c_1 + |X_a| + |X_b| + |X_d|}{2}. \tag{7}$$

In addition, we have

$$c_0 \geq |L_a| + |L_b| + |L_c| + |L_d| = |L_a| + |L_c|. \tag{8}$$

Before we bound $|R|$, consider the set S formed by all leaves in T and all vertices with exactly one child in T. Then S is column planar by Lemma 1 and $|S| = c_0 + c_1$. Whenever the greedily chosen R has size less than $c_0 + c_1$, we choose $R = S$ instead. Thus, we may assume

$$|R| \geq c_0 + c_1. \tag{9}$$

Combining (7) and (9) yields

$$|R| > n - c_0 + |X_a| + |X_b| + |X_d|; \tag{10}$$

combining (2) and (8) yields

$$c_0 \geq 6|X_a| - |I_a| + |L_c|; \tag{11}$$

and combining (3), (4), (5), and (6) yields

$$|R| \geq c_0 + 5|X_b| + 5|X_c| + 4|X_d| - |L_c| + |I_a|. \tag{12}$$

To eliminate c_0, we add inequality (10) to two times (11) and three times (12) to obtain $4|R| > n + 13|X_a| + 16|X_b| + 15|X_c| + 13|X_d| - |L_c| + |I_a|$. With equation (4), this gives $4|R| > n + 13|X_a| + 16|X_b| + 13|X_c| + 13|X_d| + |I_a| \geq n + 13(|X_a| + |X_b| + |X_c| + |X_d|)$. Together with equation (1), this yields the desired bound of $|R| > 14n/17$. □

The algorithm achieves roughly this amount on the family $(\mathcal{B}_i)_{i=0}^{\infty}$ of trees depicted in Figure 5. The tree $\mathcal{B}_i$ has $|\mathcal{B}_i| = 17i + 23$ vertices, of which the greedy algorithm selects $|R_i| = 14i + 19$. The set S_i of all vertices in $\mathcal{B}_i$ with at most one child has size $|S_i| = |R_i| = 14i + 19$. Hence the algorithm from Theorem 1 selects a fraction of $|R_i|/|\mathcal{B}_i|$ vertices, with $\lim_{i\to\infty} |R_i|/|\mathcal{B}_i| = 14/17$.

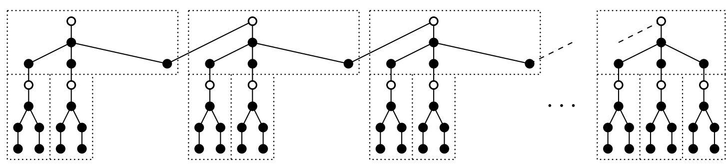

Figure 5: A family of trees for which $|R| = |S| \approx 14n/17$. The set R is colored black.

The algorithm does not, however, find a maximum column planar subset for these trees. Figure 6 depicts a column planar subset of size $15n/17$ and a corresponding drawing. The vertices u_1 and u_2 are always drawn with very large or very small y-coordinates in such a way that the edge $\{u_i, v_i\}$ does not cross the drawing of the tree rooted at v_i for $i = 1, 2$. In Figure 6, u_1 would be given a high y-coordinate and u_2 would be given a low y-coordinate. Although the bound in Theorem 1 is not tight, it is close to best possible:

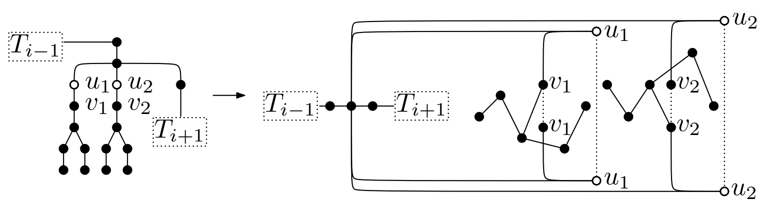

Figure 6: A larger column planar subset and corresponding drawing for the tree depicted in Figure 5.

Theorem 2 *For any $\epsilon > 0$ and any $n > 1/\epsilon$ that is a multiple of six, there exists a tree T with n vertices in which every column planar subset has less than $(5/6 + \epsilon)n$ vertices.*

Proof: Let $p = n/6$. Let T consist of p copies, $\langle T_1, T_2, \ldots, T_p\rangle$, of the tree shown in Figure 7(a) in which the root r of T_{i+1} is made a child of the rightmost leaf w of T_i, for $i = 1, \ldots, p-1$. Suppose for a contradiction that there is a column planar set R of marked vertices in T with $|R|/n \geq 5/6 + \epsilon$. We say that a copy T_i is *full* if its 6 vertices are marked. We claim that there are at least two full

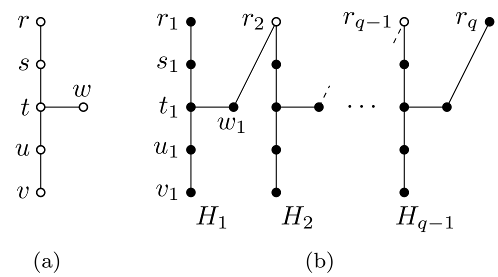

Figure 7: (a) The tree T_i and (b) H used in the proof of Theorem 2.

trees. To see this, suppose for a contradiction that there is at most one full tree. Then, using $n > 1/\epsilon$ the number of marked vertices is at most

$$5(p-1)+6 = 5p+1 < (5/6+\epsilon)n, \tag{13}$$

contrary to our assumption $|R|/n \geq 5/6+\epsilon$. Hence the claim holds and there are at least two full trees.

Let T_i and T_j be two consecutive full trees in T, that is, T_k is not full for each $i < k < j$. If there is some index $i < k < j$ such that T_k has less than 5 marked vertices, then the average number of marked vertices of the sequence $\langle T_i, T_{i+1}, \ldots, T_{j-1}\rangle$ is at most $5/6$. If this is the case between any two consecutive full trees, then in the set containing all vertices of T except for those that belong to the last full tree, the average number of marked vertices is at most $5/6$. That is, the total number of marked vertices in T is at most $5(p-1)+6$ which, using (13), is less than $(5/6+\epsilon)n$ contrary to our assumption that $|R| \geq (5/6+\epsilon)n$.

Therefore, there exists a sequence $\langle T_i, T_{i+1}, \ldots, T_j\rangle$ such that T_i and T_j are full and T_k has exactly 5 marked vertices for each $i < k < j$. Let $\ell > i$ be the smallest index such that the root of T_ℓ is marked. Since T_j is full, T_ℓ exists. Let H be the subtree induced by the root of T_ℓ and the vertices in $T_i \cup T_{i+1} \cup \cdots \cup T_{\ell-1}$. By definition, the unmarked vertices in H are exactly the roots of the subtrees $\langle T_{i+1}, T_{i+2}, \ldots, T_{\ell-1}\rangle$. We claim that the marked vertices are not column planar in H. If this claim is true, then we obtain a contradiction as the set of marked vertices is column planar in T. That is, if this claim is true, then every column planar subset of T has at most $(5/6+\varepsilon)n$ vertices as claimed by the theorem. Therefore, to complete our proof, we focus on proving that the marked vertices are not column planar in H.

In order to simplify notation, let $\langle H_1, H_2, \ldots, H_{q-1}\rangle$ be the sequence of subtrees in H and let r_q be the (marked) root of T_ℓ. Label the vertices of H_i by adding the subscript i to every vertex in Figure 7(a); see Figure 7(b). Let R' be the marked vertices in H and suppose for a contradiction that R' is ϱ-column planar in H for some injection $\varrho : R' \to \mathbb{R}$. For an edge $\{a,b\}$ in H with $a, b \in R'$, let $x(a,b) = [\varrho(a), \varrho(b)]$ be the x-interval of edge $\{a,b\}$. Note that for two edges $\{a,b\}$ and $\{c,d\}$ in H, where a, b, c, and d are distinct vertices in

R', $x(a,b) \cap x(c,d) = \emptyset$: otherwise, by choosing γ appropriately we can cause the edges to intersect within their shared x-interval. This implies, for example, that the x-interval spanned by marked vertices in one subtree does not intersect that of a different subtree.

For H_1, since $x(s_1,t_1) \cap x(u_1,v_1) = \emptyset$ and $x(t_1,u_1) \cap x(r_1,s_1) = \emptyset$, $\varrho(t_1)$ is *between* $x(r_1,s_1)$ and $x(u_1,v_1)$ (meaning that either $x(r_1,s_1) < \varrho(t_1) < x(u_1,v_1)$ or $x(u_1,v_1) < \varrho(t_1) < x(r_1,s_1)$, where $A < B$ if $a < b$ for all $a \in A$ and $b \in B$). By similar reasoning, $\varrho(w_1)$ is between $\varrho(t_1)$ and $x(u_1,v_1)$ or between $\varrho(t_1)$ and $x(r_1,s_1)$. Let us assume by renaming vertices if necessary, that $\varrho(w_1)$ is between $\varrho(t_1)$ and $x(u_1,v_1)$. See Figure 8.

To achieve a contradiction, we describe a ϱ-compatible injection γ for which the resulting partial embedding cannot be extended to a plane straight-line drawing of H. The basic idea is to choose γ so that

$$\gamma(u_i) < \gamma(s_i) < 0 \approx \gamma(w_i) < \gamma(t_i) < \gamma(v_i)$$

for all i (except when mentioned otherwise), and so that unmarked vertices are forced to be above the x-axis. To ease description, we assume that $\gamma(w_i) = 0$. However, because γ must be ϱ-compatible, we cannot allow the embeddings of the w_i's to be collinear. Thus, we perturb their value under γ so that they stay very close to 0, but the resulting embedding has no collinear triples.

We set $\gamma(u_1) < \gamma(s_1)$ to be both negative and $\gamma(t_1) < \gamma(v_1)$ to be both positive so that w_1 lies in the triangle $t_1u_1v_1$. Because $x(s_2,t_2) \cap x(t_1,u_1) = \emptyset$ and $x(s_2,t_2) \cap x(u_1,v_1) = \emptyset$, we know that $\rho(s_2)$ is not between $\rho(t_1)$ and $\rho(v_1)$. All this, together with the fact that r_2 is connected to s_2, forces the edge from w_1 to r_2 to be upward and thus r_2 to be above the x-axis (the edges $\{t_1,u_1\}$ and $\{u_1,v_1\}$ work as barriers that prevent it from going downwards). Moreover, by setting $\gamma(s_2)$ smaller than $\gamma(t_1) < \gamma(v_1)$, we can guarantee that the vertical coordinate of r_2 is larger than $\gamma(t_1)$ (r_2 must be in the convex cone with apex w_1 bounded by the rays passing through v_1 and t_1, and must connect to s_2).

Consider the order of $\varrho(s_2)$, $\varrho(t_2)$ and $x(u_2,v_2)$ and notice that $x(u_2,v_2) \cap x(s_2,t_2) = \emptyset$ as the edges $\{u_2,v_2\}$ and $\{s_2,t_2\}$ are vertex disjoint. If $\varrho(s_2)$ is

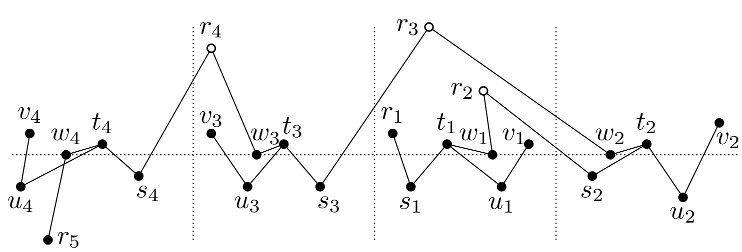

Figure 8: An example of how γ is chosen in the proof of Theorem 2 where $q = 5$. Note that forcing r_5 (bottom left) below the x-axis causes the edge $\{w_4, r_5\}$ to intersect another edge.

between $\varrho(t_2)$ and $x(u_2, v_2)$, then we set γ so that

$$\gamma(t_2) < \gamma(v_2) < 0 < \gamma(s_2) < \gamma(u_2),$$

and s_2 lies in the triangle $t_2u_2v_2$. As a result either $\{t_2, u_2\}$ or $\{u_2, v_2\}$ intersects $\{r_2, s_2\}$ (recall that $\{r_2, s_2\}$ is going downwards). Therefore, $\varrho(t_2)$ is between $\varrho(s_2)$ and $\varrho(u_2, v_2)$. Now let us consider the possible positions of $\varrho(w_2)$. If $\varrho(s_2)$ is between $\varrho(w_2)$ and $\varrho(t_2)$, then we set γ so that

$$\gamma(w_2) < \gamma(u_2) < 0 < \gamma(s_2) < \gamma(t_2),$$

and s_2 lies in the triangle $u_2t_2w_2$. As a result either $\{t_2, w_2\}$ or $\{u_2, t_2\}$ intersects $\{r_2, s_2\}$ (recall that $\{r_2, s_2\}$ is going downwards). Note that $x(u_2, v_2)$ cannot be between $\varrho(w_2)$ and $\varrho(t_2)$ since $x(u_2, v_2) \cap x(t_2, w_2) = \emptyset$. Hence, $\varrho(w_2)$ is between $\varrho(s_2)$ and $\varrho(t_2)$ or between $\varrho(t_2)$ and $x(u_2, v_2)$.

In the first case, we set γ so that

$$\gamma(u_2) < \gamma(s_2) < 0 \approx \gamma(w_2) < \gamma(t_2),$$

and w_2 lies above the line segment from s_2 to t_2. We claim that as a consequence, w_2 lies in the triangle $r_2s_2t_2$. While we do not know the exact position of r_2, since we can lower bound its vertical coordinate, we can assume that r_2 lies higher than t_2 (having $\gamma(t_2) < \gamma(t_1)$ is sufficient as r_2 lies higher than t_1). To prove our claim, let ω be the line passing through s_2 and w_2. Note that if r_2 and t_2 lie on the same closed halfplane supported by ω, then the edge $\{s_2, r_2\}$ crosses either the edge $\{w_2, t_2\}$ or the edge $\{t_2, u_2\}$. Therefore, r_2 and t_2 lie on different halfplanes supported by ω and hence, w_2 lies in the triangle $r_2s_2t_2$, as claimed. To avoid intersecting the path r_2, s_2, t_2, the edge from w_2 to r_3 is forced upward. As before for r_2, we can lower bound the vertical coordinate of r_3 because r_3 must lie in the convex cone with apex w_2 bounded by the rays passing through r_2 and t_2.

In the second case, that is, if $\varrho(w_2)$ is between $\varrho(t_2)$ and $x(u_2, v_2)$, then we set γ so that

$$\gamma(u_2) < 0 \approx \gamma(w_2) < \gamma(t_2) < \gamma(v_2),$$

and w_2 lies in the triangle $t_2u_2v_2$. Thus, to avoid intersecting the path $t_2u_2v_2$, the edge from w_2 to r_3 is forced upward. We can lower bound the vertical coordinate of r_3 because it must lie in the convex cone with apex w_2 bounded by the rays passing through v_2 and t_2.

By repeating this argument, we force all unmarked vertices as well as r_q to be above the x-axis. Since r_q is marked, we derive a contradiction by setting $\gamma(r_q) < 0$ that comes from assuming that R' was column planar in H. $\square$

3 Column planar sets in outerplanar graphs

Let $G = (V, E)$ be an outerplanar graph with $n \geq 4$ vertices. Assume without loss of generality that G is maximal outerplanar. In this section we show that G has a column planar subset of size at least $n/2$.

Every maximal outerplanar graph on $n \geq 4$ vertices has at least two nonadjacent vertices of degree 2; the internal face incident to such a vertex is called an *ear*. Let $\langle v_0, v_1, \ldots, v_{n-1}\rangle$ be the sequence of vertices of V along the unique Hamiltonian cycle of G. Consider the following *removal procedure*: Choose an arbitrary vertex of degree 2 in G and different from v_0 and v_{n-1}, remove it from the graph and repeat recursively. Note that removing a degree-2 vertex of an ear maintains maximal outerplanarity in the resulting graph. Thus, because G has at least two ears and since v_0 and v_{n-1} do not both have degree 2 as they are adjacent, the above procedure is well-defined. The *removal order* of the vertices $V \setminus \{v_0, v_{n-1}\}$ is the order in which they are removed by this procedure. For $0 \leq i < n$, let

$$V(v_i) = \{v_j \in V : v_j \text{ is removed before } v_i\}.$$

Let $N(v_i)$ be the set that consists of v_i and its neighbors in G. For $0 < i < n-1$, the *left index* ℓ_i of v_i is the smallest index such that $v_{\ell_i} \in N(v_i)$. Similarly, the *right index* r_i of v_i is the largest index with $v_{r_i} \in N(v_i)$. Naturally, $\ell_i < i < r_i$. (Note that these indices are defined with respect to the order along the Hamiltonian cycle of G, not with respect to the removal order.)

Lemma 3 *Let v_i be a vertex with $0 < i < n-1$ and suppose that there is a vertex v_j with $i \neq j$ such that $\ell_i < j < r_i$. Then all neighbors of v_j, except possibly for v_i, v_{ℓ_i} and v_{r_i}, are in $V(v_i)$.*

Proof: Let $\ell = \ell_i$ and $r = r_i$ and assume without loss of generality that $i < j$ (if $i > j$, then consider the reversed sequence $v_{n-1}, \ldots, v_0$ instead). Since $i < r-1$, the edge $\{v_i, v_r\}$ is a chord of G. See Figure 9. Hence, the removal of v_i and v_r splits G into two connected components H_1 and H_2 such that $v_j \in H_1$ and $v_0 \in H_2$. Note that v_j neighbors no vertex in H_2. Let $V_i^- = V \setminus (V(v_i) \cup \{v_i\})$ denote the vertex set after removal of v_i. We claim that all the vertices in V_i^- lie in H_2. If this claim is true, then v_j neighbors no vertex in V_i^-, expect possibly for v_r, which proves the statement.

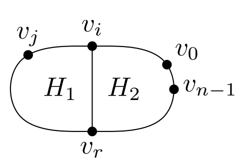

Figure 9: Connected components in the proof of Lemma 3.

Assume for a contradiction that there is a vertex $v \in V_i^-$ that belongs to H_1. Therefore, v lies after v_i in the removal order. Since (i) there is no edge between a vertex of H_1 and a vertex of H_2, (ii) H_1 contains a vertex after removing v_i (namely v), and (iii) H_2 contains a vertex after removing v_i (namely v_0), the induced subgraph $G[V_i^-]$ is either disconnected, or has v_r as a cutvertex. Regardless of the case, the graph is no longer maximal outerplanar after removal of v_i. However, the removal procedure described above preserves maximal outerplanarity: a contradiction. $\square$

Let $E_C \subset E$ be the set of all chords of G having endpoints whose removal splits G into components with at least two vertices each. Note that the chords adjacent to ears of G are not part of E_C. Let $C = (V, E_C)$ be the *chord graph* of G (Figure 10).

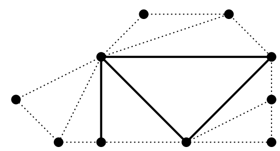

Figure 10: A maximal outerplanar graph $G = (V, E)$. The edge set E_C is drawn solid; the other edges are dotted.

Lemma 4 *Let $I \subset V$ be an independent set in C such that there is an edge of the Hamiltonian cycle of G whose endpoints are both not in I. Then I is column planar in G.*

Proof: Let $\langle v_0, v_1, \ldots, v_{n-1}\rangle$ be the sequence of vertices of V along the unique Hamiltonian cycle of G such that v_0 and v_{n-1} are not in I. To set the x-coordinate of the vertices in I, we define the injection $\varrho : I \to \mathbb{R}$ such that $\varrho(v_i) = i$.

For any ϱ-compatible injection $\gamma : I \to \mathbb{R}$, we need to show that there exists a plane straight-line embedding of G where each $v_i \in I$ is embedded at $\varphi(v_i) = (\varrho(v_i), \gamma(v_i))$.

We show that φ is a plane straight-line embedding of the graph $G[I]$. We first prove that $G[I]$ contains no edge $\{v_i, v_{i+c}\}$ with $c > 2$ (sum taken modulo n). If it has such an edge, then the removal of $\{v_i, v_{i+c}\}$ splits G into two components with at least two vertices each. Hence, by definition of C, the edge $\{v_i, v_{i+c}\}$ is in C, which contradicts the assumption that I is an independent set of C. We conclude that all edges of $G[I]$ are of the form $\{v_i, v_{i+1}\}$ or $\{v_i, v_{i+2}\}$. An edge $\{v_i, v_{i+1}\}$ in $G[I]$ cannot cross any other edge $\{v_j, v_{j+c}\}$ of $G[I]$, since this would require $j < i < i+1 < j+c$ and thus $c > 2$. Hence, any possible crossing in φ must involve two edges $\{v_{i-1}, v_{i+1}\}$ and $\{v_i, v_{i+2}\}$. However, since v_i is separated from the rest of G by the chord $\{v_{i-1}, v_{i+1}\}$, it cannot be adjacent to v_{i+2}. We conclude that φ is a plane straight-line embedding of $G[I]$.

We now describe an algorithm that places the remaining vertices of V to obtain a plane straight-line embedding of G. The algorithm is incremental and adds one vertex at a time in the given removal order.

Let X_k be the union of the vertices in I and the length-k prefix of the removal order. We never embed two vertices at the same x-coordinate. We say that the *visibility invariant* holds if each vertex of X_k that neighbors a vertex of $V \setminus X_k$ in G is *visible from below*, i.e., the ray shooting downwards from this vertex intersects no edge of the embedding of $G[X_k]$. We can see that the visibility invariant holds for $X_0 = I$ as follows. Suppose that there is a vertex v_i that is not visible from below. Then the ray from v_i downwards intersects some edge $\{v_x, v_y\}$. Since v_x and v_y are independent in C and since $x < i < y$, we must have $i = x + 1$ and $y = x + 2$. But then the only neighbors of v_i are v_x and v_y, and hence v_i does not neighbor a vertex of $V \setminus X_0$, as required.

For any $k \geq 0$, let v_i be the first vertex in $V \setminus X_k$ according to the removal order and let $X_{k+1} = X_k \cup \{v_i\}$. Note that $V(v_i) \subseteq X_k$ by definition of $V(v_i)$. Let ℓ_i and r_i be the left and right indices of v_i, respectively. We place v_i at coordinates (i, y_i), where y_i is a sufficiently small number such that all neighbors of v_i in X_k are visible from v_i. This number always exists by the visibility invariant and since we never embed two vertices with the same x-coordinate. After placing v_i and drawing the edges to its neighbors in X_k, the vertices $v_j \in X_k$ with $\ell_i < j < r_i$ (and only those) may no longer be visible from below. In particular, v_i is visible from below after embedding v_i. By Lemma 3 and the fact that $V(v_i) \subseteq X_k$, all neighbors of v_j, except maybe for v_i, v_{ℓ_i} and v_{r_i}, are in X_k, and have hence already been embedded. That is, each vertex $v_j \in X_k$ with $\ell_i < j < r_i$ that is no longer visible from below has no neighbor in $V \setminus X_{k+1}$. Therefore, the visibility invariant is preserved for X_{k+1}.

After this process completes, the only remaining vertices to embed are v_0 and v_{n-1}. Embed v_0 at $x = 0$ and v_{n-1} at $x = n - 1$. Move both down sufficiently far so that the edge $\{v_0, v_{n-1}\}$ does not intersect the rest of the drawing and so that v_0 and v_{n-1} can both see their neighbors from below. This completes the plane straight-line embedding of G. $\square$

Lemma 5 *The graph C has an independent set of size at least $(n + 2)/2$.*

Proof: Let D be the weak dual graph (the dual graph, but without a vertex for the outer face) of the maximal outerplanar graph G. Let x_i be the number of vertices of degree i in D. Note that D is a binary tree (that is, of maximum degree 3) whose leaves correspond to ears of G. Since the degree-2 vertex of an ear in G is an isolated vertex in C, we know that C has at least x_1 isolated vertices. Since D is a binary tree, we know that $x_1 = x_3 + 2$.

We describe a greedy procedure to construct an independent set I of C. The algorithm chooses a vertex of smallest degree in the current graph (initially C), adds it to I, and removes its neighbors from the graph. Clearly this procedure generates an independent set. We claim that $|I| \geq (n + 2)/2$.

Because C is outerplanar, it is 2-degenerate (every subgraph has a vertex of degree at most 2). Therefore, whenever we add a vertex to I, it has degree 0, 1, or 2. Let n_i be the number of vertices in I that had degree i at the moment they were chosen. Thus, $|I| = n_0 + n_1 + n_2$. Moreover, we know that $n_0 \geq x_1$ as isolated vertices of C will be added to I before any other vertex of C. Thus, $n_0 \geq x_1 = x_3 + 2$.

Let m be the number of bounded faces of C. Since $m \leq x_3$, we conclude that $m + 2 \leq n_0$.

Since removing vertices of degree 0 or one does not change the number of bounded faces, we remove a bounded face of the current graph exactly when we add a vertex of degree 2 to I. Thus, $m \geq n_2$. Therefore, $n_2 \leq n_0 - 2$.

Since every time our algorithm chooses a vertex of degree i we remove its i neighbors from the graph, and since only vertices of degree 0, 1 or 2 are chosen, we conclude that $n = n_0 + 2n_1 + 3n_2$. Because $|I| = n_0 + n_1 + n_2$, we infer that

$$n = n_0 + 2n_1 + 3n_2 \leq 2(n_0 + n_1 + n_2) - 2 = 2|I| - 2.$$

Consequently $|I| \geq (n+2)/2$. □

If the independent set I guaranteed by Lemma 5 does not satisfy the condition of Lemma 4, for instance when n is even and I is the set of vertices with an even index, then take any $v_i \in V \setminus I$ and remove v_{i+1} from I. Since the modified I satisfies Lemma 4, we have the following.

Theorem 3 *Every outerplanar graph on n vertices contains a column planar set of size at least $n/2$.*

4 Partially-simultaneous geometric embedding

The relation between column planarity and PSGE is expressed by the following lemma, which relates the size of column planar sets to PSGE.

Lemma 6 *Consider planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on n vertices. If R_1 is column planar in G_1 and R_2 is column planar in G_2, then G_1 and G_2 admit a k-PSGE with $k = |R_1 \cap R_2| \geq (|R_1| + |R_2| - n)$.*

Proof: Figure 11 illustrates the construction. The set $R = R_1 \cap R_2$ has size at least $|R_1|+|R_2|-|R_1 \cup R_2| \geq |R_1|+|R_2|-n$ and is column planar in both G_1 and G_2. More specifically, there exist injections $\varrho_1 : R \to \mathbb{R}$ and $\varrho_2 : R \to \mathbb{R}$ such that R is ϱ_1-column planar in G_1 and ϱ_2-column planar in G_2. By exchanging the roles of the x- and y-coordinates in the definition of column planar in G_2, we see that there exists an injection $\varrho : V \to \mathbb{R}$ such that, for all injections $\gamma : R \to \mathbb{R}$, there exists a plane straight-line embedding of G_2 that embeds each $v \in R$ at $(\gamma(v), \varrho_2(v))$. In particular, we may choose $\gamma = \varrho_1$. □

4.1 Two trees

Combining Lemma 6 and Theorem 1 immediately yields the following lower bound on the size of a PSGE of two trees.

Theorem 4 *Every two trees on a set of n vertices admit an $11n/17$-PSGE.*

There are two trees T_1 and T_2 on 226 vertices that do not admit an SGE [19]. Thus, an upper bound on the size of the common set in a PSGE of T_1 and T_2 is 225. Root T_1 arbitrarily and let T_1^k be the result of taking k copies of T_1 and connecting their roots with a path. Define T_2^k similarly. Then the size of the common set in a PSGE of T_1^k and T_2^k is at most $225k$. It follows that:

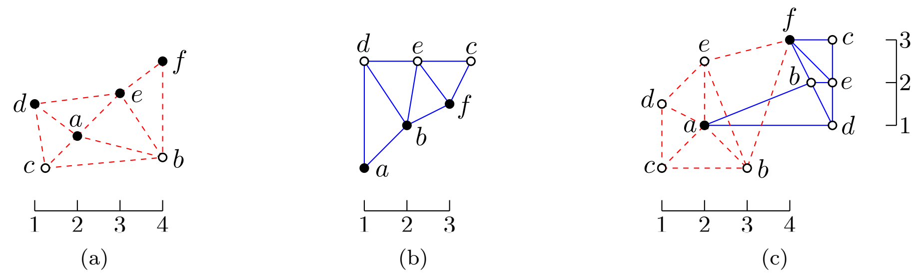

Figure 11: (a) Graph G_1 with $R_1 = \{a, d, e, f\}$ and $\varrho_1 = \{d \mapsto 1, a \mapsto 2, e \mapsto 3, f \mapsto 4\}$. (b) Graph G_2 with $R_2 = \{a, b, f\}$ and $\varrho_2 = \{a \mapsto 1, b \mapsto 2, f \mapsto 3\}$. (c) A 2-PSGE of G_1 and G_2 where vertex set $R = R_1 \cap R_2 = \{a, f\}$ is shared.

Corollary 1 *There exist two trees on a set of n vertices that admit no k-PSGE for $k > 225n/226$.*

4.2 Tree and ULP graph

If one of the two graphs in our PSGE is ULP, then the size of the common set depends only on the size of the column planar set in the other graph:

Lemma 7 *Consider a planar graph $G_1 = (V, E_1)$ and a ULP graph $G_2 = (V, E_2)$ on n vertices. If R is column planar in G_1, then G_1 and G_2 admit a $|R|$-PSGE.*

Proof: By exchanging the roles of x- and y-coordinates in the definition of column planar, we see that for all injections $\gamma : R \to \mathbb{R}$, there exists a plane straight-line embedding of G_1 with $v \in R$ at $(\gamma(v), \varrho(v))$. Since G_2 is a ULP graph, for all injections $y : V \to \mathbb{R}$, there exists an injection $x : V \to \mathbb{R}$ such that placing $v \in V$ at $(x(v), y(v))$ results in a straight-line embedding of G_2. Thus, placing the vertices $v \in R$ at $(x(v), \varrho(v))$ permits both a straight-line embedding of G_1 and G_2. □

Combining this with Theorem 1 yields the following.

Theorem 5 *A tree and a ULP graph admit a $14n/17$-PSGE.*

4.3 Two outerplanar graphs

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two outerplanar graphs, both on the same set V of n vertices. Let C_1 and C_2 be the chord graphs of G_1 and G_2, respectively. First use Lemma 5 to compute an independent set I_1 of size at least $n/2 + 1$ in C_1. Remove at most one vertex from I_1 to obtain a set R_1 of size at least $n/2$ that is column planar in G_1 by Lemma 4. Next, use Lemma 5 to compute an independent set I_2 of size at least $n/4 + 1$ in the chord graph of $G_2[R_1]$ (after adding edges to make $G_2[R_1]$ maximal outerplanar). Note that

I_2 is also independent in C_2, and hence we can remove at most one vertex from I_2 to obtain a set $R \subseteq R_1 \subseteq V$ of size at least $n/4$ that is column planar in G_2 using Lemma 4. Note that R is also column planar in G_1 since $R \subseteq R_1$. Applying Lemma 6 with R gives the following result.

Theorem 6 *Every two outerplanar graphs on a set of n vertices admit an $(n/4)$-PSGE.*

5 Concluding remarks

It is worth noting that for both trees and outerplanar graphs, our results hold true for a slightly stronger notion of column planarity. In both cases, we can prescribe the x-coordinate of each vertex of the graph and not only of the column planar set R, and still get a planar straight-line drawing of the graph for an arbitrary y-coordinate assignment for the vertices of R.

Our results leave several directions for future research. The tree drawings produced by Theorem 1 may have exponential area. It would be interesting to see whether polynomial area is sufficient. Recently, Dujmović [12] showed that every planar graph has a column planar subset of size at least $\sqrt{n/2}$. No upper bounds other than the one from Theorem 2 are known. A matching upper bound for the case of outerplanar graphs is also an open problem. Further research could be directed towards closing the gap between the lower and upper bound on the size of column planar sets for trees. Finally, another direction of research is to design efficient algorithms to compute column planar sets in planar graphs.

Acknowledgements. This work began at the 1st International Workshop on *Drawing Algorithms for Networks in Changing Environments* (DANCE 2014), April 22–25, 2014, in Langbroek, The Netherlands. The workshop was supported by the Netherlands Organisation for Scientific Research (NWO) under project no. 639.023.208. We thank the organizers and all participants for the productive and positive atmosphere.

References

[1] P. Angelini, M. Geyer, M. Kaufmann, and D. Neuwirth. On a tree and a path with no geometric simultaneous embedding. *J. Graph Alg. Appl.*, 16(1):37–83, 2012. doi:10.7155/jgaa.00250.

[2] L. Barba, M. Hoffmann, and V. Kusters. Column planarity and partial simultaneous geometric embedding for outerplanar graphs. In *Abstracts 31st European Workshop on Computational Geometry (EuroCG2015)*, pages 53–56, 2015.

[3] T. Bläsius, S. G. Kobourov, and I. Rutter. Simultaneous embedding of planar graphs. In R. Tamassia, editor, *Handbook of Graph Drawing and Visualization*. Chapman and Hall/CRC, 2013. arXiv:http://arxiv.org/abs/1204.5853.

[4] P. Bose, V. Dujmović, F. Hurtado, S. Langerman, P. Morin, and D. R. Wood. A polynomial bound for untangling geometric planar graphs. *Disc. & Comp. Geom.*, 42(4):570–585, 2009. doi:10.1007/s00454-008-9125-3.

[5] P. Brass, E. Cenek, C. A. Duncan, A. Efrat, C. Erten, D. P. Ismailescu, S. G. Kobourov, A. Lubiw, and J. S. Mitchell. On simultaneous planar graph embeddings. *Comp. Geom.*, 36(2):117–130, 2007. doi:10.1016/j.comgeo.2006.05.006.

[6] S. Cabello, M. van Kreveld, G. Liotta, H. Meijer, B. Speckmann, and K. Verbeek. Geometric simultaneous embeddings of a graph and a matching. *J. Graph Alg. Appl.*, 15(1):79–96, 2011. doi:10.7155/jgaa.00218.

[7] J. Cano, C. D. Tóth, and J. Urrutia. Upper bound constructions for untangling planar geometric graphs. *SIAM J. Discrete Math.*, 28(4):1935–1943, 2014. doi:10.1137/130924172.

[8] J. Cappos, A. Estrella-Balderrama, J. J. Fowler, and S. G. Kobourov. Simultaneous graph embedding with bends and circular arcs. *Comp. Geom.*, 42(2):173–182, 2009. doi:10.1016/j.comgeo.2008.05.003.

[9] G. Da Lozzo, V. Dujmović, F. Frati, T. Mchedlidze, and V. Roselli. Drawing planar graphs with many collinear vertices. In *Proc. 24th Internat. Sympos. Graph Drawing and Network Visualization (GD2016)*, volume 9801 of *LNCS*, pages 152–165. Springer, 2016.

[10] E. Di Giacomo, W. Didimo, G. Liotta, H. Meijer, and S. K. Wismath. Planar and quasi-planar simultaneous geometric embedding. *The Computer Journal*, 58(11):3126–3140, 2015. doi:10.1093/comjnl/bxv048.

[11] E. Di Giacomo, W. Didimo, M. van Kreveld, G. Liotta, and B. Speckmann. Matched drawings of planar graphs. *J. Graph Alg. Appl.*, 13(3):423–445, 2009.

[12] V. Dujmović. The utility of untangling. In *Proc. 23rd Internat. Sympos. Graph Drawing and Network Visualization (GD2015)*, volume 9411 of *LNCS*, pages 321–332. Springer, 2015.

[13] C. Erten and S. G. Kobourov. Simultaneous embedding of planar graphs with few bends. *J. Graph Alg. Appl.*, 9(3):347–364, 2005. URL: `http://eudml.org/doc/55411`.

[14] A. Estrella-Balderrama, J. J. Fowler, and S. G. Kobourov. Characterization of unlabeled level planar trees. *Comp. Geom.*, 42(6):704–721, 2009. `doi:10.1016/j.comgeo.2008.12.006`.

[15] A. Estrella-Balderrama, E. Gassner, M. Jünger, M. Percan, M. Schaefer, and M. Schulz. Simultaneous geometric graph embeddings. In *Proc. 16th Internat. Sympos. Graph Drawing (GD2008)*, volume 5417 of *LNCS*, pages 280–290. Springer, 2008. `doi:10.1007/978-3-540-77537-9_28`.

[16] W. Evans, V. Kusters, M. Saumell, and B. Speckmann. Column planarity and partial simultaneous geometric embedding. In *Proc. 22nd Internat. Sympos. Graph Drawing (GD2014)*, volume 8871 of *LNCS*, pages 259–271, 2014. `doi:10.1007/978-3-662-45803-7_22`.

[17] J. J. Fowler and S. G. Kobourov. Characterization of unlabeled level planar graphs. In *Proc. 15th Internat. Sympos. Graph Drawing (GD2007)*, volume 4875 of *LNCS*, pages 37–49. Springer, 2008. `doi:10.1007/978-3-540-77537-9_7`.

[18] F. Frati, M. Kaufmann, and S. G. Kobourov. Constrained simultaneous and near-simultaneous embeddings. *J. Graph Alg. Appl.*, 13(3):447–465, 2009. `doi:10.7155/jgaa.00194`.

[19] M. Geyer, M. Kaufmann, and I. Vrt'o. Two trees which are self-intersecting when drawn simultaneously. *Disc. Math.*, 309(7):1909–1916, 2009. `doi:10.1016/j.disc.2008.01.033`.

[20] X. Goaoc, J. Kratochvíl, Y. Okamoto, C.-S. Shin, A. Spillner, and A. Wolff. Untangling a planar graph. *Disc. & Comp. Geom.*, 42(4):542–569, 2009. `doi:10.1007/s00454-008-9130-6`.