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A Note on the Existence of All (g, f)-Factors

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Abstract

This paper is devoted to the problem of existence of all (g, f)-factors in a bipartite graph. We present an algorithm to test if a given bipartite graph contains all (g, f)-factors. In this way we give the affirmative answer to the question stated by Niessen. An analogous result for general graphs remains still unanswered.

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1 Introduction

Niessen in his paper [4] asks whether there exists a polynomial-time algorithm for testing if a graph has all (g, f)-factors. This work solves this problem for bipartite graphs.

The paper is organized as follows. We first present the necessary formal background on matching theory in Section 2. Then, in Section 3 we study various conditions under which a bipartite graph contains all (g, f)-factors and give a polynomial algorithm in Section 4. Finally, in Section 5 we state some open problems and conclude.

2 Preliminaries

We start with formal definitions of the central concepts. We recall some terminology of matching theory that we use in the rest of the paper. We assume that the reader is familiar with the essentials of graph theory (see [2]). We use standard terminology but for the sake of clarity we repeat the most important definitions and notations from [1].

Let G be a bipartite graph, and let (g, f) be a pair of mappings of $V(G)$ into the non-negative integers such that $0 \leq g(x) \leq f(x) \leq d_G(x)$, where $d_G(x)$ denotes the degree of the vertex x, i.e. the number of adjacent edges. A (g, f)-factor of G is a spanning subgraph F with the property that $g(x) \leq d_F(x) \leq f(x)$ hold for each $x \in V(G)$. A (g, f)-factor of G with the minimum number of edges is called a minimum (g, f)-factor. The maximum (g, f)-factor is defined analogously.

For fixed g and f, a graph is said to have all (g, f)-factors if it has an (h, h)-factor for every function h with $g(x) \leq h(x) \leq f(x)$ for all vertices x and where $h(V)$ is even, whereby $h(V) = \sum_{x \in V(G)} h(x)$.

A bipartite graph with bipartion (X, Y) has all (g, f)-factors if function h satisfies the necessary condition for the existence of h-factors, i.e. $h(X) = h(Y)$.

3 Some results concerning (g, f)-factors

It is well known that a maximum matching can be obtained from any other maximum matching by a sequence of transfers along alternating cycles and paths of even length (see, for example, Theorem 5.1.7 in [1]). The similar result holds for f-factors (see Theorem 7.2.4 in [1]).

Just as for matchings and f-factors, we can transform any (g, f)-factor into any other by a series of transformations. These transformations are based on alternating paths and cycles. We investigate some properties of factor transformation in bipartite graphs. That is, given two factors in G, we can start with one and transform to another one through the operation of symmetric difference. Combining theorems with properties we can deduce the following result.

Theorem 1 *A (g, f)-factor F of a bipartite graph G can be obtained from any other (g, f)-factor F' by a sequence of transfers along alternating cycles of even length and alternating paths of even and odd length.*

Proof: The proof of this theorem is similar to that of Theorem 7.2.4 in [1]. Thus, we only give an intuitive argument. First we must convert a (g, f)-factor problem in G to an f-factor problem in a more complex graph G^*. We construct a new graph G^* from G as follows. The construction we present is due to Tutte [5], who in 1981 gave an elegant reduction of (g, f)-factor problem to the f-factor problem.

We introduce one new vertex w to G, and join w to every vertex x of G by exactly $f(x) - g(x)$ new multiple edges. Finally we attach $\lfloor \frac{q}{2} \rfloor$ new loops to w, where q is an integer expressed as

$$q = \sum_{x \in V(G)} (f(x) - g(x))$$

and must have the same parity as the number $\sum_{x \in V(G)} f(x)$. We next define the function $f^* : V(G^*) \to \mathbb{Z}^+$ as

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G), \\ q & \text{if } x = w. \end{cases}$$

It is not difficult to check that any f^*-factor of G^* corresponds exactly to a (g, f)-factor of G (by removing from the graph the dummy vertex w with all adjacent edges).

Now let F_1 and F_2 be two (g, f)-factors of G. Each factor corresponds to an f-factor in G^*, say F_1^* and F_2^*, respectively. According to Theorem 7.2.4 in [1] F_1^* can be transformed into F_2^* by a sequence of transfers along closed alternating trails relative to F_1^*. Each of these closed trails, after removal of w, correspond to an alternating trail relative to F_1^*, and this trail is itself a union of alternating cycles of even length (when they contain only vertices of G), and union of alternating paths of even (when they lead through w) and odd length (when they contain any loop attached to w). Thus, the theorem is established. □

We now prove the theorem that if a bipartite graph G contains a (g, f)-factor with the property that if for every vertex x there exists a (g, f)-factor F such that $d_F(x) = g(x)$ and there exists a (g, f)-factor F' such that $d_{F'}(x) = f(x)$ then G contains for every vertex x and every number $d(x)$ between $f(x)$ and $g(x)$ a (g, f)-factor with degree $d(x)$ at x.

Observe that the above criterion does not mean that if G contains a (g, g)-factor and an (f, f)-factor then G contains all (g, f)-factors.

Theorem 2 *Let G be any bipartite graph with the mappings g and f as required above. For any vertex $x \in V(G)$ the degrees of x in (g, f)-factors form a sequence of consecutive integers.*

Proof: Let $F^{g(x)}$ be a (g, f)-factor such that $d_F(x) = g(x)$ and let $F^{f(x)}$ be a (g, f)-factor such that $d_F(x) = f(x)$. According to Theorem 1 we can transform $F^{g(x)}$ into $F^{f(x)}$ by a sequence of transfers along alternating cycles and alternating paths. Observe that each transfer along any alternating path does not change the degrees of internal vertices but only changes the degrees of endpoints by 1. Since any alternating path changes the degree of x by at most 1, the degrees of x in the intermediate (g, f)-factors F will cover every integer in the interval $[g(x), f(x)]$. □

Let us remark that the above theorem does not hold for general graphs. As a counter-example let us take the graph K_3 with the following mappings: $f(x_1) = g(x_1) = 1$, $f(x_2) = g(x_2) = 1$, $f(x_3) = 0$ and $g(x_3) = 2$. There exists no (g, f)-factor F with $d_F(x_3) = 1$ (see Figure 10.2.2 in [3]).

4 Algorithm

Assume now that a (g, f)-factor F in a graph G is initially found. We can classify the vertices of G (with respect to F) into the following categories:

- positive vertex, if $g(x) \leq d_F(x) < f(x)$,
- negative vertex, if $g(x) < d_F(x) \leq f(x)$,
- neutral vertex, if $g(x) = d_F(x) = f(x)$.

Observe that when a graph G with a (g, f)-factor is given the following alternating paths can be distinguished:

- an even alternating path leading from a positive vertex v and a free edge to a negative vertex w, or vice versa. In the first case the degree of v in F will be increased by 1, whereas the degree of w will be decreased by 1, or vice versa.
- an odd alternating path leading from a positive vertex v and a free edge to another positive vertex w. In this case the degree of both v and w will increase by 1.
- an odd alternating path leading from a negative vertex v and a matched edge to another negative vertex w. In this case the degree of both v and w will decrease by 1.

Observe that the operation of symmetric difference on the alternating path does not change the degrees of internal vertices. The same holds for alternating cycles.

From the above considerations we can receive an easy algorithm to test if a given bipartite graph G with pair of mappings (g, f) possesses all (g, f)-factors. First, we have to check the necessary condition. For a specific vertex x we check whether there exists a (g, f) factor F such that $d_F(x) = g(x)$ and a (g, f) factor

F' such that $d_{F'}(x) = f(x)$. Next, we compute the minimum (g, f)-factor and for every positive/negative vertex we test whether there exists an alternating path to another positive/negative vertex. Finally, we examine the conditions for the maximum (g, f)-factor. If these criteria hold for all vertices in $V(G)$ then the bipartite graph has all (g, f)-factors. Observe that the algorithm requires $O(n \cdot m)$ steps (assuming that the (g, f)-factor in G has been found), where n is the number of vertices and m is the number of edges in G.

Thus, an algorithm for testing if a graph G has all (g, f)-factors looks as follows:

1. check the necessary condition,
2. for every positive/negative vertex check if there exists an alternating path to every positive/negative vertex with respect to the minimum and maximum (g, f)-factor,
3. if yes, then the graph has all (g, f)-factors.

5 Conclusion

We have proven that bipartite (g, f)-factors have a continuity property. We know that this result does not hold for general graphs. But an interesting question is whether we can determine vertices in a general graph whose degrees are consecutive in (g, f)-factors.

Of course, one can use standard algorithms for the existence of (g, f)-factors to check whether, for every single vertex x and every number h with $g(x) \leq h \leq f(x)$, there is a (g, f)-factor F of G with $d_F(x) = h$. As there are $O(n^2)$ possibilities of picking x and h, one can easily check this property in polynomial time. However, this brute-force method is not efficient, and therefore we present an algorithm only for bipartite graphs. This means that the open problem of efficiently checking whether a graph has all (g, f)-factors still remains not solved for general case.

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