The Utility of Untangling

Vida Dujmović[1]

[1]School of Computer Science and Electrical Engineering,
University of Ottawa, Ottawa, Canada

Abstract

In this note we show how techniques developed for untangling planar graphs by Bose *et al.* [Discrete & Computational Geometry 42(4): 570-585 (2009)] and Goaoc *et al.* [Discrete & Computational Geometry 42(4): 542-569 (2009)] imply new results about some recent graph drawing models. These include column planarity, universal point subsets, and partial simultaneous geometric embeddings (with or without mappings). Some of these results answer open problems posed in previous papers.

1   Introduction

A *geometric graph* is a graph whose vertex set is a set of distinct points in the plane and each pair of adjacent vertices $\{v, w\}$ is connected by a line segment $\overline{vw}$ that intersects only the two vertices. A geometric graph is *planar* if its underlying combinatorial graph is planar. It is *plane* if no two edges cross other than in a common endpoint. A *straight-line crossing-free drawing* of a planar graph is a representation of that graph by a plane geometric graph.

Given a geometric planar graph, possibly with many crossings, to *untangle* it, means to move some of its vertices to new locations (that is, change their coordinates) such that the resulting geometric graph is plane. The goal is to do so by moving as few vertices as possible, or in other words, by keeping the locations of as many vertices as possible unchanged (that is, *fixed*). A series of papers have studied untangling of planar graphs or subclasses of planar graphs [9, 12, 15, 26, 28, 30, 32]. The best known (lower) bound for general planar graphs is due to Bose *et al.* [9] who proved that every n-vertex geometric planar graph can be untangled while keeping the locations of at least $\Omega(n^{1/4})$ vertices fixed. On the other hand, Cano *et al.* [12] showed that for all large enough n, there exists an n-vertex geometric planar graph that cannot be untangled while keeping the locations of more than $\omega(n^{0.4948})$ vertices fixed.

The purpose of this note is to highlight how the techniques developed by Bose *et al.* [9] and Goaoc *et al.* [26] can be used to establish new results on several recently studied graph drawing problems. Before presenting the new results we state the two key lemmas that are at the basis of all the results. The statements of these two lemmas are new, but their proofs are contained in and directly inferred by the work described in [9] and [26].

Let G be a plane triangulation (that is, an embedded simple planar graph each of whose faces is bounded by a 3-cycle). Canonical orderings of plane triangulations were introduced by de Fraysseix *et al.* [18]. They proved that G has a vertex ordering $\sigma = (v_1 := x, v_2 := y, v_3, \ldots, v_n := z)$, called a *canonical ordering*, with the following properties. Define G_i to be the embedded subgraph of G induced by $\{v_1, v_2, \ldots, v_i\}$. Let C_i be the subgraph of G induced by the edges on the boundary of the outer face of G_i. Then

- x, y and z are the vertices on the outer face of G.
- For each $i \in \{3, 4, \ldots, n\}$, C_i is a cycle containing xy.
- For each $i \in \{3, 4, \ldots, n\}$, G_i is biconnected and all the interior faces of G_i are bounded by 3-cycles.
- For each $i \in \{3, 4, \ldots, n\}$, v_i is a vertex of C_i with at least two neighbours in C_{i-1}, and these neighbours are consecutive on C_{i-1}.

For example, the ordering in Figure 1(a) is a canonical ordering of the depicted plane triangulation.

The following structure was defined first in Bose *et al.* [9]. Refer to Figure 1(b). Using the above notation, a *frame* $\mathcal{F}$ of G is the oriented subgraph of G with vertex set $V(\mathcal{F}) := V(G)$, where:

- Edges xy, xv_1 and v_1y are in $E(\mathcal{F})$ where xy is oriented from x to y, xv_1 is oriented from x to v_1 and v_1y is oriented from v_1 to y.

- For each $i \in \{4, 5 \ldots, n\}$ in the canonical ordering σ of G, edges pv_i and v_ip' are in $E(\mathcal{F})$, where p and p' are the first and the last neighbour, respectively, of v_i along the path in C_{i-1} from x to y not containing edge xy. Edge pv_i is oriented from p to v_i, and edge v_ip' is oriented from v_i to p'.

By definition, $\mathcal{F}$ is a directed acyclic graph with one source x and one sink y. $\mathcal{F}$ defines a partial order $<_\mathcal{F}$ on $V(\mathcal{F})$, where $v <_\mathcal{F} w$ whenever there is a directed path from v to w in $\mathcal{F}$.
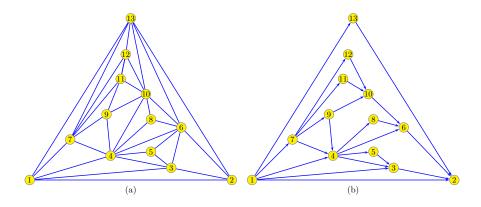

Figure 1:   (a) Canonical ordering of a plane triangulation G (b) Frame $\mathcal{F}$ of G.

Subsequently, it has been observed that a frame of G can also be obtained by taking the union of any two trees in Schnyder 3-tree-decompositions where the orientation of the edges in one of the two trees is reversed. See, for example, page 13 in Di Giacomo *et al.* [20] for this alternative formulation.

Recall that a *chain* (*antichain*) in a partial order is a subset of its elements that are pairwise comparable (incomparable). Given a partial order $(V, \leq)$ on a set of vertices V of some graph, we will often refer to a *chain* $V' \subseteq V$ (or antichain) and by that mean a subset of vertices of V that form a chain (antichain) in the given partial order $(V, \leq)$. We also say that a chain V' contains a chain V'' if V' and V'' are both chains in $(V, \leq)$ and $V'' \subseteq V'$.

Consider an n-vertex planar graph G and a set P of $k \leq n$ points in the plane together with a bijective mapping from a set V_k of k vertices in G to P. Let D be a straight-line crossing-free drawing of G. We say that D *respects* the given mapping if each vertex of V_k is represented in D by its image point as determined by the given mapping.

The following two lemmas are implicit in the work of Bose *et al.* [9] and Goaoc *et al.* [26]. Parts (b), (c) and consequently (d), in Lemma 1, are due to Goaoc *et al.* [26]. Note that, unlike here, the results of Goaoc *et al.* [26] are not expressed in terms of a chain in the frame of G but in terms of an equivalent structure: a simple path L in a plane triangulation, connecting two vertices x and y on the outer face x, y, z with the property that all chords of L lie on one

side of L and z lies on the other.

Consider a graph G, a set $S \subseteq V(G)$ and a set P of $|S|$ points in the plane together with a bijective mapping from S to P. For a vertex $v \in S$ mapped to a point $p \in P$, let $\mathsf{x}(v)$ denote the x-coordinate of p.

Lemma 1 *[9, 26] Let G be an n-vertex plane triangulation with a partial order $<_{\mathcal{F}}$ associated with a frame $\mathcal{F}$ of G. Let $C \subseteq V$ be a chain in $<_{\mathcal{F}}$. Let H be the graph induced in G by a maximal chain that contains C in $<_{\mathcal{F}}$. The embedding of H is implied by the embedding of G. Then:*

(a) H is a 2-connected outerplane graph, i.e. a 2-connected embedded outerplanar graph all of whose vertices lie on the cycle bounding the infinite face.

(b) Let $I \subseteq V(H)$ such that if $v, w \in I$ and $vw \in E(H)$ then vw lies on the outer face of H and is not edge xy. Let P be any set of $|I|$ points in the plane where no two points of P have the same x-coordinate. Given a bijective mapping from I to P such that, for every two vertices $v, w \in I$, $v <_{\mathcal{F}} w$ if and only if $\mathsf{x}(v) < \mathsf{x}(w)$, there exists a straight-line crossing-free drawing of G that respects the given mapping.

(c) There exists such a set I with at least $(V(H) + 1)/2$ vertices.

(d) There exists such a set I with at least $|C|/3$ vertices of C.

While the lower bound in part (c) is stronger than the lower bound in part (d), part (d) ensures that a fraction of vertices of C are used. That will be critical for some applications (see Theorem 2 in Section 2 and Theorem 6 in Section 4.2). Part (d) follows from (b) as follows. Consider the graph H' induced in H by the vertices of C. By part (a), H' is outerplanar. Thus its vertices can be coloured with three colours such that adjacent vertices in H' receive distinct colours. Thus there exists an independent set I in H' that contains at least $|C|/3$ vertices of C. The conditions imposed on the vertex set I in part (b) are immediate since I is an independent set in H' and H.

Note that, in an interesting recent development, Di Giacomo *et al.* [21] proved that every n-vertex plane triangulation has a frame where some chain has size at least $n^{1/3}$. Thus by part (a), $|V(H)| \geq n^{1/3}$ in that frame and consequently, every n-vertex plane triangulation has a 2-connected outerplane graph of size at least $n^{1/3}$ as an embedded induced subgraph.

The following is the second key lemma.

Lemma 2 *[9] Let G be an n-vertex plane triangulation with a partial order $<_{\mathcal{F}}$ associated with a frame $\mathcal{F}$ of G and the total order $<_{\sigma}$ associated with the corresponding canonical ordering. Let $A \subseteq V$ be an antichain in $<_{\mathcal{F}}$. Let P be any set of $|A|$ points in the plane where no two points of P have the same x-coordinate. Given a bijective mapping from A to P such that, for every two vertices $v, w \in A$, $v <_{\sigma} w$ if and only if $\mathsf{x}(v) < \mathsf{x}(w)$, then there exists a straight-line crossing-free drawing of G that respects the given mapping.*

2   Column Planarity

Given a planar graph G, a set $R \subset V(G)$ is *column planar* in G if the vertices of R can be assigned x-coordinates such that given any arbitrary assignment of y-coordinates to R, there exists a straight-line crossing free drawing of G that respects the implied mapping of vertices of R to the plane. Notions similar to column planarity were studied by Estrella-Balderrama *et al.* [24] and Di Giacomo *et al.* [19].

The column planar sets were first defined by Evans *et al.* [25]. A slightly stronger notion[1] was used earlier (although not named) in [9] (see Lemma 1 and Lemma 6 in [9]) where such sets were studied and used to prove Lemma 2 in the previous section. In particular, define a set $R \subset V(G)$ as *strongly column planar* if the following holds: there exists a total order μ on R such that

(a) given any set P of $|R|$ points in the plane where no two points have the same x-coordinate; and,

(b) given a bijective mapping from R to P such that, for every two vertices $v, w \in R$, $v <_\mu w$ if and only if $\mathsf{x}(v) < \mathsf{x}(w)$,

then there exists a straight-line crossing-free drawing of G that respects the given mapping. Being strongly column planar implies being column planar but not the converse. We use this slightly stronger notion as it is needed in the later sections.

It is implicit in the work of Bose *et al.* [9] (see the proof of Lemma 2 in [9]) that every tree has a strongly column planar set of size at least $n/2$. For column planar sets, this result is improved to $14n/17$ by Evans *et al.* [25]. Having a bound greater than $n/2$ is critical for an application of column planarity to partial simultaneous geometric embedding with mapping [25]. Barba *et al.* [6] proved that every n-vertex outerplanar graph has a column planar set of size at least $n/2$. Ravsky and Verbitsky [32] introduce a notion of *free collinear sets* in planar graphs. Lemma 1 in [9] (see also Lemma 8 in [16] for more readily applicable version) and the definition of free collinear sets imply that a set of vertices in a planar graph is strongly column planar if and only if it is a free collinear set. Thus the results of Ravsky and Verbitsky [32] on free collinear sets imply that every n-vertex outerplanar graph has a strongly column planar set of size at least $n/2$ and that every n-vertex partial 2-tree has a strongly column planar set of size at least $n/30$. This was further generalized and strengthened by Da Lozzo *et al.* [16] who proved that n-vertex planar partial 3-trees have strongly column planar sets of size at least $n/8$. This is in contrast to the fact that, for infinitely many values n, there is an n-vertex planar graph whose largest strongly column planar set has size $o(n)$, as proved by Ravsky and Verbitsky [32].

Evans *et al.* [25] pose as an open problem the question of developing any bound for column planar sets in general planar graphs. We provide here the first non-trivial (that is, better than constant) lower bound for this problem.

---

[1] with the roles of x and y coordinates reversed

Theorem 1 *Every n-vertex planar graph G has a (strongly) column planar set of size at least $\sqrt{n/2}$.*

Proof: If $|V(G)| \leq 2$, the result is trivially true. Thus without loss of generality we may assume that G is a triangulated plane graph. Let $\mathcal{F}$ be a frame of G, let $<_{\mathcal{F}}$ be its associated partial order, and let σ be the associated canonical ordering. Consider a chain in $<_{\mathcal{F}}$ of maximum size. (Hence, the chain starts with x and ends with y). Let H be the subgraph of G induced by that chain, as defined in Lemma 1. Let $I \subseteq V(H)$ be as defined in Lemma 1 (b). Consider any set P of $|I|$ points in the plane where no two points have the same x-coordinate and consider a bijective mapping from I to P such that, for every two vertices $v, w \in I$, it holds that $v <_{\mathcal{F}} w$ if and only if $\mathsf{x}(v) < \mathsf{x}(w)$. By Lemma 1 (b), there exists a straight-line crossing-free drawing of G that respects the given mapping and thus I, as ordered by $<_{\mathcal{F}}$, is a strongly column planar set. By Lemma 1 (c), $|I| \geq |V(H)|/2$. Thus if the size of the maximum chain in $<_{\mathcal{F}}$ is at least $\sqrt{2n}$, and thus $|V(H)| \geq \sqrt{2n}$, we are done. Otherwise, by Dilworth's theorem [22], $<_{\mathcal{F}}$ has a partition into at most $\sqrt{2n}$ antichains. By the pigeon-hole principle, there is an antichain in that partition with at least $n/\sqrt{2n} = \sqrt{n/2}$ vertices. Let $A \subseteq V(G)$ be the maximum antichain in $<_{\mathcal{F}}$. Consider any set P of $|A|$ points in the plane where no two points have the same x-coordinate and consider a bijective mapping from A to P such that, for every two vertices $v, w \in A$, it holds that $v <_{\sigma} w$ if and only if $\mathsf{x}(v) < \mathsf{x}(w)$. By Lemma 2, there exists a straight-line crossing-free drawing of G that respects the given mapping and thus A, as ordered by $<_{\sigma}$, is a strongly column planar set. This completes the proof since $|A| \geq \sqrt{n/2}$. $\qquad\square$

We conclude this section by proving a slightly stronger statement (with a slightly weaker bound when $S = V$) than Theorem 1. This stronger statement relies on part (d) of Lemma 1, and is a critical strengthening for some applications, such as partial simultaneous geometric embeddings with mappings (see Theorem 6 in Section 4.2).

Theorem 2 *Given any planar graph G and any subset $S \subseteq V$, there exists $R \subseteq S$ such that R is a strongly column planar set of G and $|R| \geq \sqrt{|S|/3}$.*

Proof: If $|V(G)| \leq 2$, the result is trivially true. Thus we may assume that G is a triangulated plane graph. Let $\mathcal{F}$ be a frame of G, let $<_{\mathcal{F}}$ be its associated partial order, and let σ be the associated canonical ordering. Assume first that $<_{\mathcal{F}}$ has a chain C such that $C \subseteq S$ and $|C| \geq \sqrt{3|S|}$. Let H be the subgraph of G induced by a maximal chain that contains C in $<_{\mathcal{F}}$, as defined in Lemma 1. Let $I \subseteq S$ be as defined in Lemma 1, (b) and (d). Consider any set P of $|I|$ points in the plane where no two points have the same x-coordinate and consider a bijective mapping from I to P such that, for every two vertices $v, w \in I$, it holds that $v <_{\mathcal{F}} w$ if and only if $\mathsf{x}(v) < \mathsf{x}(w)$. By Lemma 1 (b), there exists a straight-line crossing-free drawing of G that respects the given mapping and thus I, as ordered by $<_{\mathcal{F}}$, is a strongly column planar set. By Lemma 1 (d), $I \subseteq C$ and $|I| \geq |C|/3|$. Thus if $<_{\mathcal{F}}$ has a chain C such that

$C \subseteq S$ and $|C| \geq \sqrt{3|S|}$, we are done. Otherwise, by Dilworth's theorem [22], $<_{\mathcal{F}}$, when restricted to S, has a partition into at most $\sqrt{3|S|}$ antichains. By the pigeon-hole principle, there is an antichain $A \subseteq S$ in that partition that has at least $|S|/\sqrt{3|S|} = \sqrt{|S|/3}$ elements. Consider any set P of $|A|$ points in the plane where no two points have the same x-coordinate and consider a bijective mapping from A to P, such that for every two vertices $v, w \in A$, $v <_{\sigma} w$ if and only if $\mathsf{x}(v) < \mathsf{x}(w)$. By Lemma 2, there exists a straight-line crossing-free drawing of G that respects the given mapping and thus A, as ordered by $<_{\sigma}$, is a strongly column planar set. This completes the proof since $|A| \geq \sqrt{|S|/3}$ and $A \subseteq S$. $\qquad\square$

3   Universal Point Subsets

A set of points P is *universal* for a set of planar graphs if every graph from the set has a straight-line crossing-free drawing where each of its vertices maps to a distinct point in P. It is known that, for all large enough n, no universal pointset of size n exists for all n-vertex planar graphs – as first proved by de Fraysseix *et al.* [18]. The authors also proved that the $O(n) \times O(n)$ integer grid is universal for all n-vertex planar graphs and thus a universal pointsets of size $O(n^2)$ exists. Currently the best known lower bound on the size of a smallest universal pointset for n-vertex planar graphs is $1.235n - o(n)$ [29] and the best known upper bound is $n^2/4 - O(n)$ [5]. Closing the gap between $\Omega(n)$ and $O(n^2)$ is a major, and likely difficult, graph drawing problem, open since 1988 [17, 18].

This motivated the following notion introduced by Angelini *et al.* [2]. A set P of $k \leq n$ points in the plane is a *universal point subset* for all n-vertex planar graphs if the following holds: every n-vertex planar graph G has a subset $S \subseteq V(G)$ of k vertices and a bijective mapping from S to P such that there exists a straight-line crossing-free drawing of G that respects that mapping.

Angelini *et al.* [2] proved that for every n there exists a set of points of size at least $\sqrt{n}$ that is a universal point subset for all n-vertex planar graphs. Di Giacomo *et al.* [20] continued this study and showed that for every n, *every* set P of at most $(\sqrt{\log_2 n} - 1)/4$ points in the plane is a universal point subset for *all* n-vertex planar graphs. They also showed that *every* one-sided convex point set P of at most $n^{1/3}$ points in the plane is a universal point subset for *all* n-vertex planar graphs. The following theorem improves all these results.

Theorem 3 *Every set P of at most $\sqrt{n/2}$ points in the plane is a universal point subset for all n-vertex planar graphs.*

The proof of this lemma can be derived directly from Lemma 1 and Lemma 2, similarly to the proof of Theorem 1, but we will instead prove it using Theorem 1.

Proof: Rotate P to obtain a new pointset P' where no two points of P' have the same x-coordinate. By Theorem 1, every n-vertex planar graph has a strongly column planar set R of size $|P|$. Thus, by the definition of strongly column planar sets, there exists a total order μ on R such that given a bijective mapping

from R to P' where for every two vertices $v, w \in R$, $v <_\mu w$ if and only if $\mathsf{x}(v) < \mathsf{x}(w)$, there exists a straight-line crossing-free drawing of G that respects the given mapping. Such a mapping clearly exists since no two points of P' have the same x-coordinate. Rotating P' back to the original pointset completes the proof. □

It is not known if, for all n, there exist a universal point subset of size $n^{1/2+\epsilon}$ for some $\epsilon > 0$. Better, in particular linear, bounds are known for some subclasses of planar graphs. For example, every pointset of size n in general position is universal for all n-vertex outerplanar graphs [8, 14, 27]. Da Lozzo *et al.* [16] proved recently, that every set of at most $n/8$ points in the plane is a universal point subset for all n-vertex planar partial 3-trees.

4   (Partial) Simultaneous Geometric Embeddings

Simultaneous Geometric Embeddings were introduced by Braß *et al.* [11]. Initially there were two main variants of this problem, one in which the mapping between the vertices of the two graphs is given and another in which the mapping is not given. Since then there has been a plethora of work on the subject for various variants of the problem – see, for example a survey by Bläsius *et al.* [7].

4.1   Without mapping

Whether the following statement, on simultaneous geometric embeddings, is true is an open question asked by Braß *et al.* [11] in 2003 (Problem 12 in [10] asks the same question): For all n and for any two n-vertex planar graphs there exists a pointset P of size n such that each of the two graphs has a straight-line crossing-free drawing with its vertices mapped to distinct points of P. The statement is known *not* to be true when "two" is replaced by 7393 and $n = 35$ [13].

This motivates a study of (partial) geometric simultaneous embeddings – various versions of which have been proposed and studied in the literature [7]. We start with the following version.

Two graphs G_1 and G_2, where $|V(G_1)| \geq |V(G_2)|$ are said to have a *geometric simultaneous embedding with no mapping* if there exists a pointset P of size $|V(G_1)|$ such that each of the two graphs has a straight-line crossing-free drawing where all of its vertices are mapped to distinct points in P. Angelini *et al.* [3] write: "What is the largest $k \leq n$ such that every n-vertex planar graph and every k-vertex planar graph admit a geometric simultaneous embedding with no mapping? Surprisingly, we are not aware of any super-constant lower bound for the value of k."

The following theorem answers their question.

Theorem 4 *For every n and every $k \leq \sqrt{n/2}$, every n-vertex planar graph and every k-vertex planar graph admit a geometric simultaneous embedding with no mapping.*

Proof: Let G_1 and G_2 be the two given planar graphs with $|V(G_1)| = n$ and $|V(G_2)| = k$. By Fáry's theorem, G_2 has a straight-line crossing-free drawing on some set, P_2, of k points. By Theorem 3, G_1 has a straight-line crossing-free drawing where $|P_2|$ vertices of G_1 are mapped to distinct points in P_2. Consider now the set of points, P, defined by the vertices in the drawing of G_1. This set is our desired pointset as it is a set of n points such that each of G_1 and G_2 has a straight-line crossing-free drawing where all of its vertices are mapped to the points in P. □

Here is another variant of the (partial) geometric simultaneous embedding problem. For $k \leq n$, two n-vertex planar graphs G_1 and G_2 are said to have a *k-partial simultaneous geometric embedding with no mapping* (k-PSGENM) if there exists a set P of at least k points in the plane such that each of the two graphs has a straight-line crossing-free drawing where $|P|$ of its vertices are mapped to distinct points of P. Thus the (still) open question by Braß *et al.* [11] asks if every pair of n-vertex planar graphs has an n-PSGENM. Recall that Angelini *et al.* [2] proved that for every n there exists a set of points of size at least $\sqrt{n}$ that is a universal point subset for all n-vertex planar graphs. This implies that, for all n, any two n-vertex planar graphs have an $\sqrt{n}$-PSGENM. Note however that this does not imply Theorem 4. Namely, if one starts with a straight-line crossing-free drawing of the smaller graph G_2 (say on $\sqrt{n}$ vertices), there is no guarantee with this result that the bigger, n-vertex graph, G_1 can be drawn while using all the points generated by the drawing of G_2.

4.2   With Mapping

The notion of *k-partial simultaneous geometric embedding with mapping* (k-PSGE) is the same as k-PSGENM except that a bijective mapping between $V(G_1)$ and $V(G_2)$ is given and the two drawings have a further restriction that if $v \in V(G_1)$ is mapped to a point in P then the vertex w in $V(G_2)$ that v maps to, has to be mapped to the same point in P. In other words, two n-vertex planar graphs G_1 and G_2 on *the same vertex set*, V, are said to have a *k-partial simultaneous geometric embedding with mapping* (k-PSGE) if there exists a straight-line crossing free drawing D_1 of G_1 and D_2 of G_2 such that there exists a subset $V' \subseteq V$ with $|V'| \geq k$ and each vertex $v \in V'$ is represented by the same point in D_1 and D_2.

It is known that, for every large enough n, there are pairs of n-vertex planar graphs that do not have an n-partial simultaneous geometric embedding with mapping, that is, an n-PSGE [11]. In fact the same is true for simpler families of planar graphs, for example for a tree and a path [4], for a planar graph and a matching [4] and for three paths [11].

k-PSGE was introduced by Evans *et al.* [25] who proved (using their column planarity result) that any two n-vertex trees have an $11n/17$-PSGE. Barba *et*

al. [6] proved that any two n-vertex outerplanar graphs have an $n/4$-PGSE. Evans *et al.* [25] also observed that the main untangling result by Bose *et al.* [9] implies that every pair of n-vertex planar graphs has an $\Omega(n^{1/4})$-PSGE. Namely, start with a straight-line crossing-free drawing of G_1. Since the vertex sets of G_1 and G_2 are the same, the drawing of G_1 (or rather the drawing of its vertex set) defines a straight-line drawing of G_2. Untangling G_2 such that $\Omega(n^{1/4})$ of its vertices remain fixed (which is possible by [9]) gives the result.

Theorem 5 *[6] Every pair of n-vertex planar graphs has an $\Omega(n^{1/4})$-partial simultaneous geometric embedding with mapping, that is, it has an $\Omega(n^{1/4})$-PGSE.*

However, the above untangling argument fails if we try to apply it one more time. Namely, consider the following generalization of the k-PGSE problem. Given any set $\{G_1, \ldots, G_p\}$ of $p \geq 2$ n-vertex planar graphs on *the same vertex set*, V, we say that $G_1, \ldots, G_p$ have a *k-partial simultaneous geometric embedding with mapping* (k-PSGE) if there exists a straight-line crossing-free drawing D_i of each G_i, $i \in \{1, \ldots, p\}$ such that there exists a subset $V' \subseteq V$ with $|V'| \geq k$ and each vertex $v \in V'$ is represented by the same point in all drawings D_i, $i \in \{1, \ldots, p\}$.

If we try to mimic the earlier untangling argument that proves Theorem 5, it fails for $p = 3$ already since we cannot guarantee that when G_3 is untangled the set of its vertices that stays fixed has a non-empty intersection with the set that remained fixed when untangling G_2. It is here that part (d) of Lemma 1 is needed, or rather the stronger result on column planarity from Theorem 2.

Theorem 6 *Any set of $p \geq 2$ n-vertex planar graphs has an $\Omega(n^{1/4^{(p-1)}})$-partial simultaneous geometric embedding with mapping, that is, it has an $\Omega(n^{1/4^{(p-1)}})$-PGSE.*

Proof: Let $\{G_1, \ldots, G_p\}$ be the given set of p n-vertex planar graphs. The proof is by induction on p. The base case, $p = 2$, is true by Theorem 5. Let $p \geq 3$ and assume by induction that the set $\{G_1, \ldots, G_{p-1}\}$ has an $\Omega(n^{1/4^{(p-2)}})$-PGSE. Let $V' \subseteq V$ be the set from the definition of k-PSGE and let P' be the set of $|V'|$ points that V' is mapped to in the drawings $D_1, \ldots, D_{p-1}$. Thus $|V'| \in \Omega(n^{1/4^{(p-2)}})$ by induction. We may assume that no pair of points in P' has the same x-coordinate as otherwise we can just rotate the union of $D_1, \ldots, D_{p-1}$. By Theorem 2, there exists $R \subseteq V'$ that is strongly column planar in G_p and $|R| \geq \sqrt{|V'|/3}$. Since the vertices of V' are bijectively mapped to P', that mapping defines a bijective mapping from R to a subset P_R of P'. Consider the total order μ of R (the total order from the definition of strongly column planar sets) and the total order ϕ of R as defined by the x-coordinates of P_R. By the Erdős–Szekeres theorem [23, 33], there exists a subset R' of R of at least $\sqrt{|R|} \geq (|V'|/3)^{1/4}$ vertices such that the order of R' in μ is the same or reverse as the order of R' in ϕ. In the second case the union of all the drawings of $D_1, \ldots, D_p$ can be mirrored such that the order of R' in μ is the

same as the order of R' in ϕ. Since the vertices of R are bijectively mapped to P_R, this defines a bijective mapping from R' to a subset P'_R of P_R. Since R' is strongly column planar in G_p, we conclude that G_p has a straight-line crossing-free drawing D_p that respects the mapping from R' to P'_R and thus each vertex $v \in R'$ is represented by the same point in all drawings D_i, $i \in \{1, \ldots, p\}$. Since $|V'| \in \Omega(n^{1/4^{(p-2)}})$, and $|R'| \geq (|V'|/3)^{1/4}$, the lower bound holds. $\qquad\square$

Note that the definition of k-PSGE, as introduced in Evans *et al.* [25], has one additional requirement, as compared with the definition used here. Namely, the additional requirement states that if $v, w \in V$ are mapped to a same point in D_i and D_j, then $v = w$. However this additional requirement can always be met by the fact that it is possible to perturb any subset of vertices of a geometric plane graph without introducing crossings. More precisely, for any geometric plane graph there exists a value $\epsilon > 0$ such that each vertex can be moved any distance of at most ϵ, and the resulting geometric graph is also crossing-free.[2]

5   Conclusion

The main purpose of this note is to draw attention to Lemma 1 and Lemma 2 in the current form as they seem to have applications to numerous, some seemingly unrelated, graph drawing problems as evidenced by the results highlighted in the previous sections. The two lemmas appear in the current form for the first time here. Their original formulation was tailored towards specific application (untangling) and not directly applicable to any of the above mentioned problems.

Acknowledgements

---

[2]The maximum value ϵ for which this property holds is called the tolerance of the arrangement of segments. This concept, both for the geometric realization and the combinatorial meaning of the graphs was systematically studied in [1, 31].

References

[1] M. Abellanas, F. Hurtado, and P. Ramos. Tolerancia de arreglos de segmentos. In *Proc. VI Encuentros de Geometría Computacional*, pages 77–84, 1995.

[2] P. Angelini, C. Binucci, W. S. Evans, F. Hurtado, G. Liotta, T. Mchedlidze, H. Meijer, and Y. Okamoto. Universal point subsets for planar graphs. In *Proc. 23rd International Symposium of Algorithms and Computation (ISAAC)*, pages 423–432, 2012. `doi:10.1007/978-3-642-35261-4_45`.

[3] P. Angelini, W. S. Evans, F. Frati, and J. Gudmundsson. SEFE without mapping via large induced outerplane graphs in plane graphs. *Journal of Graph Theory*, 2015. `doi:10.1002/jgt.21884`.

[4] P. Angelini, M. Geyer, M. Kaufmann, and D. Neuwirth. On a tree and a path with no geometric simultaneous embedding. *Journal of Graph Algorithms and Applications*, 16(1):37–83, 2012. `doi:10.7155/jgaa.00250`.

[5] M. J. Bannister, Z. Cheng, W. E. Devanny, and D. Eppstein. Superpatterns and universal point sets. *Journal of Graph Algorithms and Applications*, 18(2):177–209, 2014. `doi:10.7155/jgaa.00318`.

[6] L. Barba, M. Hoffmann, and V. Kusters. Column planarity and partial simultaneous geometric embedding for outerplanar graphs. In *Abstracts of the 31st European Workshop on Computational Geometry (EuroCG)*, pages 53–56, 2015.

[7] T. Bläsius, S. G. Kobourov, and I. Rutter. Simultaneous embedding of planar graphs. In *In Roberto Tamassia (editor), Handbook of Graph Drawing and Visualization*, pages 349–381, 2013.

[8] P. Bose. On embedding an outer-planar graph in a point set. *Comput. Geom.*, 23(3):303–312, 2002. `doi:10.1016/S0925-7721(01)00069-4`.

[9] P. Bose, V. Dujmović, F. Hurtado, S. Langerman, P. Morin, and D. R. Wood. A polynomial bound for untangling geometric planar graphs. *Discrete & Computational Geometry*, 42(4):570–585, 2009. `doi:10.1007/s00454-008-9125-3`.

[10] F. Brandenburg, D. Eppstein, M. T. Goodrich, S. G. Kobourov, G. Liotta, and P. Mutzel. Selected open problems in graph drawing. In *Graph Drawing*, volume 2912 of *Lecture Notes in Computer Science*, pages 515–539. Springer, 2003. `doi:10.1007/978-3-540-24595-7_55`.

[11] P. Braß, E. Cenek, C. A. Duncan, A. Efrat, C. Erten, D. Ismailescu, S. G. Kobourov, A. Lubiw, and J. S. B. Mitchell. On simultaneous planar graph embeddings. *Comput. Geom.*, 36(2):117–130, 2007. `doi:10.1016/j.comgeo.2006.05.006`.

[12] J. Cano, C. D. Tóth, and J. Urrutia. Upper bound constructions for untangling planar geometric graphs. *SIAM J. Discrete Math.*, 28(4):1935–1943, 2014. `doi:10.1137/130924172`.

[13] J. Cardinal, M. Hoffmann, and V. Kusters. On universal point sets for planar graphs. *Journal of Graph Algorithms and Applications*, 19(1):529–547, 2015. `doi:10.7155/jgaa.00374`.

[14] N. Castañeda and J. Urrutia. Straight line embeddings of planar graphs on point sets. In *Proc. the 8th Canadian Conference on Computational Geometry, (CCCG)*, pages 312–318, 1996.

[15] J. Cibulka. Untangling polygons and graphs. *Discrete & Computational Geometry*, 43(2):402–411, 2010. `doi:10.1007/s00454-009-9150-x`.

[16] G. Da Lozzo, V. Dujmovic, F. Frati, T. Mchedlidze, and V. Roselli. Drawing planar graphs with many collinear vertices. *CoRR*, abs/1606.03890, 2016. URL: `http://arxiv.org/abs/1606.03890`.

[17] H. de Fraysseix, J. Pach, and R. Pollack. Small sets supporting fary embeddings of planar graphs. In *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing*, STOC '88, pages 426–433, 1988. `doi:10.1145/62212.62254`.

[18] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990. `doi:10.1007/BF02122694`.

[19] E. Di Giacomo, W. Didimo, M. van Kreveld, G. Liotta, and B. Speckmann. Matched drawings of planar graphs. *Journal of Graph Algorithms and Applications*, 13(3):423–445, 2009. `doi:10.7155/jgaa.00193`.

[20] E. Di Giacomo, G. Liotta, and T. Mchedlidze. How many vertex locations can be arbitrarily chosen when drawing planar graphs? *CoRR*, abs/1212.0804, 2012. URL: `http://arxiv.org/abs/1212.0804`.

[21] E. Di Giacomo, G. Liotta, and T. Mchedlidze. Lower and upper bounds for long induced paths in 3-connected planar graphs. *Theor. Comput. Sci.*, 636:47–55, 2016. `doi:10.1016/j.tcs.2016.04.034`.

[22] R. P. Dilworth. A decomposition theorem for partially ordered sets. *Ann. of Math. (2)*, 51:161–166, 1950.

[23] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[24] A. Estrella-Balderrama, J. J. Fowler, and S. G. Kobourov. Characterization of unlabeled level planar trees. *Comput. Geom.*, 42(6-7):704–721, 2009. `doi:10.1016/j.comgeo.2008.12.006`.

[25] W. Evans, V. Kusters, M. Saumell, and B. Speckmann. Column planarity and partial simultaneous geometric embedding. In *Proc. of 22nd International Symposium on Graph Drawing, (GD)*, pages 259–271, 2014. `doi:10.1007/978-3-662-45803-7_22`.

[26] X. Goaoc, J. Kratochvíl, Y. Okamoto, C. Shin, A. Spillner, and A. Wolff. Untangling a planar graph. *Discrete & Computational Geometry*, 42(4):542–569, 2009. `doi:10.1007/s00454-008-9130-6`.

[27] P. Gritzmann, B. Mohar, J. Pach, and R. Pollack. Embedding a planar triangulation with vertices at specified points (solution to problem e3341). *Amer. Math. Monthly*, 98:165–166, 1991.

[28] M. Kang, O. Pikhurko, A. Ravsky, M. Schacht, and O. Verbitsky. Untangling planar graphs from a specified vertex position - Hard cases. *Discrete Applied Mathematics*, 159(8):789–799, 2011. `doi:10.1016/j.dam.2011.01.011`.

[29] M. Kurowski. A 1.235 lower bound on the number of points needed to draw all n-vertex planar graphs. *Information Processing Letters*, 92(2):95–98, 2004. `doi:10.1016/j.ipl.2004.06.009`.

[30] J. Pach and G. Tardos. Untangling a polygon. *Discrete & Computational Geometry*, 28(4):585–592, 2002. `doi:10.1007/s00454-002-2889-y`.

[31] P. Ramos. *Tolerancia de estructuras geométricas y combinatorias*. PhD thesis, Universidad Politécnica de Madrid, Madrid, Spain, 1995.

[32] A. Ravsky and O. Verbitsky. On collinear sets in straight-line drawings. In *Proc. 37th International Workshop on Graph-Theoretic Concepts in Computer Science, (WG)*, pages 295–306, 2011. `doi:10.1007/978-3-642-25870-1_27`.

[33] J. M. Steele. Variations on the monotone subsequence theme of Erdös and Szekeres. In *Discrete probability and algorithms*, pages 111–131. Springer, 1995. `doi:10.1007/978-1-4612-0801-3_9`.