BOOK REVIEW

The Kepler Problem: Group Theoretical Aspects, Regularization and Quantization, with Application to the Study of Perturbation, by Bruno Cordani, Birkhäuser, Basel, 2002, xvii + 439 pp. + CD included, ISBN 3-7643-6902-7

Different versions of the famous Kepler problem are important for a wide variety of applications in mathematical and physical problems. In particular, the last three decades have seen an explosion of applications to quantum mechanics (geometric quantization, magnetic monopoles, etc.) and celestial mechanics. So, the need of a systematic exposition of all its classical and modern aspects is rather tangible.

In some broader sense the book offers a comprehensive treatment of the Kepler problem, i.e. the two body problem but it should be remarked that it does not provide neither encyclopedic collection of various aspects of the problem nor presents a historical survey. An adequate idea of its spirit can be found in the following extract from the *Introduction:*

" *The aim of this book is rather ambitious: to gather and order in a logical way most of the important ideas published on the various aspects of the Kepler problem...* "

One might trace three main directions of contemporary development in the studies of two body problem:

1. the existence of a symplectomorphism between the phase space of the regularized Kepler problem and the cotangent bundle of the n-dimensional sphere with its zero section removed [7]

2. the existence of a symplectomorphism between the cotangent bundle of the n-dimensional sphere and the coadjoint orbit of the $SO(n+1, 2)$ Lie group [5, 6]

3. the interpretation of the Kepler motion as a result of an extension of the canonical transformation corresponding to the geodesic motion on the sphere [2].

Taking into account the above list one sees that the book under review is both a survey and a research monograph. The author's contributions to the subject are mostly related to the third item above.

The book consists of an *Introductory Survey* and four basic *Parts*. The *Introductory Survey* starts with a sketch of some basic facts like separation of variables for the Kepler problem, its quantization, regularizations and symmetries. This includes the analytical and geometrical properties of conics, description of the first integrals of the problem (the total energy, the momentum and the eccentricity vector), the three anomalies (true, eccentric and mean), etc.

Part I (*Chapters 2 ÷ 5*) gives a survey on the elementary theory of the Kepler motion starting with the basic fact about it (*Chapter 2*). The separation of variables of the Hamilton-Jacobi equation is reviewed in *Chapter 3*. *Chapter 4* deals with the quantization of the Kepler problem. Starting with the Schrödinger quantization the author describes the solutions of the Schrödinger equation for the Kepler problem in those four coordinate systems (spherical, parabolic, elliptic and spheroconical) that admit separation of variables. Next the Pauli quantization technique is described. This chapter ends with the Fock quantization scheme. Note that the modern treatment of the Kepler problem began with the Fock's paper [3]. An interesting property of the Fock method is that it can be generalized to higher-dimensional case [1]. In the next *Chapter 5* the author describes the regularizations and symmetry of the Kepler problem. The three most important regularization procedures [7, 9, 10] are reviewed. E.g., Moser [7] has shown that the Kepler flow of the regularized n-dimensional Kepler problem for negative energy is equivalent to the geodesic flow on the n-dimensional sphere $\mathbb{S}^n$ (the $SO(n+1)$-symmetry). Originally it has been proved using an appropriate extension of the stereographic projection of $\mathbb{S}^n$ to its cotangent bundle. While the Moser method works in the configurational manifold the Souriau method [9] works in the phase space. The chapter ends with the so-called Kuastaanheimo-Stiefel transformations (KS transformations) [10]. This transformation brings the Hamiltonian of the initial three-dimensional Kepler problem into the Hamiltonian of the four-dimensional harmonic oscillator. Some generalizations of these results in the case of the n-dimensional spheres $\mathbb{S}^n$ and hyperboloids can be found in [4].

Part II of the book (*Chapters 6 ÷ 10*) entitled *Group-Geometric Theory* starts with a chapter about the conformal regularization of the Kepler problem. The action of the conformal group on the Minkowski space $\mathcal{M}^{n,1}$ is described and its generators are found. It is shown that the conformal group is isomorphic to the pseudo-orthogonal group $O(n+1, 2)$. The momentum maps related to all relevant sections of the null cone are constructed. Here also the relation between

the Fock and Bacry-Guörgyi parameters are clarified and the equivalence of the Fock-Moser and KS regularizations are proved for any value of the energy. *Chapter 7* is about the spinorial regularization of the Kepler problem. The point is that the well-known Levi-Civita and KS transformations are examples of spinorial regularization ($n = 2, 3$, respectively) [6]. For example if $n = 3$ the relevant symmetry group of the corresponding Kepler problem is $SO(4, 2)$ which admits as a *Spin* group the pseudo-unitary group $SU(2, 2)$. The possibility for extension of the KS transformations to higher dimensions is also investigated. It is shown also that while these generalizations are valid for arbitrary n, an explicit formulas can be obtained only for $n = 2, 3, 5$. *Chapter 8* returns to the problem of separation of variables. Three general theorems about separability are proved: the Stäckel theorem, which gives the necessary and sufficient conditions for a natural orthogonal system to be separable, the Eisenhart theorem, which provides a strategy of searching for orthogonal systems separating some n-dimensional problems and the Robertson theorem, which gives necessary and sufficient conditions for the separability of the Schrödinger equation if the corresponding Hamilton-Jacobi equation is separable. The *ninth Chapter* describes the geometric quantization of the Kepler Problem. Speaking in general the geometric quantization of a Hamiltonian system consists of two stages [5, 8]. In the first the phase space is identified as a co-adjoint orbit of some Lie group and then an irreducible unitary representation of the dynamical group is constructed by means of Hermitian operators acting in the Hilbert space $\mathcal{L}_2$ of the square-integrable functions defined over the base space of a polarization of the co-adjoint orbit. The author describes in details the quantization of the geodesic flow on the n-dimensional sphere $\mathbb{S}^n$. The last *Chapter 10* of the second part start with an appropriate modification of the construction of the momentum map in the case of presence of a magnetic monopole. This is applied to MICZ (McIntosh-Cisneros-Zwanziger) and the Taub-NUT Kepler problems.

Part III (*Chapters 11 ÷ 13*) gives a treatment of the theory of perturbations for the Kepler motion. It starts with a review on general perturbation theory for Hamiltonian systems. After that the formal machinery of the Lie series is also presented. The core of the formal problem of the perturbations is the so-called homological equation. Solving it up to a given order one gets an approximate Hamiltonian which gives an integrable system if the unperturbed Hamiltonian is nondegenerate. The subtle problem of the convergence of the perturbed schemes is solved using the famous KAM (Kolmogorov-Arnold-Moser) theorem: if the unperturbed Hamiltonian is non-degenerate, picking out a torus such that the corresponding frequencies are "sufficiently irrational" it will not be destroyed but only slightly deformed by a sufficiently small perturbation. The properties of the Cantor sets of

the conserved and destroyed tori under perturbations are also discussed. Central attention in this occasion is paid to the Nekhoroshev theorem, which states that the variation of the action is of the order of the perturbation value (thus very small) for the time intervals increasing exponentially with the reciprocal of the perturbation parameter. *Chapter 12* studies the perturbations of the Kepler problem. The traditional method here requires the usage of action-angle variables (usually Delaunay and Poincaré variables in our case). Unfortunately their inverses cannot be found in a closed form, since Delaunay and Poincaré variables are transcendental functions of the physical coordinates and momenta. Besides, they are not a global coordinates (some orbits are missing in these parametrizations). The later is related to the fact that the global topology of the phase space of the (regularized) Kepler problem is that of $T^+\mathbb{S}^3$ (the cotangent bundle of the three-dimensional sphere without zero section) while the action-angle variables describe a manifold with the topology of $\mathbb{T}^3 \times \mathbb{R}^3$ (here $\mathbb{T}^3$ is the three-dimensional torus). In order to avoid these obstacles the global coordinates given by the Fock parametrization are appropriately adopted. So the author no longer deals with a six-dimensional symplectic manifold, but with eight-dimensional (rank six) Poisson manifold. Using the Moser theorem the author proves the existence of four periodic orbits (corresponding to the critical points of the related Hamiltonian vector field on the reduced manifold $\mathbb{S}^2 \times \mathbb{S}^2$) for the complete (non-truncated) Hamiltonian. Then using the Kolmogorov theorem it is shown that almost all three-dimensional orbits (corresponding to those orbits on $\mathbb{S}^2 \times \mathbb{S}^2$ surrounding one of the elliptic critical points) can be continued to the full Hamiltonian as well. *Chapter 13* is called *Perturbations with Axial Symmetry* and the basic idea here is to reduce the manifold of the orbits, which has the topology of $\mathbb{S}^2 \times \mathbb{S}^2$ by taking into account the residual axial symmetry. The result is a symplectic manifold homeomorphic to $\mathbb{S}^2$, so the level surfaces of the reduced Hamiltonian intersecting this symplectic manifold lead to a family of curves describing the perturbed motion. Three concrete examples are demonstrated in this framework: the Lunar problem (the Moon moving around the Earth and perturbed by the Sun), combined Stark and Zeeman effect (the Hydrogen atom in constant electric and magnetic fields that are mutually orthogonal) and the problem of an artificial satellite moving around an oblate primary.

Part IV contains four appendices on differential geometry, Lie groups and Lie algebras, Lagrangian and Hamiltonian mechanics. The first *Appendix* on differential geometry starts with reminding some basic facts from topology, differentiable manifolds, tensors and fiber bundles. The second one gives a brief survey on Lie groups and Lie algebras. The (co)adjoint representation is defined using the

natural linear action of the corresponding Lie group on the (co)tangent space at the identity. The classification of the real simple Lie groups is given. The third *Appendix* reminds Lagrangian dynamics and the fourth one is about Hamiltonian dynamics. The last *Appendix* contains a brief review of the theory of canonical transformations and the Hamilton-Jacobi integration method paying a special attention on their geometrical meaning. After defining the momentum map the Noether theorem is extended to the case of a generic Lie group acting on a symplectic manifold. This *Appendix* ends with the Liouville-Arnold theorem about complete integrability.

The CD included to the book contains couple of programs which illustrate the methods presented in the text. The first one, called KEPLER allows one to calculate the effects of a generic perturbation of the Kepler problem, i.e. a point moving under the gravitational force of a fixed point. The second program by the name EULER (in fact a Maple routine) plots the orbit (in Cartesian coordinates) of a point moving under the gravitational forces of two fixed points, one placed at the origin and having unit mass, the other placed at the point (0,2) with mass m in the interval $[0, 1]$.

The class of potential readers of the present book is rather wide - from advanced undergraduate students to graduate and postgraduate students and researchers who work in mathematical physics. The selection of the topics in the book allows one to extract at least three students courses (together with a suitable excerpts from the *Appendices*) concerning various aspects of the Kepler problem:

1. group-theoretical essence of the Kepler problem

2. the power of the geometric quantization scheme

3. perturbation theory of the Keplerian orbits.

Summarizing the main features of the book under review, namely that:

- the book contains a lucid and comprehensive introduction to the Kepler problem including all of its classical and modern aspects

- the author's original results fit very well in the general scheme of the book

it is clear that this book is undoubtedly a nice benefit for the whole mathematical and physical community and should be present at every university or research institute library.

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