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MULTIDIMENSIONAL POISSON BRACKETS OF HYDRODYNAMIC TYPE AND FLAT PENCILS OF METRICS

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Abstract. We give an exposition of some recent crucial achievements in the theory of multidimensional Poisson brackets of hydrodynamic type. In particular, we solve the well-known Dubrovin–Novikov problem posed as long ago as 1984 in connection with the Hamiltonian theory of systems of hydrodynamic type, namely, the classification problem for multidimensional Poisson brackets of hydrodynamic type. In contrast to the one-dimensional case, in the general case, a nondegenerate multidimensional Poisson bracket of hydrodynamic type cannot be reduced to a constant form by a local change of coordinates. We obtain the classification of all nonsingular nondegenerate multidimensional Poisson brackets of hydrodynamic type for any number N of components and for any dimension n by differential-geometric methods. This problem is equivalent to the classification of a special class of flat pencils of metrics. A key role in the solution of this problem was played by the theory of compatible metrics that had been earlier constructed by the present author.

1. Introduction

In this paper we study *multidimensional Poisson brackets of hydrodynamic type*, i.e., field-theoretic Poisson brackets of the form

$$\{u^i(x), u^j(y)\} = \sum_{\alpha=1}^{n} \Big(g^{ij\alpha}(u(x))\delta_\alpha(x-y) + b_k^{ij\alpha}(u(x))u_\alpha^k(x)\delta(x-y)\Big) \quad (1)$$

where $u = (u^1, \dots, u^N)$ are local coordinates on a certain smooth N-dimensional manifold M or in a domain of $\mathbb{R}^N$, $x = (x^1, \dots, x^n)$, $y = (y^1, \dots, y^n)$ are independent variables, the coefficients $g^{ij\alpha}(u)$, $b_k^{ij\alpha}(u)$ are smooth functions of the local coordinates $(u^1, \dots, u^N)$, $1 \le i, j, k \le N$, $1 \le \alpha \le n$, $u(x) = (u^1(x), \dots, u^N(x))$ are smooth functions (N-component fields) of n independent variables $x^1, \dots, x^n$ with values in the manifold M, $u_\alpha^k(x) = \partial u^k/\partial x^\alpha$ and $\delta(x)$ is the Dirac delta-function and $\delta_\alpha(x-y) = \partial\delta(x-y)/\partial x^\alpha$.