BEYOND DELAUNAY SURFACES

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Abstract. An interesting class of axially symmetric surfaces, which generalizes Delaunay's unduloids and provides solutions of the shape equation is described in explicit parametric form. This class provide the first analytical examples of surfaces with periodic curvatures studied by K. Kenmotsu and leads to some unexpected relationships among Jacobian elliptic functions and their integrals.

1. Introduction

It is well-known that in aqueous solution, amphiphilic molecules (e.g., phospholipids) form spontaneously bilayers so that the hydrophilic heads of these molecules are located in both outer sides of the bilayer which are in contact with the liquid, while their hydrophobic tails remain at the interior. In many cases, the bilayer form a closed membrane, which is called a vesicle. Vesicles constitute well-defined and sufficiently simple model systems for studying basic physical properties of the more complex biological cells.

In 1973, Helfrich [3] had proposed the so-called spontaneous curvature model according to which the equilibrium shapes of a lipid vesicle are determined by the extremals of the curvature (shape) energy

$$\mathcal{F}_c = \frac{k_c}{2} \int_{\mathcal{S}} (2H - \mathbb{h})^2 \, \mathrm{d}A + k_G \int_{\mathcal{S}} K \mathrm{d}A$$

under the constraints of fixed total enclosed volume V and area A of its middle surface $\mathcal{S}$. In the above equation H and K denote the mean, respectively the Gaussian curvature of the surface $\mathcal{S}$ while $\mathbb{h}$, k_c and k_G are real constants representing the spontaneous curvature, bending and Gaussian rigidity of the membrane. Using two Lagrange multipliers λ and p, this yields the functional

$$\mathcal{F} = \mathcal{F}_c + \lambda \int_{\mathcal{S}} \mathrm{d}A + p \int \mathrm{d}V.$$