BOOK REVIEW

Symplectic Geometry of Integrable Hamiltonian Systems, by Michele Audin, Ana Cannas da Silva and Eugene Lerman, Birkhäuser, Basel, 2003, viii + 225pp, ISBN 3-7643-2167-9

A modern mathematician, or mathematical physicist, could hardly imagine the contemporary mathematics and mathematical physics without differential geometry with all its directions both in pure studies and in applications. The joint work of mathematicians and physicists has resulted in a tremendous advance in many of its branches, for example recall the Riemannian geometry and general relativity during the last century. Another, also very appropriate example is the connection-curvature look on the integrability of partial differential equations in the frame of *fibre bundle theory* and the physical Yang-Mills theory and its later development as general *gauge theory*, where the mathematical *connection* field becomes the carrier of some physical interactions. Physics has always stimulated mathematics to develop new branches, and mathematics has always been very responsive and warm-hearted to the needs of physics. Starting with Newton and Leibniz the greatest mathematicians of all times have contributed considerably to deeply understanding and solving difficult physical problems. One of the most fascinating and impressive examples seems to be the interrelations between *Hamiltonian Mechanics* and *Symplectic Geometry*. Although Hamilton published his view on dynamical equations in 1835, the essence and structure of his approach was adequately understood just in twentieth century, and this would not happen without the work of people like S. Poisson (1781-1840), L. Euler (1707-1783), J. Liouville (1809-1882), J. Lagrange (1736-1813), H. Poincare (1854-1912), and many others among which I would mention S. Lie and E. Cartan with their great achievements in differential geometry and the theory of continuous groups (Lie groups). In this way symplectic geometry was born, i.e. geometry of even dimensional smooth manifolds endowed with a non-degenerate closed two-form ω, $\mathrm{d}\omega = 0$, and the main subject of Hamiltonian mechanics appeared as finding integral trajectories of Hamiltonian vector fields X, i.e. those vector fields which give closed one-forms of the kind $i(X)\omega$. Locally, all such one-forms are exact, i.e., $i(X)\omega = \mathrm{d}H$ and

the corresponding function H is called a hamiltonian. Every such hamiltonian H represents an integral of the corresponding system of equations, i.e. on every trajectory H is a constant. The problem of integrability of a given hamiltonian vector field consists in a full description of all of its trajectories, and this problem can be solved if we find sufficiently many such integrals. In the general case this is not possible, and if this is possible, the corresponding system is called integrable. And studying some aspects of the corresponding geometry in such a case is the main subject of the book under consideration. The book consists of three parts, every part is written by a single author. The first part (*Part A*) of the book is written by Michele Audin, and is divided into two chapters (plus exercises). *Chapter 1* deals mainly with the space of complex numbers, considered as a vector space on one hand, and as a manifold on the other hand. The basic concepts of the corresponding symplectic geometry, Lagrangian submanifolds, special Lagrangian submanifolds and integrability are introduced and discussed. The text is clear, well ordered and illustrated. The Lagrangian submanifolds are introduced through Lagrangian immersions. The Gauss map view is also presented. In the Appendices some brief introductory facts, concerning the topology of the symplectic group, integrability and Poisson brackets are considered. In Exercise I.17 a short introduction to Lagrangian cobordisms is given. *Chapter 2* considers general symplectic manifolds and gives the basics of Lagrangian and special Lagrangian submanifolds in this more general situation. Tubular neighborhoods of Lagrangian submanifolds and examples of Calabi-Yau manifolds are defined and discussed. Finally, the concept of mirror symmetry is briefly considered. Exercise II.7 is devoted to Calabi-Yau surfaces. The Bibliography consists of 35 entries and gives the most important papers and some of the books, which the reader could make a good use of.

The second part (*Part B*) of the book, named *Symplectic Toric Manifolds* is written by Ana Canneas da Silva, the author of the well known *Lectures on Symplectic Manifolds*, Springer LNM, 2001 (could be found also on Internet), and is also divided into two chapters: 1. Symplectic Viewpoint, 2. Algebraic Viewpoint. The basic working tool used by the author is the *moment(um) map*, given in Definition I.1.11. In author's words toric manifolds provide examples of extremely symmetric and completely integrable hamiltonian spaces. The definition of a toric manifold is given in subsection I.1.6, i.e., it is a compact connected symplectic manifold equipped with an effective hamiltonian action of a torus T of dimension half the dimension of the manifold, and with a choice of a corresponding moment map. In section I.3.3 some homology properties of toric manifolds are presented. The second chapter of this part presents the algebraic viewpoint trying to explain

toric manifolds as a special class of projective varieties. After some preparation material the corresponding *algebraic* view of a toric manifold is given in Definition II.1.16. The subsections II.2 and II.3 are devoted respectively to classification considerations and introduction to *moment polytopes* in the algebraic context. Finally, subsection II.3.6 gives comparative consideration of the symplectic and algebraic viewpoints. Similarly to the above mentioned book on symplectic geometry of the author, the text of this second part of the presented book is very well written and illustrated, and may serve as a very good introduction to the subject, so, it can be strongly recommended to all interested readers. The Bibliography presents 50 entries. There is also *Index* pages.

The last *Part C* of the book is written by Eugene Lerman and named *Geodesic Flows and Contact Toric Manifolds*. It is devoted to a more specific topic: topological study of contact toric manifolds. The author introduces the basic concepts of *symplectic cone* and *contact manifold*. A symplectic cone is a symplectic manifold with a chosen vector field X on it, such that the Lie derivative of the symplectic form with respect to X does not change it, and the generated by X one-parameter transformation group acts properly on the manifold. A contact manifold is a manifold endowed with a contact one-form, and a contact one-form is everywhere non-zero one-form, such that its differential gives a symplectic structure on the vector bundle which fiber at the point p coincides with the kernel of this one-form. The third chapter of this *Part C* is devoted to the proof of a theorem (Theorem I.38) with topological meaning. This theorem establishes the homotopy non-equivalence between a compact connected toric manifold endowed with appropriate action of a torus and the direct product of n-torus and $(n-1)$-sphere. As it was mentioned, this text follows mainly some topological purpose, so it can be recommended to topologically oriented geometers. The *Appendix* deals with hypersurfaces of contact type. The Bibliography gives 25 entries.

As a whole, (together with the exercises presented) the book is well written, very well ordered and the exposition is very clear. It could serve as a good stand-by introduction to this part of symplectic geometry and integrability problems of hamiltonian systems. In the spirit of many modern mathematical and mathematical physics studies it incorporates different important areas and directions of mathematics aiming to get valuable results in the difficult problem of integrability of hamiltonian systems.

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