**BOOK REVIEW**

*Differential Geometry*: *Bundles, Connections, Metrics and Curvature*, by Clifford Henry Taubes, Oxford University Press, Oxford, xiii + 298 pp, ISBN: 978-0-19-960588-0 (hbk), 978-0-19-960587-3 (pbk).

This book is a textbook for beginners on the topics mentioned in its title, which play a central role in the contemporary differential geometry. According to the author, the audience is expected to consist mainly of first-year graduate students and advanced undergraduate students. The text resembles indeed accompanying notes of a lecture course. The language is a bit less formal than usual and there is no numbering of formulae for reference purposes. Only in several cases a formula is denoted by ($*$) in order to be cited in the course of some long proof. If at a certain point some earlier construction or example is needed, the author usually does not hesitate to repeat it briefly. Relatively a few assertions have the honour to be distinguished as Theorems, Lemmas, Propositions and Corollaries, 64 according to the list at the end of the book, which are in average of one in four and a half pages. The situation with the definitions is quite similar. The book has no list of references. Instead of that each chapter ends with a section "Additional reading" which contains titles of books covering the topic of the chapter. In the very rare cases an article or a book is cited, the reference is given inside the text. A nice feature of the book is that it contains many examples. I cannot resist mentioning the extensive use of 𝔉𝔯𝔞𝔨𝔱𝔲𝔯 for notational purposes. It would be hardly an exaggeration to say that the number of objects denoted by 𝔉𝔯𝔞𝔨𝔱𝔲𝔯 letters and by standard latin letters are almost the same. So, when finishing the book you will not only know a lot about bundles, connections, metrics and curvature, but also feel like reading German novels printed before World War I.

Let me now describe the contents of the nineteen chapters. Chapter 1 introduces the notions of smooth manifold, smooth map between manifolds, submanifold, immersion, submersion and partition of unity. Although the author warns that the reader is expected to be familiar with smooth manifolds and no proofs will be given, out of the one lemma and four theorems (the inverse function theorem, the implicit function theorem, Sard's theorem and the contraction mapping theorem)

all but Sard's theorem are proved, albeit incompletely (with the exception of the contraction mapping theorem). In this chapter there is an eccentric definition of the notion of locally finite atlas. This is what in standard terms is called point finite atlas. The notion appears several times in the first half of the book and is basically harmless, since nothing is lost if the words "locally finite" are ignored. There is one notable exception though and this is Appendix 1.2 on partitions of unity. This section is a complete failure. There are at least three different reasons why the given construction of a smooth partition of unity is not correct. And even if those problems did not exist, due to the peculiar meaning of "locally finite atlas" the resulting partition of unity would be, in standard terminology, point finite and not locally finite. Thus it would not be able to do the work a smooth partition of unity is supposed to do: to glue together locally defined smooth functions to obtain a globally defined smooth function. Because of that and because there are also some other inaccuracies in this chapter, the reader better follows the advice of the author and consults also some other source about the notions considered here. In Chapter 2 the notion of Lie group is defined and the classical matrix groups are considered. I hope the reader will easily get over the claims that the $n \times n$ hermitian matrices form a *complex* vector space and that the dimension of this space is $\frac{1}{2}n(n+1)$. Chapter 3 introduces the notions of vector bundle, tangent and cotangent bundle, vector bundle homomorphism, section of a vector bundle, vector fields, one-forms and the differential of a function. It should be mentioned that when defining the vector fields as derivations of the algebra of smooth functions on the manifold they are required only to be additive maps and to satisfy the product rule. The requirement of homogeneity with respect to the multiplication with real numbers has been omitted, but one needs it to prove that the directional derivative along a vector field of *every* constant function is zero, which in its turn is needed when proving that every vector field can be localized. In Chapter 4 various algebraic operations with vector bundles are considered. Chapter 5 is on pull-backs of vector bundles, pull-back of differential forms and push-forward of tangent vectors, invariant forms and vector fields on Lie groups and the exponential map on some matrix groups. The reader better examines carefully and corrects the claims in the section about $\mathrm{GL}(n, \mathbb{C})$ and its subgroups. The chapter concludes with a section about immersions, submersions and transversality. Chapter 6 is on complex vector bundles, covering what in the previous three chapters has been done for real vector bundles. It contains also many examples. Chapter 7 is a short chapter on metrics on vector bundles, including also the definition of the notion of Riemannian metric. Chapters 8 and 9 are on geodesics, the exponential map and Gaussian coordinates (i.e., geodesic normal coordinates) on a Riemannian manifold. The geodesics are

not defined through the Levi-Civita connection but through their equations in local coordinates. The exponential map is defined only for geodesically complete manifolds but later in the book it is also used for non-complete manifolds without any comment. The proofs in these chapters are neither flawless nor very clearly presented. For example, the existence of Gaussian coordinates is proved using the local length-minimizing property of geodesics and the local length-minimizing property of geodesics is proved by using Gaussian coordinates. Chapter 10 is on principal bundles, associated vector bundles and quotients of Lie groups by subgroups. In Chapter 11 the central notions of covariant derivative on a vector bundle and connection on a principal bundle are defined and the relation between them is explained. The covariant derivative of a section $s$ of a vector bundle $E$ over a manifold $M$ is defined to be a section $\nabla s$ of $E \otimes T^*M$ and the author carefully avoids the use of covariant derivatives of sections with respect to given vectors or vector fields. In fact, expressions of the type $\nabla_u s$ appear only once in the whole book. For example, when in Chapter 16 the Jacobi equation along a geodesic $\gamma$ is considered, the author prefers to introduce the notation $\nabla_t$ for the covariant derivative along $\gamma$ instead of using $\nabla_{\dot{\gamma}(t)}$. It should be also noted that the fundamental lemma of differential geometry (Lemma 11.1) that a map that sends sections of one bundle $E$ over a manifold $M$ to sections of another bundle $E'$ over $M$ and is $C^\infty(M)$-linear is represented by a section of the bundle of homomorphisms from $E$ to $E'$ is neither properly formulated nor properly proved. Another remark is that in the whole book each time a computation involving the Lie algebra of a Lie group is done the group is assumed to be a matrix group. A general warning to the reader about that fact would have been very useful. Chapter 12 introduces the notions of exterior derivative, de Rham cohomology, Lie derivative, exterior covariant derivative, curvature of a covariant derivative and curvature of a connection. The definition of the exterior derivative is by requiring weaker properties than what is usually done. This demands much more work to show that the exterior derivative is indeed defined on all differential forms and that it is local. No word is said about that though and in fact the author silently assumes the local character of the exterior derivative. There are similar problems with the definition of the exterior covariant derivative. On the other hand, the Lie derivative is defined only for compact manifolds. The reason is not very clear, it is probably because the local one-parameter group of a vector field has not been defined, although the global one-parameter group has not been explained properly either. The curvature of a covariant derivative is defined as the square of the exterior covariant derivative. The classical formula expressing the curvature through commutators of covariant derivatives with respect to vector fields is left to the reader to prove and is almost never used later in the book. Chapter 13 includes the Frobenius theorem and the classification theorem for principal

bundles with flat connections over a given manifold. The notion of holonomy of a connection is also considered here. The holonomy group is not defined. The holonomy is viewed as an assignment to a smooth map from the unit circle to the manifold of an element of the structure group of the principal bundle. Unfortunately, this definition does not possess the properties needed for the proof of the above mentioned classification theorem, for example the concatenation property. Instead, one needed to define holonomy for piecewise smooth loops. The chapter has also an Appendix about smoothing of continuous maps. In Chapter 14 the Bianchi identity is proved and characteristic classes are introduced. There are also many examples where the Chern classes of complex vector bundles are computed. It is somewhat confusing that the author takes the functions that define the characteristic classes to be either smooth or real analytic. In fact, in order that the things work one needs to take them holomorphic. Similarly, in the Appendix on the ad-invariant functions on the space of $n \times n$ complex matrices all the functions are to be taken to be complex polynomials. There are also appendices on integration on manifolds and on the degree of a map. The integration is defined only for smooth forms on compact manifolds but later the author feels free to integrate on open sets (even forms without compact support), on compact sets, on sets that are neither open nor compact and the forms integrated are sometimes only continuous. The proof of the proposition about the degree of a map needs also to be corrected at numerous places. Chapter 15 defines the notions of metric compatible connection, the torsion of a connection, the Levi-Civita connection of a Riemannian manifold, its curvature tensor, the Ricci tensor and the scalar curvature. A peculiar feature of the exposition is that when defining a connection on the tensor bundles over a manifold the basic bundle to start from is the cotangent bundle and not the tangent bundle. For example, the torsion is defined for a covariant derivative on the cotangent bundle and the classical formula expressing the torsion through the covariant derivative of vector fields as $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ never appears. (The torsion in the book corresponds to that expression taken with minus sign). Similarly, the Levi-Civita connection is defined by a covariant derivative on the cotangent bundle and the classical Koszul formula is not given. Another extravagant move in this chapter is that in addition to the exterior covariant derivative $d_\nabla$, for a covariant derivative $\nabla$ on the cotangent bundle the author denotes in the same way the operator which is up to a sign $\nabla$ followed by antisymmetrization. And even more, at least twice later in the book these two operators are used in the same sentence without any comment. In Chapter 16 examples of manifolds with constant sectional curvature (positive, negative or zero) are given, the Gauss-Bonnet theorem for compact oriented surfaces is proved (although the genus of a surface

is never clearly defined in the book) and conformal changes of a Riemannian metric and the corresponding changes of the curvature tensor, the Ricci tensor and the scalar curvature are described. Also, the theorem that any metric on a compact oriented surface can be transformed via a diffeomorphism and a conformal change into a constant curvature metric is formulated and a short review is made of the proved by Perelman Geometrization conjecture of Thurston for compact three-dimensional manifolds. Further, the Jacobi equation is considered and Jacobi vector fields along a geodesic are defined, and the theorem that the universal cover of a manifold with nonpositive sectional curvature is diffeomorphic to a euclidean space and the classification theorem for the universal cover of a manifold with constant sectional curvature are proved. In these last two results the requirement that the manifold is complete and connected is forgotten which is probably due to the fact that their proofs are left incomplete. Finally, it is shown that the bi-invariant metric on a simple compact matrix Lie group is Einstein. Chapter 17 is on almost complex, complex and Kähler manifolds. The Newlander-Nirenberg theorem is also formulated. It is proved in an Appendix that for each non-degenerate two-form on an even dimensional manifold there exists a compatible almost complex structure. An unlucky choice of conventions leads to the situation that if one starts with an almost complex structure $j$ and a hermitian metric and constructs the corresponding Kähler form $\omega$, then not $j$ but rather $-j$ is compatible with $\omega$. Chapter 18 is on complex submanifolds of a complex manifold, holomorphic vector bundles over complex manifolds and holomorphic sections of holomorphic vector bundles. In the short last Chapter 19 the Hodge star is defined and the Hodge decomposition theorem and its corollary that de Rham cohomology classes are represented by harmonic forms are formulated. The notions of self-dual and anti-self-dual form and the signature of a manifold are also introduced.

Now it is time for some critical remarks. I have mentioned some already but now I will put it straightforward: I have never seen a book with such a huge number of mistakes. As an illustration, there are about $505$ formulae that have the honour to be on a separate line and in about $159$ of them there is something wrong. It is clear that mistakes cannot be avoided completely but what happens here exceeds any good measure. Many are misprints but a good deal are not. Of course, a very careful reader will be able to repair himself a considerable amount of them, some immediately and some after reading several sentences or paragraphs and returning back. But looking back to recall a formula is a risky undertaking. Another thing to remark is that the author's attempt to make the presentation self-contained has gone totally wrong. Here are three examples. In the first chapter the inverse function theorem, the implicit function theorem and the contraction mapping theorem

are recalled. Of them only the proof of the contraction mapping theorem is complete. But this theorem is formulated only for closed balls in a euclidean space and nevertheless later used for the space of continuous maps defined on a compact interval with the supremum norm. In Chapter 8 the so called vector field theorem is stated and a proof is given in an Appendix. This is the local existence and uniqueness theorem for autonomous systems of ordinary differential equations. The smooth dependence of the solution on the initial data is also formulated but in fact not proved. Later this vector field theorem is used as a universal reference, without any additional comment, about everything related to ordinary differential equations: local existence and uniqueness of solutions of non-autonomous systems, global existence and uniqueness of solutions of linear systems, smooth dependence of the solutions of a system on parameters that are not the initial data. In the Appendix of Chapter 14 about integration on manifolds the author takes on the ambitious task to present on two pages the definition of the integral of a compactly supported function on $\mathbb{R}^n$ and its properties with proofs. The integral is defined for smooth functions, although already in the previous chapter continuous functions have been integrated in order to be smoothed. The section on the integral not only comes some 30 pages later on but also important and future needed properties are not mentioned, for example the computation of the integral by reducing it to multiple one-dimensional integrals (Fubini's theorem). In my opinion it would have been much better to drop the proofs of the background material and instead of that concentrate on giving a precise and thorough formulation of the facts needed. Another thing one notices is that recalling something considered earlier in the book often comes with some change of notations. And indices jump everywhere, up and down, left and right. I cannot help longing for the classical conventions, where the positions of the indices are instructive of what is going on. All said above makes one think that the author has written the book in one breath and never looked back.

In conclusion, I think that the big number of mistakes makes the book inappropriate for the majority of its targeted audience of beginners in the field of differential geometry. It would be very useful though for those of them that are ready to take their time and work through, writing down in the process their own, corrected version. But my impression is that nowadays the young people inclined to do that are rather rare.

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