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FRAMING CURVES IN EUCLIDEAN AND MINKOWSKI SPACE

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Abstract. We give a unified picture of the Frenet-Serret equations in Euclidean space and their analogues in Minkowski space that provides further insight into how and why the Minkowski versions differ from the Euclidean.

1. Introduction

The Frenet apparatus for a curve in Euclidean space $\mathbb{E}^3$ whose curvature vanishes nowhere is a standard part of the undergraduate introduction to differential geometry, and a description can be found in any introductory text, such as that of Pressley [3]. The usual naming convention for this orthonormal triad is $\{T, N, B\}$, where T is (proportional to) the tangent vector, N is proportional to the derivative of T, and B completes a right-handed orthonormal basis. These are related by the Frenet-Serret equations

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Analogous framings can be found in the three dimensional Minkowski space $\mathbb{M}^{1,2}$, with signature $(-,+,+)$ and inner product $\langle .\, , . \rangle$ for curves which are everywhere timelike, everywhere spacelike, or everywhere null. The equations describing these framings can be derived in a similar way once a cross product has been defined and are similar to those of the Frenet-Serret equations frame, but with some changes of sign arising from the indefinite nature of the metric, as in Lopez [2].

For example, for a timelike curve, one obtains

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where the sign of κ in the second row changes because of the indefinite inner product. For curves of other causal characters there is a different pattern of signs, which can be found by an explicit calculation similar to that in Pressley [3].