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TOPOLOGY, GEOMETRY AND PHYSICS: BACKGROUND FOR THE WITTEN CONJECTURE II.

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Abstract. The profound, beautiful and, at times, rather mysterious symbiosis between mathematics and physics has always been a source of wonder, but, in the past twenty years, the intensity of the mutual interaction between these two has become nothing short of startling. Our objective here is to provide an introduction, in terms as elementary as possible, to one small aspect of this relationship. Toward this end we shall tell a story. Although we make no attempt to relate it chronologically, the story can be said to begin with the efforts of Yang and Mills to construct a nonabelian generalization of classical electromagnetic theory, and to culminate in a remarkable conjecture of Witten concerning the Donaldson invariants of a smooth four-manifold.

6. Equivariant Localization

Motivated by our discussion of the Witten Lagrangian, its equivariant symmetries, and the localization of the corresponding partition function to the anti-self-dual moduli space to yield the Donaldson invariant, we return now to the finite-dimensional context and consider the general phenomenon of "equivariant localization." In order to provide a relatively complete treatment and because it describes the finite-dimensional analogue of Witten's partition function (the zero-dimensional Donaldson invariant) we will restrict our attention to the simplest ("discrete") equivariant localization theorem.

We consider a compact, oriented, smooth manifold M of dimension $n = 2k$ and denote by $\boldsymbol{\nu}$ a volume form on M. Suppose $H : M \to \mathbb{R}$ is a Morse function on M, i.e., a smooth function whose critical points p $(\mathrm{d}H(p) = 0)$ are all nondegenerate (this means that the Hessian $\mathcal{H}_p : T_p(M) \times T_p(M) \to \mathbb{R}$, defined by $\mathcal{H}_p(V_p, W_p) = V_p(W(H))$, where $V_p, W_p \in T_p(M)$ and W is a vector field on M with $W(p) = W_p$, is a nondegenerate bilinear form). Finally, let T denote some real parameter. We consider the integral