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ON THE COMPOSITION OF FINITE ROTATIONS IN $\mathbb{E}^4$

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Abstract. We achieve compositions rules for the geometric parameters of the composed rotations, which is in a certain sense analogous to the well known Rodrigues formula. We also obtain a necessary and sufficient condition for a composition of two simple rotations in $\mathbb{E}^4$ to be a simple rotation.

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1. Introduction

In this note we are studying compositions of finite rotations in the Euclidean space $\mathbb{E}^4$, the four dimensional real vector space with the standard scalar product.

There are several significant differences between rotations in $\mathbb{E}^4$ and in the three dimensional space. The group of rotations in $\mathbb{E}^3$ essentially comprises one type, that is, a rotation about an axis, while in $\mathbb{E}^4$ there are three types of rotations: i) *A simple rotation*, it leaves a plane (two dimensional subspace of $\mathbb{E}^4$) point-wise fixed and induces a two dimensional rotation in the orthogonal plane; ii) *A Clifford translation (an isoclinic rotation)*, here each vector in $\mathbb{E}^4$ turns through the same angle; iii) *A double rotation*, here $\mathbb{E}^4$ is decomposed into two orthogonal planes and points in the first plane rotate through an angle α, while points in the second plane rotate through an angle, $\beta \neq \alpha$.

Here we derive two properties of the compositions of rotations in $\mathbb{E}^4$. The first one gives analytical and geometrical characterizations of the subgroup of simple rotations. The second result deals with compositions formulas for double rotations, that is, we obtained formulas that enable the calculations of the orthogonal planes and angles of the composed rotation in terms of the corresponding characteristics of each one of the rotations in $\mathbb{E}^4$.

Our main tool is the quaternionic representation

$$x \mapsto axb \tag{1}$$

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