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ELASTIC BENDING ENERGY: A VARIATIONAL APPROACH

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Abstract. Geometric continuum models for fluid lipid membranes are considered using classical field theory, within a covariant variational approach. The approach is cast as a higher-derivative Lagrangian formulation of continuum classical field theory, and it can be seen as a covariant version of the field theoretical variational approach that uses the height representation. This novel Lagrangian formulation is presented first for a generic reparametrization invariant geometric model, deriving its equilibrium equation, or shape equation, and its linear and angular stress tensors, using the classical Canham-Helfrich elastic bending energy for illustration. The robustness of the formulation is established by extending it to the presence of external forces, and to the case of heterogenous lipid membranes, breaking reparametrization invariance. In addition, a useful and compact general expression for the second variation of the free energy is obtained within the Lagrangian formulation, as a first step towards the study of the stability of membrane configurations. The simple structure of the expressions derived for the basic entities that appear in the mechanics of a lipid membrane is a direct consequence of the well established power of a Lagrangian variational approach. The paper is self-contained, and it is meant to provide, besides a new framework, also a convenient introduction to the mechanics of lipid membranes.

MSC: 74B20, 35A15, 70S05
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