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BOOK REVIEW

Cycle Spaces of Flag Domains – A Complex Geometric Viewpoint, by Gregor Fels, Alan Huckleberry and Joseph A. Wolf, Birkhäuser, Boston · Basel · Berlin, 2006, xx + 329 pp., ISBN 0-8176-4391-5.

The research monograph under review constructs cycle spaces of complex flag domains and studies their complex-analytic properties. The results are expected to through a bridge between the representation theory of semisimple Lie groups and the variations of Hodge structure of compact complex manifolds. Throughout the book, G is a complex simple Lie group and Q is a parabolic subgroup of G. Then the projective algebraic manifold $Z = G/Q$ is called a flag manifold. If G_0 is a real form of G then any open G_0-orbit $D = G_0(z_0)$ of the base point $z_0 \in Z$ is referred to as a flag domain. For any maximal compact subgroup K_0 of G_0, there is a unique K_0-orbit C_0 of D, which is a complex submanifold of Z and C_0 is known as the base cycle of D. Let us denote by $\mathcal{C}_q(X)$ the space of the q-dimensional cycles of a complex space X. Then $\mathcal{C}_q(Z)$ is a locally finite-dimensional complex space with compact projective irreducible components. In the case of $q := \dim_{\mathbb{C}} C_0$ consider the connected component $\mathcal{C}$ of $\mathcal{C}_q(D)$, containing C_0, and its subspace $\mathcal{M}_D$, defined as the connected component of $G(C_0) \cap \mathcal{C}_q(D)$ through C_0. Let K be the complexification of K_0 and $\Omega = G/K$ be the corresponding affine homogeneous space. Except for some noncompact Hermitian symmetric D, the cycle space $\mathcal{M}_D$ turns to be isomorphic to the maximal, G_0-invariant, Kobayashi-hyperbolic, Stein domain in Ω through the base point $\check{o}_\Omega$, corresponding to C_0. If D is a bounded symmetric domain then $\mathcal{M}_D$ coincides with D, its complex conjugate $\overline{D}$ or their Cartesian product $D \times \overline{D}$. The monograph establishes that $\mathcal{C}$ is smooth and provides a complete representation-theoretic description of the Zariski tangent space $T_{[C_0]}\mathcal{C}$, viewed as a K-module. For an arbitrary G_0-orbit γ on Z, the authors define a cycle space $\mathcal{C}(\gamma)$, realized as an open subset of a G-orbit on $\mathcal{C}_q(Z)$. In particular, $\mathcal{C}(D) = \mathcal{M}_D$ for the open G_0-orbit D on Z. It turns out that except for some well understood hermitian symmetric cases, $\mathcal{C}(\gamma)$ are isomorphic to some explicitly constructed universal domains $\mathcal{U}$, introduced by Akhiezer and Gindikin in 1990. More precisely, let $G_0 = K_0A_0N_0$ be the Iwasawa decomposition, $\Sigma(\mathfrak{g}(G_0), \mathfrak{g}(A_0))$ be the set of the roots of the Lie algebra $\mathfrak{g}(G_0)$ with respect to its Cartan subalgebra $\mathfrak{g}(A_0)$ and ω_0 be the connected component of $0 \in \mathfrak{g}(A_0)$ in the polyhedron $\left\{\xi \in \mathfrak{g}(A_0)\,;\, \alpha(\xi) \leq \frac{\pi}{2} \text{ for all } \alpha \in \Sigma(\mathfrak{g}(G_0), \mathfrak{g}(A_0))\right\}$. Then $\mathcal{U} := G_0 \exp(\mathrm{i}\omega_0)(z_0)$. Towards a differential-geometric description of $\mathcal{U}$, let us recall that any geodesic $c : \mathbb{R} \to G_0/K_0$ of the Riemannian symmetric space G_0/K_0 induces a map $\mathrm{d}c : T\mathbb{R} \to T\,(G_0/K_0)$ of the corresponding tangent bundles. An integrable complex structure on a domain in $T\,(G_0/K_0)$ is said to be adapted if under the identification $T\mathbb{R} = \mathbb{C}$, the maps $\mathrm{d}c : T\mathbb{R} \to T\,(G_0/K_0)$ are holomorphic for all (G_0/K_0)-geodesics c. The universal domain $\mathcal{U}$ is shown to be biholomorphic to the maximal domain of existence for the adapted complex structure on $T\,(G_0/K_0)$, which is invariant under scalar multiplications. The cycle space $\mathcal{M}_D$ is described also by the means of the Schubert incidence geometry. More precisely, let us say that B is an Iwasawa-Borel subgroup of G if B is a Borel subgroup of G and there is an Iwasawa decomposition $G_0 = K_0A_0N_0$, such that the solvable group A_0N_0 is contained in B. The closure of a B-orbit $\mathcal{O}_B$ in Z is called a Schubert variety $S = \mathrm{cl}(\mathcal{O}_B)$. For any cycle $C \in \mathcal{M}_D$ there is a Schubert variety S, intersecting C. Moreover, $S \cap C$ is a finite subset of $\mathcal{O}_B$. For any Iwasawa-Borel subgroup $B \subset G$ there is a Schubert variety $S = \mathrm{cl}(\mathcal{O}_B)$, such that $Y_S := \mathrm{cl}(\mathcal{O}_B) \setminus \mathcal{O}_B$ is associated with a B-invariant incidence variety $I_{Y_S} := \{C \in \mathcal{C}_q(Z)\,;\, C \cap Y_S \neq \emptyset\} \subseteq \mathcal{C}_q(Z) \setminus \mathcal{M}_D$ with unique lifting to a hypersurface H_S in $\Omega = G/K$. The envelope $\mathcal{E}_{H_S}(\mathcal{M}_D)$ of $\mathcal{M}_D$ is defined as the connected component of $\Omega \setminus (\cup_{g \in G_0} g(H_S))$ through the base point $\check{o}_\Omega \in \Omega$. If $q := \dim_{\mathbb{C}} C_0$ then the connected component of the base point in the intersection of the envelopes $\mathcal{E}_{H_S}(\mathcal{M}_D)$ for all the q-codimensional Schubert varieties S with $S \cap C_0 \neq \emptyset$ is called the Schubert domain and denoted by S_D. The cycle space $\mathcal{M}_D$ is shown to coincide with the Schubert domain S_D. The book studies also the so called double fibration transform $\mathcal{P}$, which is a map from the q-cohomology of a homogeneous holomorphic vector bundle $\mathbb{E}$ on D to a space of sections of a canonically derived homogeneous holomorphic vector bundle $\mathbb{E}'$ on $\mathcal{M}_D$. It establishes the injectivity of $\mathcal{P}$ for sufficiently negative $\mathbb{E}$. The final chapters of the book describe the K-module structure of the Zariski tangent space $T_{[C]}\mathcal{C}(Z)$ at a closed K-orbit C on Z, after decomposing into K-submodules $T_{[C]}\mathcal{C}(Z) = T_{[C]}G([G]) \oplus V$.

The book is organized as follows. Most of the first chapter is devoted to the background on the structure of the semisimple Lie groups and Lie algebras, as well as on the theory of their finite-dimensional representations. *Chapter 2* recalls the Bruhat decomposition and applies it to the study of the isotropy subgroups of

G_0 on Z. As a result, one gets a finite bound on the number of the G_0-orbits on Z, the existence of an open G_0-orbit D on Z, a codimension formula for a G_0-orbit on Z and others. The third chapter deals with the orbit structure of a Hermitian real form G_0 on its dual Hermitian symmetric space Z of compact type. *Chapter 4* comprises some preliminaries on the cycle spaces $\mathcal{M}_D$ of the open G_0-orbits D on Z. Namely, the closed K-orbits on Z are shown to be contained in a unique open G_0-orbit D on Z where they are the unique complex K_0-orbits on D. That suggests the study of the cycle spaces on D or Z. All holomorphic functions on a flag domain turn to be pulled-back from its subordinate bounded symmetric domain. A flag domain D, which admits an invariant pseudo-Kähler metric is said to be measurable. Any measurable flag domain D is proved to support a canonical exhaustion function, measuring the holomorphic convexity or concavity of D. As a result are derived explicit cohomology vanishing theorems for holomorphic vector bundles on measurable flag domains D. *Chapter 5* starts with the definition of the cycle space $\mathcal{M}_D$. The canonical exhaustion function of a measurable flag domain D is shown to be lifted to a strictly plurisubharmonic exhaustion function on $\mathcal{M}_D$. Thus, $\mathcal{M}_D$ turns to be a Stein manifold. For an arbitrary flag domain D of non-Hermitian type, the same result is derived in *Chapter 11* by identifying $\mathcal{M}_D$ with the universal domain $\mathcal{U}$. The fifth chapter establishes also that the cycle space $\mathcal{M}_D$ of a noncompact Hermitian symmetric space D coincides with D, $\overline{D}$ or with the Cartesian product $D \times \overline{D}$ where $\overline{D}$ is the complex conjugate of D.

Substantial amount of the text is devoted to the characterization of the so called universal domain $\mathcal{U}$. The arguments rely both on complex-analytic and group theoretic techniques. After recalling Akhiezer-Gindikin's definition of a universal domain $\mathcal{U} := G_0 \exp(\mathrm{i}\omega_0)(z_0)$, *Chapter 6* establishes that $\mathcal{U}$ is biholomorphic to the maximal domain of existence for the adapted complex structure on $T(G_0/K_0)$, which is invariant under scalar multiplications. Then a G_0-invariant function on $\mathcal{U}$ is proved to be strictly plurisubharmonic if and only if its pull-back to the restricted root polytope ω_0 is strictly convex. This is established by a detailed analysis of the Cauchy-Riemann structure of the G_0-orbits on $\mathcal{U}$ and of the Levi form of the G_0-invariant hypersurfaces in $\mathcal{U}$. *Chapter 7* introduces the Iwasawa-Borel subgroups of G, the Schubert varieties $S = \mathrm{cl}(\mathcal{O}_B)$ of codimension $q := \dim_{\mathbb{C}} C_0$ and the incidence varieties I_{Y_S} with $Y_S = S \setminus \mathcal{O}_B$. For any S with $S \cap C_0 \neq \emptyset$ the intersection $I_{Y_S} \cap \mathcal{M}_Z$ is shown to be a B-invariant analytic hypersurface in the complement of $\mathcal{M}_D$ in $\mathcal{M}_Z$. The Schubert slices Σ, associated with the Schubert varieties S of codimension $q = \dim_{\mathbb{C}} C_0$ with $S \cap C_0 \neq \emptyset$, are defined as the connected components of $S \cap D$. The intersection $\Sigma \cap C_0$ is shown to be transversal and consisting of a single point z, such that

$\Sigma = A_0N_0(z)$. *Chapter 8* concentrates on Mitsuki's duality between the K-orbits κ on Z and the G_0-orbits γ on Z. Namely, κ and γ correspond to each other if and only if $\kappa \cap \gamma$ is a K_0-orbit. More precisely, if G_u is a compact real form of G then an arbitrary G_u-invariant Kähler form on Z gives rise to a K_0-moment map $\mu_{K_0} : Z \to (\mathfrak{g}(K_0))^*$. Consider the energy function $E := ||\mu_{K_0}||$, computed with respect to a K_0-invariant Killing form, and its gradient field ∇E with respect to the associated Kähler metric. The critical set $\mathrm{Crit}_Z = \{z \in Z\,;\,(\nabla E)(z) = 0\}$ turns to be a finite union of K_0-orbits. The basic duality theorem shows the presence of bijective correspondences among the sets of the K-orbits and G_0-orbits on Z, as well as the set of the K_0-orbits on Crit_Z. Namely, for any K_0-orbit $\kappa_0 \subset \mathrm{Crit}_Z$ the orbits $\kappa = K(z)$ and $\gamma = G_0(z)$ turn to be independent on the choice of $z \in \kappa_0$. Moreover, the flow of ∇E, respectively, $-\nabla E$ realizes κ_0 as a strong deformation retract of γ, respectively, κ. *Chapter 9* establishes that through any boundary point $z \in \partial D$ of an open G_0-orbit D on Z there is a $(q+1)$-dimensional Schubert variety, contained in the complement $Z \setminus D$. That suffices for the coincidence of $\mathcal{M}_D$ with D, $\overline{D}$ or, respectively, $D \times \overline{D}$, in the case of a Hermitian symmetric D. If G_0 is not of Hermitian type, the cycle space $\mathcal{M}_D$ coincides with the Schubert domain S_D, so that $\mathcal{M}_D$ turns to be a Stein domain. The considerations from *Chapter 10* prepare the results of the next *Chapter 11*. This concerns, primarily, the invariant theory of the G_0-action on the boundary $\partial\mathcal{U}$ of the universal domain $\mathcal{U}$. It includes a characterization of the closed orbits and a limiting procedure for attaining a point from the closed orbit in the closure of any non-closed orbit. A boundary point $z \in \partial\mathcal{U}$ is generic if the closed orbit in the closure of $G_0(z)$ is of the form $G_0(\exp(i\xi))$ with ξ from some maximal dimensional face of the polyhedron ω_0. The set $\partial^{\mathrm{gen}}\mathcal{U}$ of the generic boundary points turns to be open and dense in $\partial\mathcal{U}$, as far as its complement $\partial\mathcal{U} \setminus \partial^{\mathrm{gen}}\mathcal{U}$ is of codimension ≥ 1. For any $z \in \partial^{\mathrm{gen}}\mathcal{U}$ from a non-closed G_0-orbit on $\partial\mathcal{U}$, the authors construct a semisimple three-dimensional subgroup S of G_0, such that the orbit $S(z)$ closes up in a controlled way to a point from the closed orbit in the closure $\mathrm{cl}(G_0(z))$. *Chapter 11* concludes the proof of $\mathcal{M}_D \simeq \mathcal{U}$ for a non-Hermitian symmetric D, by showing that $\mathcal{U}$ is the unique maximal, Kobayashi hyperbolic, G_0-invariant Stein domain in G/K, up to the choice of a base point. The next *Chapter 12* discusses the cycle spaces $C\{\gamma\}$ of arbitrary (not necessarily open) G_0-orbits $\gamma \subset Z$. In order to define $C\{\gamma\}$, let us consider the dual K-orbit κ of γ. If $q := \dim_{\mathbb{C}} \kappa$ then the closure $\mathrm{cl}(\kappa)$ of κ in Z belongs to $\mathcal{C}_q(Z)$ together with $g(\mathrm{cl}(\kappa))$ for all $g \in G$. The cycle space $C\{\gamma\}$ is the connected component of $\mathrm{cl}(\kappa)$ in the set of $g(\mathrm{cl}(\kappa)) \in \mathcal{C}_q(Z)$, which intersects γ in a smooth compact submanifold $g(\mathrm{cl}(\kappa)) \cap \gamma$ of γ. The complete description of $C\{\gamma\}$ makes use of the fact that for Schubert varieties S with $Y_S \subset (Z \setminus \gamma)$, the incidence varieties I_{Y_S} are contained in the complement of $C\{\gamma\}$. *Chapter 13* discusses several specific examples. The universal domains for $\mathrm{SU}(p,q)$, $\mathrm{Sp}(p,q)$ and $\mathrm{SO}(p,q)$ are endowed with specific matrix realizations.

Chapter 14 introduces the double fibration transform

$$\mathcal{P} : H^q(D, \mathcal{O}_D(\mathbb{E})) \longrightarrow H^0(\mathcal{M}_D, \mathcal{O}_{\mathcal{M}_D}(\mathbb{E}'))$$

where $\mathcal{O}_D(\mathbb{E})$ is the sheaf of germs of the holomorphic sections of a homogeneous holomorphic vector bundle $\mathbb{E} \to D$ and $\mathbb{E}' \to \mathcal{M}_D$ is a derived homogeneous holomorphic vector bundle. For sufficiently negative $\mathbb{E} \to D$ is derived the injectivity of $\mathcal{P}$, making use of the topological triviality of the Schubert fibration $\mathcal{M}_D \to \Sigma$, defined by intersecting cycles with a Schubert slice $\Sigma = A_0 N_0(z_0)$. The next *Chapter 15* reviews the theory of the moduli of the integrable complex structures on a fixed compact orientable manifold. It emphasizes on Griffiths' embeddings of these moduli spaces in flag domains D, called variations of Hodge structure. The images of the non-surjective variations of Hodge structure are transversal to the cycles C on D. In fact, D is a homogeneous space $D = G_0/L_0$ for a compact subgroup $L_0 \subset G_0$, so that any maximal compact subgroup $K_0 \subset G_0$, containing L_0, determines a homogeneous fibration $\varphi : G_0/L_0 \to G_0/K_0$. The fibers of φ are the cycles on D and the base G_0/K_0 is a riemannian symmetric space. *Chapter 16* outlines the fact that the moduli space of the marked $K3$-surfaces is an open $\mathrm{SO}(3,19)$-orbit in the 20-dimensional quadric $Z = G/Q$ in $\mathbb{P}^{21}$, defined by the complex bilinear form on $\mathbb{C}^{22}$, induced from a hermitian form of signature $(3,19)$. The cycles on Z are quadratic curves, obtained by intersection of Z with a projective hyperplane in $\mathbb{P}^{21}$. The cycles defined over $\mathbb{R}$ correspond to the Ricci-flat Calabi-Yau metrics on the $K3$-surface.

The last four chapters study the K-module structure of the Zariski tangent space $T_{[C]}\mathcal{C}(Z) = T_{[C]}G([C]) \oplus V$. Since $\mathcal{C}(Z)$ is shown to be smooth at $[C]$, the non-zero transversal complement V to $T_{[C]}G([C])$ represent additional deformation parameters, which are not due to the symmetry group G. The book calculates explicitly the highest weights of the irreducible sub-representations for the K-action on V. Let $\mathfrak{g}(G) = \mathfrak{g}(K) + \mathfrak{s}$ be the Cartan decomposition of $\mathfrak{g}(G)$ and $\theta : \mathfrak{g}(G) \to \mathfrak{g}(G)$ be the Cartan involution. *Chapter 17* reduces the problem to the calculation of certain cohomology groups of the homogeneous holomorphic vector bundle $\mathbb{E}$ on C, associated with the $(K \cap Q)$-module $F := (\mathfrak{g}(Q) + \theta\mathfrak{g}(Q)) \cap \mathfrak{s}$. Since the group $K \cap Q$ is not reductive, in *Chapter 18* the authors introduce a filtration $\{F^j\}$ of F, defined by the unipotent radical of $K \cap Q$. The quotients F^{j+1}/F^j are acted by the reductive part of $K \cap Q$, so that the cohomologies of their corresponding homogeneous holomorphic vector bundles can be computed by the algorithm of Borel-Bott-Weil Theorem. This works out for obtaining the cohomologies of the original $\mathbb{E} \to C$. The elegant algorithm for the computation of the K-module structure of V, developed in *Chapter 18*, is applied in the last two chapters towards specific tables in the case of $\mathrm{rank}(\mathfrak{g}(K)) < \mathrm{rank}(\mathfrak{g}(G))$, respectively, $\mathrm{rank}(\mathfrak{g}(K)) = \mathrm{rank}(\mathfrak{g}(G))$.

The book is recommended to the researchers and graduate students in the theory of the finite-dimensional representations of Lie groups and Lie algebras, as well as to those, working on moduli problems and variations of Hodge structure of compact complex manifolds.

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