Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 2, Article 22, 2004

ON BELONGING OF TRIGONOMETRIC SERIES TO ORLICZ SPACE

S. TIKHONOV

DEPARTMENT OF MECHANICS AND MATHEMATICS
MOSCOW STATE UNIVERSITY
VOROB'YOVY GORY, MOSCOW 119899 RUSSIA.
tikhonov@mccme.ru

Received 18 July, 2003; accepted 27 December, 2003
Communicated by A. Babenko

ABSTRACT. In this paper we consider trigonometric series with the coefficients from R_0^+BVS class. We prove the theorems on belonging to these series to Orlicz space.

Key words and phrases: Trigonometric series, Integrability, Orlicz space.

2000 *Mathematics Subject Classification.* 42A32, 46E30.

1. INTRODUCTION

We will study the problems of integrability of formal sine and cosine series

$$g(x) = \sum_{n=1}^{\infty} \lambda_n \sin nx, \tag{1.1}$$

$$f(x) = \sum_{n=1}^{\infty} \lambda_n \cos nx. \tag{1.2}$$

First, we will rewrite the classical result of Young, Boas and Heywood for series (1.1) and (1.2) with monotone coefficients.

Theorem 1.1 ([1], [2], [11]). *Let* $\lambda_n \downarrow 0$.
If $0 \le \alpha < 2$, *then*

$$\frac{g(x)}{x^\alpha} \in L(0,\pi) \iff \sum_{n=1}^{\infty} n^{\alpha-1}\lambda_n < \infty.$$

ISSN (electronic): 1443-5756

This work was supported by the Russian Foundation for Fundamental Research (grant no. 03-01-00080) and the Leading Scientific Schools (grant no. NSH-1657.2003.1).

002-04

If $0 < \alpha < 1$, *then*

$$\frac{f(x)}{x^{\alpha}} \in L(0,\pi) \Longleftrightarrow \sum_{n=1}^{\infty} n^{\alpha-1}\lambda_n < \infty.$$

Several generalizations of this theorem have been obtained in the following directions: more general weighted functions $\gamma(x)$ have been considered; also, integrability of $g(x)\gamma(x)$ and $f(x)\gamma(x)$ of order p have been examined for different values of p; finally, more general conditions on coefficients $\{\lambda_n\}$ have been considered.

Igari ([3]) obtained the generalization of Boas-Heywood's results. The author used the notation of a slowly oscillating function.

A positive measurable function $S(t)$ defined on $[D;+\infty)$, $D>0$ is said to be slowly oscillating if $\lim_{t\to\infty} \frac{S(\varkappa t)}{S(t)} = 1$ holds for all $x > 0$.

Theorem 1.2 ([3]). *Let* $\lambda_n \downarrow 0$, $p \geq 1$, *and let* $S(t)$ *be a slowly oscillating function.*
If $-1 < \theta < 1$, *then*

$$\frac{g^p(x) S\left(\frac{1}{x}\right)}{x^{p\theta+1}} \in L(0,\pi) \Longleftrightarrow \sum_{n=1}^{\infty} n^{p\theta+p-1} S(n)\lambda_n^p < \infty.$$

If $-1 < \theta < 0$, *then*

$$\frac{f^p(x) S\left(\frac{1}{x}\right)}{x^{p\theta+1}} \in L(0,\pi) \Longleftrightarrow \sum_{n=1}^{\infty} n^{p\theta+p-1} S(n)\lambda_n^p < \infty.$$

Vukolova and Dyachenko in [10], considering the Hardy-Littlewood type theorem found the sufficient conditions of belonging of series (1.1) and (1.2) to the classes L_p for $p > 0$.

Theorem 1.3 ([10]). *Let* $\lambda_n \downarrow 0$, *and* $p > 0$. *Then*

$$\sum_{n=1}^{\infty} n^{p-2}\lambda_n^p < \infty \Longrightarrow \psi(x) \in L^p(0,\pi),$$

where a function $\psi(x)$ *is either a* $f(x)$ *or a* $g(x)$.

In the same work it is shown that the converse result does not hold for cosine series.

Leindler ([5]) introduced the following definition. A sequence $\mathbf{c} := \{c_n\}$ of positive numbers tending to zero is of rest bounded variation, or briefly R_0^+BVS, if it possesses the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(\mathbf{c})\, c_m$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$. In [5] it was shown that the class R_0^+BVS was not comparable to the class of quasi-monotone sequences, that is, to the class of sequences $\mathbf{c} = \{c_n\}$ such that $n^{-\alpha}c_n \downarrow 0$ for some $\alpha \geq 0$. Also, in [5] it was proved that the series (1.1) and (1.2) are uniformly convergent over $\delta \leq x \leq \pi - \delta$ for any $0 < \delta < \pi$. In the same paper the following was proved.

Theorem 1.4 ([5]). *Let* $\{\lambda_n\} \in R_0^+BVS$, $p \geq 1$, *and* $\frac{1}{p} - 1 < \theta < \frac{1}{p}$. *Then*

$$\frac{\psi^p(x)}{x^{p\theta}} \in L(0,\pi) \Longleftrightarrow \sum_{n=1}^{\infty} n^{p\theta+p-2}\lambda_n^p < \infty,$$

where a function $\psi(x)$ *is either a* $f(x)$ *or a* $g(x)$.

Very recently Nemeth [8] has found the sufficient condition of integrability of series (1.1) with the sequence of coefficients $\{\lambda_n\} \in R_0^+BVS$ and with quite general conditions on a weight function. The author has used the notation of almost monotonic sequences.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (decreasing) if there exists constant $C := C(\gamma) \geq 1$ such that

$$C\gamma_n \geq \gamma_m \quad (\gamma_n \leq C\gamma_m)$$

holds for any $n \geq m$.

Here and further, C, C_i denote positive constants that are not necessarily the same at each occurrence.

Theorem 1.5 ([8])**.** *If $\{\lambda_n\} \in R_0^+BVS$, and the sequence $\gamma := \{\gamma_n\}$ such that $\{\gamma_n n^{-2+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n}\lambda_n < \infty \Longrightarrow \gamma(x)g(x) \in L(0,\pi).$$

Here and in the sequel, a function $\gamma(x)$ is defined by the sequence γ in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n$, $n \in \mathbf{N}$ and there exist positive constants A and B such that $A\gamma_{n+1} \leq \gamma(x) \leq B\gamma_n$ for $x \in (\frac{\pi}{n+1}, \frac{\pi}{n})$.

We will solve the problem of finding of sufficient conditions, for which series (1.1) and (1.2) belong to the weighted Orlicz space $L(\Phi, \gamma)$. In particular, we will obtain sufficient conditions for series (1.1) and (1.2) to belong to weighted space L_γ^p.

Definition 1.1. A locally integrable almost everywhere positive function $\gamma(x) : [0,\pi] \to [0,\infty)$ is said to be a weight function. Let $\Phi(t)$ be a nondecreasing continuous function defined on $[0,\infty)$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = +\infty$. For a weight $\gamma(x)$ the weighted Orlicz space $L(\Phi,\gamma)$ is defined by (see [9], [12])

$$L(\Phi,\gamma) = \left\{ h : \int_0^\pi \gamma(x)\Phi(\varepsilon\,|h(x)|)dx < \infty \quad \text{for some} \quad \varepsilon > 0 \right\}. \tag{1.3}$$

If $\Phi(x) = x^p$ for $1 \leq p < \infty$, when the weighted Orlicz space $L(\Phi,\gamma)$ defined by (1.3) is the usual weighted space $L_\gamma^p(0,\pi)$.

We will denote (see [6]) by $\triangle(p,q)$ $(0 \leq q \leq p)$ the set of all nonnegative functions $\Phi(x)$ defined on $[0,\infty)$ such that $\Phi(0) = 0$ and $\Phi(x)/x^p$ is nonincreasing and $\Phi(x)/x^q$ is nondecreasing. It is clear that $\triangle(p,q) \subset \triangle(p,0)$ $(0 < q \leq p)$. As an example, $\triangle(p,0)$ contains the function $\Phi(x) = \log(1+x)$.

2. RESULTS

The following theorems provide the sufficient conditions of belonging of $f(x)$ and $g(x)$ to Orlicz spaces.

Theorem 2.1. *Let $\Phi(x) \in \triangle(p,0)$ $(0 \leq p)$. If $\{\lambda_n\} \in R_0^+BVS$, and sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}\Phi(n\lambda_n) < \infty \Longrightarrow \psi(x) \in L(\Phi,\gamma),$$

where a function $\psi(x)$ is either a sine or cosine series.

For the sine series it is possible to obtain the sufficient condition of its belonging to Orlicz space with more general conditions on the sequence $\{\gamma_n\}$ but with stronger restrictions on the function $\Phi(x)$.

Theorem 2.2. *Let $\Phi(x) \in \triangle(p,q)$ $(0 \le q \le p)$. If $\{\lambda_n\} \in R_0^+BVS$, and sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-(1+q)+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}} \Phi(n^2\lambda_n) < \infty \Longrightarrow g(x) \in L(\Phi,\gamma).$$

Remark 2.3. If $\Phi(t) = t$, then Theorem 2.2 implies Theorem 1.5, and if $\Phi(t) = t^p$ with $0 < p$ and $\{\gamma_n = 1, n \in \mathbf{N}\}$, then Theorem 2.1 is a generalization of Theorem 1.3. Also, if $\Phi(t) = t^p$ with $1 \le p$ and $\{\gamma_n = n^\alpha S(n), n \in \mathbf{N}\}$ with corresponding conditions on α and $S(n)$, then Theorems 2.1 and 2.2 imply the sufficiency parts ($\Longleftarrow$) of Theorems 1.2 and 1.4.

3. AUXILIARY RESULTS

Lemma 3.1 ([4])**.** *If $a_n \ge 0,\ \lambda_n > 0$, and if $p \ge 1$, then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=1}^{n} a_\nu\right)^p \le C \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{\nu=n}^{\infty} \lambda_\nu\right)^p.$$

Lemma 3.2 ([6])**.** *Let $\Phi \in \triangle(p,q)$ $(0 \le q \le p)$ and $t_j \ge 0$, $j = 1,2,\dots,n, \quad n \in \mathbf{N}$. Then*

(1): $\theta^p \Phi(t) \le \Phi(\theta t) \le \theta^q \Phi(t), \quad 0 \le \theta \le 1,\ t \ge 0,$

(2): $\Phi\left(\sum_{j=1}^{n} t_j\right) \le \left(\sum_{j=1}^{n} \Phi^{\frac{1}{p^*}}(t_j)\right)^{p^*}, \quad p^* = \max(1,p).$

Lemma 3.3. *Let $\Phi \in \triangle(p,q)$ $(0 \le q \le p)$. If $\lambda_n > 0,\ a_n \ge 0$, and if there exists a constant K such that $a_{\nu+j} \le K a_\nu$ holds for all $j,\nu \in \mathbf{N}, j \le \nu$, then*

$$\sum_{k=1}^{\infty} \lambda_k \Phi\left(\sum_{\nu=1}^{k} a_\nu\right) \le C \sum_{k=1}^{\infty} \Phi(k a_k) \lambda_k \left(\frac{\sum_{\nu=k}^{\infty} \lambda_\nu}{k\lambda_k}\right)^{p^*},$$

where $p^* = \max(1,p)$.

Proof. Let ξ be an integer such that $2^\xi \le k < 2^{\xi+1}$. Then

$$\sum_{\nu=1}^{k} a_\nu \le \sum_{m=0}^{\xi-1} \sum_{\nu=2^m}^{2^{m+1}-1} a_\nu + \sum_{\nu=2^\xi}^{k} a_\nu \le C_1 \left(\sum_{m=0}^{\xi-1} 2^m a_{2^m} + 2^\xi a_{2^\xi}\right) \le C_1 \sum_{m=0}^{\xi} 2^m a_{2^m}.$$

Lemma 3.2 implies

$$\begin{aligned}\Phi\left(\sum_{\nu=1}^{k} a_\nu\right) &\le \Phi\left(C_1 \sum_{m=0}^{\xi} 2^m a_{2^m}\right)\\ &\le C_1^p \Phi\left(\sum_{m=0}^{\xi} 2^m a_{2^m}\right)\\ &\le C\left(\sum_{m=0}^{\xi} \Phi^{\frac{1}{p^*}}(2^m a_{2^m})\right)^{p^*}\\ &\le C\left(\sum_{m=1}^{k} \frac{\Phi^{\frac{1}{p^*}}(m a_m)}{m}\right)^{p^*}.\end{aligned}$$

By Lemma 3.1, we have

$$\sum_{k=1}^{\infty} \lambda_k \Phi\left(\sum_{\nu=1}^{k} a_\nu\right) \leq C \sum_{k=1}^{\infty} \lambda_k \left(\sum_{m=1}^{k} \frac{\Phi^{\frac{1}{p^*}}(ma_m)}{m}\right)^{p^*} \leq C \sum_{k=1}^{\infty} \Phi(ka_k)\,\lambda_k \left(\frac{\sum_{\nu=k}^{\infty} \lambda_\nu}{k\lambda_k}\right)^{p^*}.$$

□

Note that this Lemma was proved in [7] for the case $0 < p \leq 1$.

4. PROOFS OF THEOREMS

Proof of Theorem 2.1. Let $x \in (\frac{\pi}{n+1}, \frac{\pi}{n}]$. Applying Abel's transformation we obtain

$$|f(x)| \leq \sum_{k=1}^{n} \lambda_k + \left|\sum_{k=n+1}^{\infty} \lambda_k \cos kx\right| \leq \sum_{k=1}^{n} \lambda_k + \sum_{k=n}^{\infty} \left|(\lambda_k - \lambda_{k+1})\, D_k(x)\right|,$$

where $D_k(x)$ are the Dirichlet kernels, i.e.

$$D_k(x) = \frac{1}{2} + \sum_{n=1}^{k} \cos nx,\ k \in \mathbf{N}.$$

Since $|D_k(x)| = O\left(\frac{1}{x}\right)$ and $\lambda_n \in R_0^+ BVS$, we see that

$$|f(x)| \leq C\left(\sum_{k=1}^{n} \lambda_k + n\sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}|\right) \leq C\left(\sum_{k=1}^{n} \lambda_k + n\lambda_n\right).$$

The following estimates for series (1.2) can be obtained in the same way:

$$\begin{aligned} |g(x)| &\leq \sum_{k=1}^{n} \lambda_k + \left|\sum_{k=n+1}^{\infty} \lambda_k \sin kx\right| \\ &\leq \sum_{k=1}^{n} \lambda_k + \sum_{k=n}^{\infty} \left|(\lambda_k - \lambda_{k+1})\, \widetilde{D}_k(x)\right| \\ &\leq C\left(\sum_{k=1}^{n} \lambda_k + n\sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+1}|\right) \\ &\leq C\left(\sum_{k=1}^{n} \lambda_k + n\lambda_n\right), \end{aligned}$$

where $\widetilde{D}_k(x)$ are the conjugate Dirichlet kernels, i.e. $\widetilde{D}_k(x) := \sum_{n=1}^{k} \sin nx,\ k \in \mathbf{N}$.

Therefore,

$$|\psi(x)| \leq C\left(\sum_{k=1}^{n} \lambda_k + n\lambda_n\right),$$

where a function $\psi(x)$ is either a $f(x)$ or a $g(x)$.

One can see that if $\{\lambda_n\} \in R_0^+ BVS$, then $\{\lambda_n\}$ is almost decreasing sequence, i.e. there exists a constant $K \geq 1$ such that $\lambda_n \leq K\lambda_k$ holds for any $k \leq n$. Then

$$|\psi(x)| \leq C\left(\sum_{k=1}^{n} \lambda_k + \lambda_n \sum_{k=1}^{n} 1\right) \leq C\sum_{k=1}^{n} \lambda_k. \tag{4.1}$$

We will use (4.1) and the fact that $\{\lambda_k\}$ is almost decreasing sequence; also, we will use Lemmas 3.2 and 3.3:

$$\int_0^\pi \gamma(x)\Phi\left(|\psi(x)|\right)dx \le \sum_{n=1}^\infty \Phi\left(C_1\sum_{k=1}^n \lambda_k\right)\int_{\pi/(n+1)}^{\pi/n}\gamma(x)dx$$
$$\le C_1^p \pi B \sum_{n=1}^\infty \frac{\gamma_n}{n^2}\Phi\left(\sum_{k=1}^n \lambda_k\right)$$
$$\le C\sum_{k=1}^\infty \Phi\left(k\lambda_k\right)\frac{\gamma_k}{k^2}\left(\frac{k}{\gamma_k}\sum_{\nu=k}^\infty \frac{\gamma_\nu}{\nu^2}\right)^{p^*},$$

where $p^* = \max(1,p)$. Since there exists a constant $\varepsilon > 0$ such that $\{\gamma_n n^{-1+\varepsilon}\}$ is almost decreasing, then

$$\sum_{\nu=k}^\infty \frac{\gamma_\nu}{\nu^2} \le C\frac{\gamma_k}{k^{1-\varepsilon}}\sum_{\nu=k}^\infty \nu^{-\varepsilon-1} \le C\frac{\gamma_k}{k}.$$

Then

$$\int_0^\pi \gamma(x)\Phi\left(|\psi(x)|\right)dx \le C\sum_{k=1}^\infty \frac{\gamma_k}{k^2}\Phi\left(k\lambda_k\right).$$

The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. While proving Theorem 2.2 we will follow the idea of the proof of Theorem 2.1.

Let $x \in (\frac{\pi}{n+1}, \frac{\pi}{n}]$. Then

$$|g(x)| \le \sum_{k=1}^n kx\lambda_k + \left|\sum_{k=n+1}^\infty \lambda_k \sin kx\right| \tag{4.2}$$
$$\le \sum_{k=1}^n kx\lambda_k + \sum_{k=n}^\infty \left|\left(\lambda_k - \lambda_{k+1}\right)\widetilde{D}_k(x)\right|$$
$$\le C\left(\frac{1}{n}\sum_{k=1}^n k\lambda_k + n\lambda_n\right)$$
$$\le C\left(\frac{1}{n}\sum_{k=1}^n k\lambda_k + \frac{1}{n}\lambda_n\sum_{k=1}^n k\right) \le C_1\frac{1}{n}\sum_{k=1}^n k\lambda_k.$$

Using Lemma 3.2, Lemma 3.3 and the estimate (4.2), we can write

$$\int_0^\pi \gamma(x)\Phi\left(|g(x)|\right)dx \le \sum_{n=1}^\infty \Phi\left(C_1\frac{1}{n}\sum_{k=1}^n k\lambda_k\right)\int_{\pi/(n+1)}^{\pi/n}\gamma(x)dx$$
$$\le C_1^p \pi B\sum_{n=1}^\infty \frac{\gamma_n}{n^{2+q}}\Phi\left(\sum_{k=1}^n k\lambda_k\right)$$
$$\le C_2\sum_{k=1}^\infty \Phi\left(k^2\lambda_k\right)\frac{\gamma_k}{k^{2+q}}\left(\frac{k^{1+q}}{\gamma_k}\sum_{\nu=k}^\infty \frac{\gamma_\nu}{\nu^{2+q}}\right)^{p^*},$$

where $p^* = \max(1,p)$.

By the assumption on $\{\gamma_n\}$,

$$\int_0^{\pi} \gamma(x)\Phi\left(|g(x)|\right) dx \le C \sum_{k=1}^{\infty} \frac{\gamma_k}{k^{2+q}} \Phi\left(k^2 \lambda_k\right),$$

and the proof of Theorem 2.2 is complete. □

REFERENCES

[1] R.P. BOAS JR, Integrability of trigonometric series. III, *Quart. J. Math. Oxford Ser.*, **3**(2) (1952), 217–221.

[2] P. HEYWOOD, On the integrability of functions defined by trigonometric series, *Quart. J. Math. Oxford Ser.*, **5**(2) (1954), 71–76.

[3] S. IGARI, Some integrability theorems of trigonometric series and monotone decreasing functions, *Tohoku Math. J.*, **12**(2) (1960), 139–146.

[4] L. LEINDLER, Generalization of inequalities of Hardy and Littlewood, *Acta Sci. Math.*, **31** (1970), 279–285.

[5] L. LEINDLER, A new class of numerical sequences and its applications to sine and cosine series, *Anal. Math.*, **28** (2002), 279–286.

[6] M. MATELJEVIC AND M. PAVLOVIC, L^p-behavior of power series with positive coefficients and Hardy spaces, *Proc. Amer. Math. Soc.*, **87** (1983), 309-316.

[7] J. NEMETH, Note on the converses of inequalities of Hardy and Littlewood, *Acta Math. Acad. Paedagog. Nyhazi. (N.S.)*, **17** (2001), 101–105.

[8] J. NEMETH, Power-Monotone Sequences and integrability of trigonometric series, *J. Inequal. Pure and Appl. Math.*, **4**(1) (2003). [ONLINE http://jipam.vu.edu.au]

[9] M.M. RAO AND Z.D. REN, *Theory of Orlicz spaces*, M. Dekker, Inc., New York, 1991.

[10] T.M. VUKOLOVA AND M.I. DYACHENKO, On the properties of sums of trigonometric series with monotone coefficients, *Moscow Univ. Math. Bull.*, **50** (3), (1995), 19–27; translation from *Vestnik Moskov. Univ. Ser. I. Mat. Mekh.*, **3** (1995), 22–32.

[11] W. H. YOUNG, On the Fourier series of bounded variation, *Proc. London Math. Soc.*, **12**(2) (1913), 41–70.

[12] A. ZYGMUND, *Trigonometric series*, Volumes I and II, Cambridge University Press, 1988.