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ON SOME GENERALISATIONS OF STEFFENSEN'S INEQUALITY AND RELATED RESULTS

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ABSTRACT. Steffensen's inequality is generalised to allow bounds involving any two subintervals rather than restricting them to include the end points. Further results are obtained involving an identity related to the generalised Chebychev functional in which the difference of the mean of the product of functions and the product of means of functions over different intervals is utilised. Bounds involving one subinterval are also presented.

Key words and phrases: Steffensen's Inequality, Chebychev functional.

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1. INTRODUCTION

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f, g; a, b) := \mathcal{M}(fg, a, b) - \mathcal{M}(g; a, b)\,\mathcal{M}(f; a, b), \tag{1.1}$$

where the integral mean is given by

$$\mathcal{M}(f; a, b) = \frac{1}{b-a}\int_a^b f(x)\,dx, \tag{1.2}$$

provided that the involved integrals exist.

The following inequality is well known in the literature as the Grüss inequality [10]

$$|T(f, g; a, b)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.3}$$

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provided that $m \leq f \leq M$ and $n \leq g \leq N$ a.e. on $[a,b]$, where m, M, n, N are real numbers. The constant $\frac{1}{4}$ in (1.3) is the best possible.

Another inequality of this type is due to Chebychev (see for example [14, p. 207]). Namely, if f, g are absolutely continuous on $[a,b]$ and $f', g' \in L_\infty[a,b]$ with $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|$, then

$$|T(f,g;a,b)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2 \tag{1.4}$$

and the constant $\frac{1}{12}$ is the best possible.

Finally, let us recall a result by Lupaş ([11], see also [14, p. 210]), which states that:

$$|T(f,g;a,b)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a), \tag{1.5}$$

provided f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible here.

For other Grüss type inequalities, see the books [13] and [14], and the papers [4]-[10], where further references are given.

Recently, Cerone and Dragomir [3] have pointed out generalisations of the above results for integrals defined on two different intervals $[a,b]$ and $[c,d]$. They defined a generalised Chebychev functional involving the mean of the product of two functions, and the product of the means of each of the functions, where one is over a different interval by

$$\mathcal{T}(f,g;a,b,c,d) := \mathcal{M}(fg,a,b) - \mathcal{M}(g;a,b)\mathcal{M}(f;c,d), \tag{1.6}$$

with $\mathcal{M}(\cdot,\cdot,\cdot)$ as defined in (1.2). They proved the following theorem.

Theorem 1.1. *Let $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a,b]$, $[c,d] \subset I$. In addition, let $m_1 \leq f \leq M_1$ and $n_1 \leq g \leq N_1$ a.e. on $[a,b]$ with $n_2 \leq f \leq N_2$ a.e. on $[c,d]$. Then the following inequalities hold*

$$\begin{aligned} (1.7)\quad & |\mathcal{T}(f,g;a,b,c,d)| \\ & \leq \left[T(g;a,b) + M^2(g;a,b)\right]^{\frac{1}{2}} \\ & \quad \times \left[T(f;a,b) + T(f;c,d) + (\mathcal{M}(f;a,b) - \mathcal{M}(f;c,d))^2\right]^{\frac{1}{2}} \\ & \leq \left[\left(\frac{N_1 - n_1}{2}\right)^2 + \mathcal{M}^2(g;a,b)\right]^{\frac{1}{2}} \\ & \quad \times \left[\left(\frac{M_1 - m_1}{2}\right)^2 + \left(\frac{M_2 - m_2}{2}\right)^2 + (\mathcal{M}(f;a,b) - \mathcal{M}(f;c,d))^2\right]^{\frac{1}{2}}, \end{aligned}$$

where $T(f;a,b) \equiv T(f,f;a,b)$ which is as given by (1.1) and $\mathcal{M}(f;a,b)$ by (1.2).

Proof. The proof was based on the identity

$$\mathcal{T}(f,g;a,b,c,d) = \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d g(x)(f(x) - f(y))\,dydx, \tag{1.8}$$

and the Cauchy-Buniakowski-Schwartz inequality for double integrals to give

$$\begin{aligned}
&\left|\mathcal{T}\left(f,g;a,b,c,d\right)\right|^{2} \\
&\qquad = \left[\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{a}^{b}\int_{c}^{d}g\left(x\right)\left(f\left(x\right)-f\left(y\right)\right)dydx\right]^{2} \\
&\qquad \leq \left(\frac{1}{b-a}\int_{a}^{b}g^{2}\left(x\right)dx\right)\left(\int_{a}^{b}\int_{c}^{d}\left(f\left(x\right)-f\left(y\right)\right)^{2}dydx\right) \\
&\qquad = \mathcal{M}^{2}\left(g;a,b\right)\mathcal{T}\left(f,f,a,b,c,d\right).
\end{aligned}$$

□

They noted that equivalent results to the second inequality in (1.7) could be obtained if (1.4) and (1.5) relating to the Chebyshev and Lupaş inequalities were used in the first inequality in (1.7).

The following inequality is due to Steffensen ([15], see also [14, p. 181]).

Theorem 1.2. *Let* $f,g:[a,b]\rightarrow\mathbb{R}$ *be integrable mappings on* $[a,b]$ *such that* f *is nonincreasing and* $0\leq g\left(t\right)\leq 1$ *for* $t\in[a,b]$. *Then*

$$\int_{b-\lambda}^{b}f\left(t\right)dt\leq\int_{a}^{b}f\left(t\right)g\left(t\right)dt\leq\int_{a}^{a+\lambda}f\left(t\right)dt, \tag{1.9}$$

where

$$\lambda=\int_{a}^{b}g\left(t\right)dt. \tag{1.10}$$

Hayashi obtains a similar result [14, p. 182] which may ostensibly be obtained from Theorem 1.2 be replacing $g\left(t\right)$ by $\frac{g(t)}{A}$ where A is some positive constant.

For Steffensen type inequalities with integrals over a measure space, see the work of Gauchman [9].

It may be noted that both the generalised Chebyshev functional (1.6) and Steffensen's inequality (1.9) – (1.10) involve integrals of functions and of products of functions. The current article aims at investigating the relationship further.

2. STEFFENSEN TYPE RESULTS FOR GENERAL SUBINTERVALS

The following lemma will be useful for the results that follow.

Lemma 2.1. *Let* $f,g:[a,b]\rightarrow\mathbb{R}$ *be integrable mappings on* $[a,b]$. *Further, let* $[c,d]\subseteq[a,b]$ *with* $\lambda=d-c=\int_{a}^{b}g\left(t\right)dt$. *Then the following identities hold. Namely,*

$$\begin{aligned}
&\int_{c}^{d}f\left(t\right)dt-\int_{a}^{b}f\left(t\right)g\left(t\right)dt \\
&\qquad =\int_{a}^{c}\left(f\left(d\right)-f\left(t\right)\right)g\left(t\right)dt+\int_{c}^{d}\left(f\left(t\right)-f\left(d\right)\right)\left(1-g\left(t\right)\right)dt \\
&\qquad\qquad +\int_{d}^{b}\left(f\left(d\right)-f\left(t\right)\right)g\left(t\right)dt
\end{aligned} \tag{2.1}$$

and

$$\int_a^b f(t)\,g(t)\,dt - \int_c^d f(t)\,dt = \int_a^c (f(t)-f(c))\,g(t)\,dt + \int_c^d (f(c)-f(t))\,(1-g(t))\,dt + \int_d^b (f(t)-f(c))\,g(t)\,dt. \tag{2.2}$$

Proof. Let

$$S(c,d;a,b) = \int_c^d f(t)\,dt - \int_a^b f(t)\,g(t)\,dt,\ a \le c < d \le b, \tag{2.3}$$

then

$$\begin{aligned} S(c,d;a,b) &= \int_c^d (1-g(t))\,f(t)\,dt - \left[\int_a^c f(t)\,g(t)\,dt + \int_d^b f(t)\,g(t)\,dt\right] \\ &= \int_c^d (1-g(t))\,(f(t)-f(d))\,dt + f(d)\int_c^d (1-g(t))\,dt \\ &\quad + \int_a^c (f(d)-f(t))\,g(t)\,dt - f(d)\int_a^c g(t)\,dt \\ &\quad + \int_d^b (f(d)-f(t))\,g(t)\,dt - f(d)\int_d^b g(t)\,dt. \end{aligned}$$

The identity (2.1) is readily obtained on noting that

$$f(d)\left[\int_c^d dt - \int_a^b g(t)\,dt\right] = 0.$$

Identity (2.2) follows immediately from (2.1) and (2.3) on realising that (2.2) is $S(d,c;b,a)$ or, equivalently, $-S(c,d;a,b)$. □

Remark 2.2. If $c = a$ in (2.1) and $d = b$ in (2.2) then the identities obtained by Mitrinović [12] using an idea of Apéry, are recaptured.

Theorem 2.3. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable mappings on $[a,b]$ and let f be nonincreasing. Further, let $0 \le g(t) \le 1$ and $\lambda = \int_a^b g(t)\,dt = d_i - c_i$, where $[c_i, d_i] \subset [a,b]$ for $i = 1,2$ and $d_1 \le d_2$.*
Then the result

$$\int_{c_2}^{d_2} f(t)\,dt - r(c_2,d_2) \le \int_a^b f(t)\,g(t)\,dt \le \int_{c_1}^{d_1} f(t)\,dt + R(c_1,d_1), \tag{2.4}$$

holds where,

$$r(c_2,d_2) = \int_{d_2}^b (f(c_2)-f(t))\,g(t)\,dt \ge 0$$

and

$$R(c_1,d_1) = \int_a^{c_1} (f(t)-f(d_1))\,g(t)\,dt \ge 0.$$

Proof. From (2.1) and (2.3) of Lemma 2.1

$$\begin{aligned} S(c_1, d_1; a, b) &+ \int_a^{c_1} (f(t) - f(d_1)) g(t)\, dt \\ &= \int_{c_1}^{d_1} (f(t) - f(d_1))(1 - g(t))\, dt + \int_{d_1}^{b} (f(d_1) - f(t)) g(t)\, dt \geq 0 \end{aligned}$$

by the assumptions of the theorem.

Hence, from (2.3)

$$\int_{c_1}^{d_1} f(t)\, dt + \int_a^{c_1} (f(t) - f(d_1)) g(t)\, dt - \int_a^b f(t) g(t)\, dt \geq 0$$

and thus the right inequality is valid.

Now, from (2.2) and (2.3) of Lemma 2.1

$$\begin{aligned} -S(c_2, d_2; a, b) &+ \int_{d_2}^{b} (f(c_2) - f(t)) g(t)\, dt \\ &= \int_a^{c_2} (f(t) - f(c_2)) g(t)\, dt + \int_{c_2}^{d_2} (f(c_2) - f(t))(1 - g(t))\, dt \geq 0 \end{aligned}$$

from the assumptions.

Thus, from (2.3)

$$\int_a^b f(t) g(t)\, dt - \left[\int_{c_2}^{d_2} f(t)\, dt - \int_{d_2}^{b} (f(c_2) - f(t)) g(t)\, dt\right] \geq 0,$$

giving the left inequality.

Both $r(c_2, d_2)$ and $R(c_1, d_1)$ are nonnegative since f is nonincreasing and g is nonnegative. The theorem is now completely proved. □

Remark 2.4. If in Theorem 2.3 we take $c_1 = a$ and so $d_1 = a + \lambda$, then $R(a, a + \lambda) = 0$. Further, taking $d_2 = b$ so that $c_2 = b - \lambda$ gives $r(b - \lambda, b) = 0$. The Steffensen inequality (1.9) is thus recaptured. Since (1.10) holds, then $c_2 \geq a$ and $d_1 \leq b$ giving $[c_i, d_i] \subset [a, b]$. Theorem 2.3 may thus be viewed as a generalisation of the Steffensen inequality as given in Theorem 1.2, to allow for two equal length subintervals that are not necessarily at the ends of $[a, b]$.

It may be advantageous at times to gain coarser bounds that may be more easily evaluated. The following corollary examines this aspect.

Corollary 2.5. *Let the conditions of Theorem 2.3 hold. Then*

$$\begin{aligned} \int_{c_2}^{b} f(t)\, dt - (b - d_2) f(c_2) &\leq \int_a^b f(t) g(t)\, dt \\ &\leq \int_a^{d_1} f(t)\, dt - (c_1 - a) f(d_1). \end{aligned} \tag{2.5}$$

Proof. From Theorem 2.3 on using the fact that $0 \leq g(t) \leq 1$, gives

$$\begin{aligned} 0 &\leq r(c_2, d_2) = \int_{d_2}^{b} (f(c_2) - f(t)) g(t)\, dt \\ &\leq \int_{d_2}^{b} (f(c_2) - f(t))\, dt = (b - d_2) f(c_2) - \int_{d_2}^{b} f(t)\, dt \end{aligned}$$

and so

$$\int_{c_2}^{d_2} f(t)\,dt - r(c_2,d_2) \geq \int_{c_2}^{d_2} f(t)\,dt - (b-d_2)\,f(c_2) + \int_{d_2}^{b} f(t)\,dt.$$

Combining the two integrals produces the left inequality of (2.5). Similarly,

$$0 \leq R(c_1,d_1) = \int_a^{c_1} (f(t) - f(d_1))\,g(t)\,dt \leq \int_a^{c_1} f(t)\,dt - (c_1-a)\,f(d_1)\,,$$

producing

$$\int_{c_1}^{d_1} f(t)\,dt + R(c_1,d_1) \leq \int_a^{d_1} f(t)\,dt - (c_1-a)\,f(d_1)$$

giving the right inequality. □

Remark 2.6. If we take $c_1 = a$ and so $d_1 = a+\lambda$ and $d_2 = b$ such that $c_2 = b-\lambda$ then (2.5) again recaptures Steffensen's inequality as given in Theorem 1.2.

The following lemma produces alternative identities to those obtained in Lemma 2.1. The current identities involve the integral mean of $f(\cdot)$ over the subinterval $[c,d]$.

Lemma 2.7. *Let $f,g:[a,b]\to\mathbb{R}$ be integrable mappings on $[a,b]$. Define*

$$G(x) = \int_a^x g(t)\,dt$$

and

$$\lambda = G(b) = d-c$$

where $[c,d]\subset[a,b]$.
The following identities hold

$$\int_a^b f(x)\,g(x)\,dx - \int_c^d f(y)\,dy = \lambda\left[f(b) - \mathcal{M}(f;c,d)\right] - \int_a^b G(x)\,df(x) \tag{2.6}$$

and

$$\int_c^d f(y)\,dy - \int_a^b f(x)\,g(x)\,dx = \lambda\left[\mathcal{M}(f;c,d) - f(a)\right] - \int_a^b \left[\lambda - G(x)\right] df(x)\,, \tag{2.7}$$

where $\mathcal{M}(f;c,d)$ is the integral mean of $f(\cdot)$ over $[c,d]$.

Proof. Consider

$$L := \int_a^b f(x)\,g(x)\,dx - \int_c^d f(y)\,dy$$

then from the postulates $\frac{G(b)}{d-c} = 1$ giving

$$L = \int_a^b f(x)\,g(x)\,dx - \frac{1}{d-c}\int_a^b g(x)\,dx \int_c^d f(y)\,dy.$$

Combining the integrals gives

$$L = \int_a^b g(x)\left[f(x) - \mathcal{M}(f;c,d)\right] dx, \tag{2.8}$$

where $\mathcal{M}(f;c,d)$ is the integral mean of f over $[c,d]$ as given by (1.2).

Integration by parts from (2.8) gives

$$L = G(x)\left[f(x) - \mathcal{M}(f;c,d)\right]\Big]_a^b - \int_a^b G(x)\,df(x)$$

and so

$$L = \lambda [f(b) - \mathcal{M}(f; c, d)] - \int_a^b G(x)\, df(x)$$

since $G(b) = \lambda$ and $G(a) = 0$.

Now for the second identity.

Let

$$U := \int_c^d f(y)\, dy - \int_a^b f(x)\, g(x)\, dx$$

then, from (2.6),

$$U = -L = -\lambda [f(b) - \mathcal{M}(f; c, d)] + \int_a^b G(x)\, df(x).$$

Hence

$$U = \lambda [\mathcal{M}(f; c, d) - f(a)] - \lambda [f(b) - f(a)] + \int_a^b G(x)\, df(x)$$

and so combining the last two terms gives (2.7). □

Theorem 2.8. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable mappings on $[a, b]$ and let f be nonincreasing. Further, let $g(t) \geq 0$ and $G(x) = \int_a^x g(t)\, dt$ with $\lambda = G(b) = d_i - c_i$ where $[c_i, d_i] \subset [a, b]$ for $i = 1, 2$ and $d_1 < d_2$. Then*

$$\int_{c_2}^{d_2} f(y)\, dy - \lambda [\mathcal{M}(f; c_2, d_2) - f(b)] \leq \int_a^b f(x)\, g(x)\, dx \leq \int_{c_1}^{d_1} f(y)\, dy + \lambda [f(a) - \mathcal{M}(f; c_1, d_1)] \tag{2.9}$$

where $d_2 > d_1$.

Proof. From (2.6) together with the facts that f is nonincreasing and $g(t) \geq 0$ then

$$-\int_a^b G(x)\, df(x) \geq 0$$

gives

$$\int_a^b f(x)\, g(x)\, dx - \left[\int_{c_2}^{d_2} f(y)\, dy + \lambda [f(b) - \mathcal{M}(f; c_2, d_2)] \right] \geq 0$$

and so the left inequality is obtained.

Similarly, from (2.7) and the postulates we have

$$-\int_a^b [\lambda - G(x)]\, df(x) \geq 0,$$

which gives

$$\int_{c_1}^{d_1} f(y)\, dy + \lambda [\mathcal{M}(f; c_1, d_1) - f(a)] - \int_a^b f(x)\, g(x)\, dx \geq 0.$$

□

Remark 2.9. The lower and upper inequalities in (2.9) may be simplified to $\lambda f(b)$ and $\lambda f(a)$ respectively since

$$\int_c^d f(y)\, dy = \lambda \mathcal{M}(f; c, d).$$

That is,

$$\lambda f(b) \leq \int_a^b f(x) g(x) \, dx \leq \lambda f(a) \,. \tag{2.10}$$

The result should not be overly surprising since it may be obtained directly from the postulates since

$$\inf_{x \in [a,b]} f(x) \int_a^b g(x) \, dx \leq \int_a^b f(x) g(x) \, dx \leq \sup_{x \in [a,b]} f(x) \int_a^b g(x) \, dx.$$

As a referee suggested, the result (2.10) readily follows on noting that

$$\int_a^b g(x) [f(x) - f(b)] \, dx \geq 0$$

and

$$\int_a^b f(x) [f(a) - f(x)] \, dx \geq 0$$

.

The motivation behind Lemma 2.7 and Theorem 2.8 was to obtain a Steffensen like inequality and it was not predictable in advance that the result would reduce to (2.10).

3. STEFFENSEN AND THE GENERALISED CHEBYSHEV FUNCTIONAL

Bounds will be obtained for the difference between the integral of the product of two functions from the integral over a subinterval of one of the functions.

Theorem 3.1. *Let* $f, g : [a,b] \to \mathbb{R}$ *be integrable mappings on* $[a,b]$ *such that* f *is nonincreasing and* $0 \leq g(t) \leq 1$ *for* $t \in [a,b]$*. Further, let* $[c,d] \subseteq [a,b]$ *with* $\lambda = d - c = \int_a^b g(t) \, dt$*, then the following inequality holds. Namely,*

$$\begin{aligned} |\mathcal{S}| &:= \left| \int_a^b f(x) g(x) \, dx - \int_c^d f(y) \, dy \right| \\ &\leq (b-a) \left[\frac{1}{4} + \left(\frac{\lambda}{b-a} \right)^2 \right]^{\frac{1}{2}} \times \left[\left(\frac{f(a) - f(b)}{2} \right)^2 \right. \\ &\quad \left. + \left(\frac{f(c) - f(d)}{2} \right)^2 + \left(\mathcal{M}(f;a,b) - \mathcal{M}(f;c,d) \right)^2 \right]^{\frac{1}{2}}, \end{aligned} \tag{3.1}$$

where $\mathcal{M}(f;a,b)$ *is the integral mean.*

Proof. Since $d - c = \int_a^b g(t) \, dt$, then

$$\mathcal{S} = \int_a^b f(x) g(x) \, dx - \frac{1}{d-c} \int_a^b g(x) \, dx \int_c^d f(y) \, dy$$

that is

$$\mathcal{S} = \int_a^b f(x) g(x) \, dx - \mathcal{M}(f;c,d) \int_a^b g(x) \, dx, \tag{3.2}$$

where $\mathcal{M}(f;c,d)$ is as defined by (1.2).

Thus, from (3.2), $\mathcal{S}$ may be expressed in terms of the generalised Chebyshev functional as defined in (1.6), namely

$$\mathcal{S} = (b-a) \mathcal{T}(f,g;a,b,c,d) \tag{3.3}$$

and so, from (1.7)

$$\begin{aligned}(3.4)\qquad |\mathcal{S}| &= (b-a)\left|\mathcal{T}(f,g;a,b,c,d)\right| \\ &\leq (b-a)\left[T(g;a,b)+\left(\frac{\lambda}{b-a}\right)^2\right]^{\frac{1}{2}} \\ &\quad \times\left[T(f;a,b)+T(f;c,d)+(\mathcal{M}(f;a,b)-\mathcal{M}(f;c,d))^2\right]^{\frac{1}{2}},\end{aligned}$$

where

$$T(f;a,b)=\mathcal{M}\left(f^2;a,b\right)-(\mathcal{M}(f;a,b))^2$$

and $\mathcal{M}(g;a,b)=\frac{\lambda}{b-a}$.

Hence, using the second inequality from (1.7) and (3.4) produces the stated result (3.1) upon using (1.3) and the facts that f is nonincreasing and $0 \leq g(t) \leq 1$. □

Remark 3.2. If we had more stringent conditions on f' and g' such that the Chebyshev and Lupaş results (1.4) and (1.5) could be utilised in (3.4), then bounds in terms of the $\|\cdot\|_\infty$ and $\|\cdot\|_2$ norms of the derivatives would result. This will not be pursued further here however.

The following theorem expresses $\mathcal{S}$ as a double integral over a rectangular region to obtain bounds for the Steffensen functional.

Theorem 3.3. *Let the conditions of Theorem 3.1 hold. The following inequality is then valid. Namely,*

$$\begin{aligned}(3.5)\qquad |\mathcal{S}| &: =\left|\int_a^b f(x)g(x)\,dx-\int_c^d f(y)\,dy\right| \\ &\leq (a+b+c+d)\,\mathcal{M}(f;c,d)-\frac{4}{d-c}\mu(f;c,d) \\ &\quad +\int_a^c f(x)\,dx-\int_d^b f(x)\,dx \\ &\leq (c-a)f(a)-(b-d)f(b)+(a+b+c+d)f(c) \\ &\quad -2(d+c)f(d),\end{aligned}$$

where $\mathcal{M}(f;c,d)$ *is the integral mean and*

$$(3.6)\qquad \mu(f;c,d)=\int_c^d xf(x)\,dx.$$

Proof. From (3.3) and (1.8)

$$\begin{aligned}(3.7)\qquad |\mathcal{S}| &= \frac{1}{d-c}\left|\int_a^b\int_c^d g(x)(f(x)-f(y))\,dydx\right| \\ &\leq \frac{\|g\|_\infty}{d-c}\int_a^b\int_c^d |f(x)-f(y)|\,dydx,\end{aligned}$$

where $\|g\|_\infty := ess \sup_{x\in[a,b]} |g(x)|=1$, from the postulates.

Thus,

$$(3.8)\qquad |\mathcal{S}| \leq \frac{1}{d-c}\int_a^b\int_c^d |f(x)-f(y)|\,dydx := I.$$

Now, using the fact that f is nonincreasing and that $[c,d] \subseteq [a,b]$, we have

$$\begin{aligned}(d-c)\,I &= \int_a^c \int_c^d (f(x)-f(y))\,dydx + \int_c^d \int_c^x (f(y)-f(x))\,dydx \\ &\quad + \int_c^d \int_x^d (f(x)-f(y))\,dydx + \int_d^b \int_c^d (f(y)-f(x))\,dydx \\ &= (d-c)\int_a^c f(x)\,dx - (c-a)\int_c^d f(y)\,dy \\ &\quad + \int_c^d \int_c^x f(y)\,dydx - \int_c^d (x-c)\,f(x)\,dx \\ &\quad + \int_c^d (d-x)\,f(x)\,dx - \int_c^d \int_x^d f(y)\,dydx \\ &\quad + (b-d)\int_c^d f(y)\,dy - (d-c)\int_d^b f(x)\,dx \\ &= (d-c)\left[\int_a^c f(x)\,dx - \int_d^b f(x)\,dx\right] + \int_c^d [a+b+c+d-4x]\,f(x)\,dx.\end{aligned}$$

Here we have used the facts that

$$\int_c^d \int_c^x f(y)\,dydx = \int_c^d (d-x)\,f(x)\,dx$$

and

$$\int_c^d \int_x^d f(y)\,dydx = \int_c^d (x-c)\,f(x)\,dx.$$

Some elementary simplification gives

$$I = \int_a^c f(x)\,dx - \int_d^b f(x)\,dx + \frac{4}{d-c}\int_c^d \left(\frac{a+b+c+d}{4} - x\right) f(x)\,dx \tag{3.9}$$

and hence the first inequality results.

The coarser inequality is obtained using the fact that f is nonincreasing, giving, from (3.9)

$$I \le (c-a)\,f(a) - (b-d)\,f(b) + (a+b+c+d)\,f(c) - \frac{4}{d-c} f(d) \int_c^d x dx,$$

which upon simplification gives the second inequality in (3.5). The theorem is thus completely proved. □

Corollary 3.4. *Let the conditions of Theorem 3.1 hold. Then*

$$-2c\mathcal{M}(f;c,d) - \phi(c,d) \le \int_a^b f(x)\,g(x)\,dx \le 2d\mathcal{M}(f;c,d) + \phi(c,d)\,, \tag{3.10}$$

where

$$\phi(c,d) = (a+b)\,\mathcal{M}(f;c,d) + \int_a^c f(x)\,dx - \int_d^b f(x)\,dx - \frac{4}{d-c}\mu(f;c,d) \tag{3.11}$$

with $\mathcal{M}(f;c,d)$ being the integral mean and $\mu(f;c,d)$ the mean of f over the subinterval $[c,d]$ given by (1.2) and (3.6) respectively.

Proof. From (3.5) and (3.8) we have that

$$-I \leq \int_a^b f(x)\, g(x)\, dx - \int_c^d f(y)\, dy \leq I$$

so that from (3.9)

$$I = \phi(c, d) - \int_c^d f(x)\, dx$$

giving the result as stated after some minor algebra. □

Remark 3.5. Equation (3.10) gives bounds for $\int_a^b f(x)\, g(x)\, dx$ in terms of information known over the subinterval $[c, d]$. Let $[c_1, d_1]$ and $[c_2, d_2]$ be two such subintervals with $d_1 < d_2$ and $d_i - c_i = \lambda$. Then

$$m \leq \int_a^b f(x)\, g(x)\, dx \leq M, \tag{3.12}$$

where

$$M = \min\{2d_1\mathcal{M}(f; c_1, d_1) + \phi(c_1, d_1), 2d_2\mathcal{M}(f; c_2, d_2) + \phi(c_2, d_2)\}$$

and

$$m = \max\{-2c_1\mathcal{M}(f; c_1, d_1) - \phi(c_1, d_1), -2c_2\mathcal{M}(f; c_2, d_2) - \phi(c_2, d_2)\},$$

with $\phi(\cdot, \cdot)$ being as given in (3.11).

Particularising the result (3.12) on taking $d_2 = b$ and hence $c_2 = b - \lambda$, $c_1 = a$ and so $d_1 = a + \lambda$, produces bounds in terms of subintervals at the ends of $[a, b]$. That is,

$$m_e \leq \int_a^b f(x)\, g(x)\, dx \leq M_e, \tag{3.13}$$

where

$$\begin{aligned} M_e &= \min\{2(a+\lambda)\mathcal{M}(f; a, a+\lambda) + \phi(a, a+\lambda), \\ &\qquad 2b\mathcal{M}(f; b-\lambda, b) + \phi(b-\lambda, b)\} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} m_e &= \max\{-2a\mathcal{M}(f; a, a+\lambda) - \phi(a, a+\lambda), \\ &\qquad -2(b-\lambda)\mathcal{M}(f; b-\lambda, b) - \phi(b-\lambda, b)\} \end{aligned}$$

with $\phi(\cdot, \cdot)$ defined in (3.11).
For (3.14) on using (3.11), (1.2) and (3.6) gives

$$\phi(a, a+\lambda) = (a+b)\mathcal{M}(f; a, a+\lambda) - \int_{a+\lambda}^b f(x)\, dx - \frac{4}{\lambda}\int_a^{a+\lambda} xf(x)\, dx$$

and

$$\phi(b-\lambda, b) = (a+b)\mathcal{M}(f; b-\lambda, b) + \int_a^{b-\lambda} f(x)\, dx - \frac{4}{\lambda}\int_{b-\lambda}^b xf(x)\, dx.$$

It may be possible that M_e can either be tighter or coarser than the $\int_a^{a+\lambda} f(x)\, dx$ bound in (1.9) and similarly with m_e and $\int_{b-\lambda}^b f(x)\, dx$.

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