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A NOTE ON THE UPPER BOUNDS FOR THE DISPERSION

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Abstract. In this note we provide upper bounds for the standard deviation $(\sigma(X))$, for the quantity $\sigma^2(X) + (x - \mathbb{E}(X))^2$ and for the L^p absolute deviation of a random variable. These improve and extend current results in the literature.

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1. Introduction

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ be the probability density function (p.d.f.) of the random variable X and $\mathbb{E}(X)$ and $\sigma(X)$ the mean and standard deviatin respectively.

In [2], the authors gave upper bounds for the dispersion $\sigma(X)$ and for $\sigma^2(X) + (x - \mathbb{E}(X))^2$ in terms of the p.d.f. for a continuous random variable. Recently, Agbeko [1] also obtained some upper bounds for the same two quantities, namely,

$$\sigma(X) \le \min\{\max\{|a|, |b|\}, (b - a)\}, \tag{1.1}$$

and

$$\sqrt{\sigma^2(X) + (x - \mathbb{E}(X))^2} \le 2\min\{\max\{|a|, |b|\}, (b - a)\}, \tag{1.2}$$

for all $x \in [a, b]$.

Both the work of Barnett et al. [2] and Agbeko [1] exclude the discrete case. In this present note, we remove this exclusion and improve the bounds of (1.1) and (1.2) in Section 2. In Section 3, we consider the L^p absolute deviation. The symbolism 1_A denotes the indicator function on the set A.

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2. STANDARD DEVIATION

Let $\mathbb{P}$ be the distribution of the random variable X, then the expectation of X can be denoted by

$$\mathbb{E}(X) = \int_a^b x d\mathbb{P}(x)$$

and the dispersion or standard deviation of X by

$$\sigma(X) = \int_a^b x^2 d\mathbb{P}(x) - \left(\int_a^b x d\mathbb{P}(x)\right)^2.$$

Theorem 2.1. *Let X be a random variable with $a \le X \le b$. Then we have*

$$\sigma(X) \le \min\{\max\{|a|,|b|\}, M(a,b,\mathbb{E}(X))(b-a)\}, \tag{2.1}$$

where

$$M(a,b,\mathbb{E}(X)) = \frac{1}{\sqrt{2}}\sqrt{2 - \exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} - \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\}}.$$

Proof. Since $\sigma^2(X) = \mathbb{E}(X-\mathbb{E}(X))^2$, by the Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}(X-\mathbb{E}(X))^2 &= \mathbb{E}\int_0^{(X-\mathbb{E}(X))^2} ds \\ &= \mathbb{E}\int_0^{\infty} 1_{\{(X-\mathbb{E}(X))^2 \ge s\}} ds \\ &= \int_0^{\infty} \mathbb{P}\left((X-\mathbb{E}(X))^2 \ge s\right) ds \\ &= \int_0^{(b-\mathbb{E}(X))^2} \mathbb{P}(X-\mathbb{E}(X) \ge \sqrt{s}) ds \\ &\quad + \int_0^{(\mathbb{E}(X)-a)^2} \mathbb{P}\left(-(X-\mathbb{E}(X)) \ge \sqrt{s}\right) ds. \end{aligned} \tag{2.2}$$

For any $\lambda > 0$, we have

$$\begin{aligned} \mathbb{E}e^{\lambda(X-\mathbb{E}(X))} &\le e^{-\lambda\mathbb{E}(X)}\left(\frac{b-\mathbb{E}(X)}{b-a}e^{\lambda a} + \frac{\mathbb{E}(X)-a}{b-a}e^{\lambda b}\right) \\ &= e^{\lambda(a-\mathbb{E}(X))}\left(1 - \frac{\mathbb{E}(X)-a}{b-a} + \frac{\mathbb{E}(X)-a}{b-a}e^{\lambda(b-a)}\right) \\ &\triangleq e^{L(\lambda)}, \end{aligned}$$

where

$$L(\lambda) = \lambda(a-\mathbb{E}(X)) + \log\left(1 - \frac{\mathbb{E}(X)-a}{b-a} + \frac{\mathbb{E}(X)-a}{b-a}e^{\lambda(b-a)}\right).$$

The first two derivatives of $L(\lambda)$ are

$$\begin{aligned} L'(\lambda) &= a - \mathbb{E}(X) + \frac{(\mathbb{E}(X)-a)e^{\lambda(b-a)}}{\left(1 - \frac{\mathbb{E}(X)-a}{b-a} + \frac{\mathbb{E}(X)-a}{b-a}e^{\lambda(b-a)}\right)} \\ &= a - \mathbb{E}(X) + \frac{\mathbb{E}(X)-a}{\left((1 - \frac{\mathbb{E}(X)-a}{b-a})e^{-\lambda(b-a)} + \frac{\mathbb{E}(X)-a}{b-a}\right)}, \end{aligned}$$

$$L^{''}(\lambda) = \frac{(b-\mathbb{E}(X))(\mathbb{E}(X)-a)e^{-\lambda(b-a)}}{\left((1-\frac{\mathbb{E}(X)-a}{b-a})e^{-\lambda(b-a)}+\frac{\mathbb{E}(X)-a}{b-a}\right)^2}.$$

Noting that

$$L^{''}(\lambda) \leq \frac{(b-a)^2}{4},$$

then by Taylor's formula, we have

$$L(\lambda) \leq L(0) + L^{'}(0)\lambda + \frac{(b-a)^2}{8}\lambda^2 = \frac{(b-a)^2}{8}\lambda^2.$$

Therefore from Markov's inequality, we have

$$\begin{aligned}
\mathbb{P}\left(X-\mathbb{E}(X) \geq \sqrt{s}\right) &\leq \inf_{\lambda>0} e^{-\sqrt{s}\lambda}\mathbb{E}e^{\lambda(X-\mathbb{E}(X))} \\
&\leq \inf_{\lambda>0} \exp\left\{-\sqrt{s}\lambda + \frac{(b-a)^2}{8}\lambda^2\right\} \\
&= \exp\left\{-\frac{2s}{(b-a)^2}\right\}.
\end{aligned} \tag{2.3}$$

Similarly,

$$\mathbb{P}(-(X-\mathbb{E}(X)) \geq \sqrt{s}) \leq \exp\left\{-\frac{2s}{(b-a)^2}\right\}.$$

From (2.2), it follows that

$$\begin{aligned}
&\mathbb{E}(X-\mathbb{E}(X))^2 \\
&= \int_0^{(b-\mathbb{E}(X))^2} \mathbb{P}\left((X-\mathbb{E}(X)) \geq \sqrt{s}\right) ds + \int_0^{(\mathbb{E}(X)-a)^2} \mathbb{P}\left(-(X-\mathbb{E}(X)) \geq \sqrt{s}\right) ds \\
&\leq \int_0^{(b-\mathbb{E}(X))^2} \exp\left\{-\frac{2s}{(b-a)^2}\right\} ds + \int_0^{(\mathbb{E}(X)-a)^2} \exp\left\{-\frac{2s}{(b-a)^2}\right\} ds \\
&= \frac{(b-a)^2}{2}\left(2 - \exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} - \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\}\right).
\end{aligned} \tag{2.4}$$

Furthermore,

$$\mathbb{E}(X-\mathbb{E}(X))^2 = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \leq \mathbb{E}(X^2) \leq \max\{|a|^2, |b^2|\},$$

which, by (2.4), implies the result. □

Remark 2.2. In fact, by,

$$\begin{aligned}
M(a,b,\mathbb{E}(X)) &= \frac{1}{\sqrt{2}}\sqrt{2-\exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} - \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\}} \\
&\leq \frac{1}{\sqrt{2}}\sqrt{2-e^{-2}} < 1,
\end{aligned}$$

$$\min\{\max\{|a|,|b|\}, M(a,b,\mathbb{E}(X))(b-a)\} \leq \min\{\max\{|a|,|b|\},(b-a)\},$$

and is, therefore, a tighter bound than (1.1). If $\mathbb{E}(X) = (b+a)/2$, then

$$\begin{aligned}
M(a,b,\mathbb{E}(X)) &= \frac{1}{\sqrt{2}}\sqrt{2-\exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} - \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\}} \\
&= \sqrt{1-e^{-1/2}} < \frac{1}{\sqrt{2}},
\end{aligned}$$

which means that under some assumptions, this bound is better than the result of [2] for the case $f \in L_1([a,b])$.

Corollary 2.3. *Let X be a random variable with $a \le X \le b$, then for any $x \in [a,b]$,*

$$\sqrt{\sigma^2(X) + (x - \mathbb{E}(X))^2} \le \min\{2\max\{|a|,|b|\}, N(a,b,\mathbb{E}(X))(b-a)\}, \tag{2.5}$$

where

$$N^2(a,b,\mathbb{E}(X)) = 2 - \frac{1}{2}\left(\exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} + \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\}\right).$$

Proof. It is clear that $(x-\mathbb{E}(X))^2 \le (b-a)^2$ and from the proof of Theorem 2.1, we know that

$$\sigma^2(X)+(x-\mathbb{E}(X))^2 \le \left(2 - \frac{1}{2}\left(\exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} + \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\}\right)\right)(b-a)^2.$$

Furthermore,

$$\sigma^2(X) + (x-\mathbb{E}(X))^2 = \mathbb{E}(X-x)^2 \le \max\{(x-y)^2, x, y \in [a,b]\}.$$

The remainder of the proof follows as Theorem 1.2 in Agbeko [1], giving

$$\max\{|x-y|, x, y \in [a,b]\} \le 2\min\{\max\{|a|,|b|\}, (b-a)\}.$$

□

3. L^p ABSOLUTE DEVIATION

In fact by the method in Section 2, we could extend the case of L^p absolute deviation, i.e., the following quantity,

$$\sigma_p(X) = \mathbb{E}\left(|X-\mathbb{E}(X)|^p\right)^{1/p}.$$

We have the following

Theorem 3.1. *Let X be a random variable and $p \ge 1$. Assume that $\mathbb{E}|X|^p < \infty$, then its L^p absolute deviation has the following estimation for $p > 2$,*

$$(\sigma_p(X))^p \le \min\{|b-a|^p, M_1(a,b,p,X)\},$$

where

$$\begin{aligned} M_1(a,b,p,X) = {} & \frac{(b-a)^2}{2}\left(1 - \exp\left\{-\frac{2(|b-\mathbb{E}(X)|^p \wedge 1)}{(b-a)^2}\right\}\right) \\ & + (|b-\mathbb{E}(X)|^p - |b-\mathbb{E}(X)|^p \wedge 1) \\ & \quad \times \exp\left\{-\frac{2(|b-\mathbb{E}(X)|^p \wedge 1)^{2/p}}{(b-a)^2}\right\} \\ & + \frac{(b-a)^2}{2}\left(1 - \exp\left\{-\frac{2(|a-\mathbb{E}(X)|^p \wedge 1)}{(b-a)^2}\right\}\right) \\ & + (|a-\mathbb{E}(X)|^p - |a-\mathbb{E}(X)|^p \wedge 1) \\ & \quad \times \exp\left\{-\frac{2(|a-\mathbb{E}(X)|^p \wedge 1)^{2/p}}{(b-a)^2}\right\}. \end{aligned} \tag{3.1}$$

If $1 \le p < 2$, then we have

$$(\sigma_p(X))^p \le \min\{|b-a|^p, M_2(a,b,p,X)\},$$

where

$$(3.2)\quad M_2(a,b,p,X)$$
$$= |b-\mathbb{E}(X)|^p \wedge 1 + \frac{(b-a)^2}{2}\left(\exp\left\{-\frac{2|b-\mathbb{E}(X)|^p\wedge 1}{(b-a)^2}\right\} - \exp\left\{-\frac{2|b-\mathbb{E}(X)|^p}{(b-a)^2}\right\}\right)$$
$$+ |a-\mathbb{E}(X)|^p \wedge 1 + \frac{(b-a)^2}{2}\left(\exp\left\{-\frac{2|a-\mathbb{E}(X)|^p\wedge 1}{(b-a)^2}\right\} - \exp\left\{-\frac{2|a-\mathbb{E}(X)|^p}{(b-a)^2}\right\}\right).$$

Proof. On one hand, by the Fubini's theorem, we have

$$(3.3)\qquad \begin{aligned}\mathbb{E}|X-\mathbb{E}(X)|^p &= \mathbb{E}\int_0^{|X-\mathbb{E}(X)|^p} ds\\ &= \mathbb{E}\int_0^{\infty} 1_{\{|X-\mathbb{E}(X)|^p\geq s\}}ds\\ &= \int_0^{\infty}\mathbb{P}(|X-\mathbb{E}(X)|^p\geq s)ds\\ &= \int_0^{|b-\mathbb{E}(X)|^p}\mathbb{P}\left((X-\mathbb{E}(X))\geq s^{1/p}\right)ds\\ &\qquad + \int_0^{|\mathbb{E}(X)-a|^p}\mathbb{P}\left(-(X-\mathbb{E}(X))\geq s^{1/p}\right)ds.\end{aligned}$$

From the estimation (2.3), we have

$$\begin{aligned}\mathbb{E}|X-\mathbb{E}(X)|^p &= \int_0^{|b-\mathbb{E}(X)|^p}\mathbb{P}(X-\mathbb{E}(X)\geq s^{1/p})ds + \int_0^{|\mathbb{E}(X)-a|^p}\mathbb{P}(-(X-\mathbb{E}(X))\geq s^{1/p})ds\\ &\leq \int_0^{|b-\mathbb{E}(X)|^p}\exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\}ds + \int_0^{|\mathbb{E}(X)-a|^p}\exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\}ds.\end{aligned}$$

(*Case:* $p>2$.) By simple calculating, we have

$$\begin{aligned}&\int_0^{|b-\mathbb{E}(X)|^p}\exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\}ds\\ &\leq \int_0^{|b-\mathbb{E}(X)|^p\wedge 1}\exp\left\{-\frac{2s}{(b-a)^2}\right\}ds + \int_{|b-\mathbb{E}(X)|^p\wedge 1}^{|b-\mathbb{E}(X)|^p}\exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\}ds\\ &\leq \frac{(b-a)^2}{2}\left(1-\exp\left\{-\frac{2(|b-\mathbb{E}(X)|^p\wedge 1)}{(b-a)^2}\right\}\right)\\ &\qquad + (|b-\mathbb{E}(X)|^p - |b-\mathbb{E}(X)|^p\wedge 1)\exp\left\{-\frac{2(|b-\mathbb{E}(X)|^p\wedge 1)^{2/p}}{(b-a)^2}\right\}\end{aligned}$$

and with the same reason

$$\begin{aligned}\int_0^{|\mathbb{E}(X)-a|^p}\exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\}ds &\leq \frac{(b-a)^2}{2}\left(1-\exp\left\{-\frac{2(|a-\mathbb{E}(X)|^p\wedge 1)}{(b-a)^2}\right\}\right)\\ &\qquad + (|a-\mathbb{E}(X)|^p - |a-\mathbb{E}(X)|^p\wedge 1)\exp\left\{-\frac{2(|a-\mathbb{E}(X)|^p\wedge 1)^{2/p}}{(b-a)^2}\right\}.\end{aligned}$$

(*Case* $1 \le p < 2$) In this case, we have

$$\begin{aligned}
&\int_0^{|b-\mathbb{E}(X)|^p} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds \\
&\le \int_0^{|b-\mathbb{E}(X)|^p\wedge 1} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds + \int_{|b-\mathbb{E}(X)|^p\wedge 1}^{|b-\mathbb{E}(X)|^p} \exp\left\{-\frac{2s}{(b-a)^2}\right\} ds \\
&\le |b-\mathbb{E}(X)|^p \wedge 1 + \frac{(b-a)^2}{2}\left(\exp\left\{-\frac{2|b-\mathbb{E}(X)|^p\wedge 1}{(b-a)^2}\right\} - \exp\left\{-\frac{2|b-\mathbb{E}(X)|^p}{(b-a)^2}\right\}\right)
\end{aligned}$$

and with the same reason

$$\begin{aligned}
\int_0^{|\mathbb{E}(X)-a|^p} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds &\le |a-\mathbb{E}(X)|^p \wedge 1 \\
&\quad + \frac{(b-a)^2}{2}\left(\exp\left\{-\frac{2|a-\mathbb{E}(X)|^p\wedge 1}{(b-a)^2}\right\} - \exp\left\{-\frac{2|a-\mathbb{E}(X)|^p}{(b-a)^2}\right\}\right).
\end{aligned}$$

On the other hand, for any $p \ge 1$, it is clear that

$$\mathbb{E}|X-\mathbb{E}(X)|^p \le (b-a)^p.$$

From the above discussion, the desired results are obtained. □

Remark 3.2. $M_1(a,b,p,X)$ and $M_2(a,b,p,X)$ are sometimes better than $(b-a)^p$, e.g., if $p > 2$, taking $\mathbb{E}(X) = (a+b)/2$ and letting $1/2 < (b-a)/2 < 1$, then

$$\begin{aligned}
M_1(a,b,p,X) &= (b-a)^2\left(1-\exp\left\{-2^{1-p}(b-a)^{p-2}\right\}\right) \\
&< (b-a)^p\left(1-\exp\left\{-2^{1-p}(b-a)^{p-2}\right\}\right) \\
&< (b-a)^p.
\end{aligned}$$

Further, if $1 \le p < 2$, taking $\mathbb{E}(X) = (a+b)/2$ and letting $(b-a)/2 < 1$, then

$$M_2(a,b,p,X) = 2\left(\frac{(b-a)^p}{2^p}\right) = 2^{1-p}(b-a)^p \le (b-a)^p.$$

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