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THE STABILITY OF SOME LINEAR FUNCTIONAL EQUATIONS

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Abstract

In this note, we deal with the Baker's superstability for the following linear functional equations

$$\sum_{i=1}^{m} f(x+y+a_i) = f(x)f(y), \qquad x, y \in G,$$

$$\sum_{i=1}^{m} [f(x+y+a_i) + f(x-y-a_i)] = 2f(x)f(y), \qquad x, y \in G,$$

where G is an abelian group, $a_1, \dots, a_m$ $(m \in \mathbb{N})$ are arbitrary elements in G and f is a complex-valued function on G.

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Contents

1. Introduction

Let G be an abelian group. The main purpose of this paper is to generalize the results obtained in [4] and [5] for the linear functional equations

$$\sum_{i=1}^{m} f(x+y+a_i) = f(x)f(y), \qquad x, y \in G, \tag{1.1}$$

$$\sum_{i=1}^{m} [f(x+y+a_i) + f(x-y-a_i)] = 2f(x)f(y), \qquad x, y \in G, \tag{1.2}$$

where $a_1, \dots, a_m$ ($m \in \mathbb{N}$), are arbitrary elements in G and f is a complex-valued function on G. In the case where G is a locally compact group, the form of $L^\infty(G)$ solutions of (1.1) (resp. (1.2)) are determined in [2] (resp. [6]). Some particular cases of these linear functional equations are:

- The linear functional equations

$$f(x+y+a) = f(x)f(y), \qquad x, y \in G, \tag{1.3}$$

$$f(x+y+a) + f(x-y-a) = 2f(x)f(y), \qquad x, y \in G, \tag{1.4}$$

$$f(x+y+a) - f(x-y+a) = 2f(x)f(y), \qquad x, y \in G, \tag{1.5}$$

$$f(x+y+a) + f(x-y+a) = 2f(x)f(y), \qquad x, y \in G, \tag{1.6}$$

  see [1], [2], [6], [7] and [8].

- Cauchy's functional equation

$$f(x+y) = f(x)f(y), \qquad x, y \in G, \tag{1.7}$$

- D'Alembert's functional equation

$$f(x+y) + f(x+y) = 2f(x)f(y), \qquad x, y \in G. \tag{1.8}$$

To complete our consideration, we give some applications.

We shall need the results below for later use.

2. General Properties

Proposition 2.1. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$\left| \sum_{i=1}^{m} f(x+y+a_i) - f(x)f(y) \right| \leq \delta, \quad x, y \in G, \tag{2.1}$$

then one of the assertions is satisfied

i) If f is bounded, then

$$|f(x)| \leq \frac{m + \sqrt{m^2 + 4\delta}}{2}, \qquad x \in G. \tag{2.2}$$

ii) If f is unbounded, then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in G such that $f(z_n) \neq 0$ and $\lim_n |f(z_n)| = +\infty$ and that the convergence of the sequences of functions

$$x \to \frac{1}{f(z_n)} \sum_{i=1}^{m} f(z_n + x + a_i), \quad n \in \mathbb{N}, \tag{2.3}$$

to the function

$$x \to f(x),$$

$$x \to \frac{1}{f(z_n)} \sum_{i=1}^{m} f(z_n + x + y + a_j + a_i), \tag{2.4}$$

$$n \in \mathbb{N},\ 1 \leq j \leq m,\ y \in G,$$

to the function

$$x \to f(x + y + a_j),$$

is uniform.

Proof. i) Let $X = \sup |f|$, then for all $x \in G$ we have

$$|f(x)f(x)| \leq mX + \delta$$

from which we obtain that

$$X^2 - mX - \delta \leq 0$$

hence

$$X \leq \frac{m + \sqrt{m^2 + 4\delta}}{2}.$$

ii) Since f is unbounded then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in G such that $f(z_n) \neq 0$ and $\lim_n |f(z_n)| = +\infty$. Using (2.1) one has

$$\left| \frac{1}{f(z_n)} \sum_{i=1}^{m} f(z_n + x + a_i) - f(x) \right| \leq \frac{\delta}{|f(z_n)|}, x \in G, \; n \in \mathbb{N},$$

by letting $n \to \infty$, we obtain

$$\lim_n \frac{1}{f(z_n)} \sum_{i=1}^{m} f(z_n + x + a_i) = f(x)$$

and

$$\lim_n \frac{1}{f(z_n)} \sum_{i=1}^{m} f(z_n + x + y + a_j + a_i) = f(x + y + a_j).$$

□

Proposition 2.2. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$\left| \sum_{i=1}^{m} [f(x + y + a_i) + f(x - y - a_i)] - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G, \tag{2.5}$$

then one of the assertions is satisfied

i) If f is bounded, then

$$|f(x)| \leq \frac{m + \sqrt{m^2 + 2\delta}}{2}, \ x \in G. \tag{2.6}$$

ii) If f is unbounded, then there exists a sequence $(z_n)_{n\in\mathbb{N}} \in G$ such that $f(z_n) \neq 0$ and $\lim_n |f(z_n)| = +\infty$ and that the convergence of the sequences of functions

$$x \to \frac{1}{f(z_n)} \sum_{i=1}^{m} [f(z_n + x + a_i) + f(z_n - x - a_i)], \ \ n \in \mathbb{N}, \tag{2.7}$$

to the function

$$x \to 2f(x),$$

$$x \to \frac{1}{f(z_n)} \sum_{i=1}^{m} [f(z_n + x + y + a_j + a_i) + f(z_n - x - y - a_j - a_i)], \quad n \in \mathbb{N}, \ 1 \le j \le m, y \in G, \tag{2.8}$$

to the function

$$x \to 2f(x + y + a_j),$$

$$x \to \frac{1}{f(z_n)} \sum_{i=1}^{m} [f(z_n + x - y - a_j + a_i) + f(z_n - x + y + a_j - a_i)], \quad n \in \mathbb{N}, \ 1 \le j \le m, y \in G, \tag{2.9}$$

to the function

$$x \to 2f(x - y - a_j)$$

is uniform.

Proof. The proof is similar to the proof of Proposition 2.1.

i) Let $X = \sup |f|$, then for all $x \in G$ we have

$$X^2 - mX - \frac{\delta}{2} \le 0$$

hence

$$X \le \frac{m + \sqrt{m^2 + 2\delta}}{2}.$$

ii) Follows from the fact that

$$\left|\frac{1}{f(z_n)}\sum_{i=1}^{m}[f(z_n+x+a_i)+f(z_n-x-a_i)]-2f(x)\right| \leq \frac{\delta}{|f(z_n)|},\ x\in G,\ n\in\mathbb{N}.$$

□

3. The Main Results

The main results are the following theorems.

Theorem 3.1. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$\left|\sum_{i=1}^{m} f(x+y+a_i) - f(x)f(y)\right| \le \delta, \quad x, y \in G, \tag{3.1}$$

then either

$$|f(x)| \le \frac{m + \sqrt{m^2 + 4\delta}}{2}, \; x \in G, \tag{3.2}$$

or

$$\sum_{i=1}^{m} f(x+y+a_i) = f(x)f(y), \quad x, y \in G. \tag{3.3}$$

Proof. The idea is inspired by the paper [3].
If f is bounded, then from (2.2) we obtain the first case of the theorem. For the remainder, we get by using the assertion ii) in Proposition 2.1, for all $x, y \in G,\ n \in \mathbb{N}$

$$\left|\sum_{j=1}^{m} \frac{1}{f(z_n)} \sum_{i=1}^{m} f(z_n + x + y + a_j + a_i) - f(x)\frac{1}{f(z_n)} \sum_{j=1}^{m} f(z_n + y + a_j)\right|$$

$$\leq \sum_{j=1}^{m} \left| \frac{1}{f(z_n)} \left\{ \sum_{i=1}^{m} f(z_n + x + y + a_j + a_i) - f(x)f(z_n + y + a_j) \right\} \right|$$
$$\leq \frac{m\delta}{|f(z_n)|},$$

since the convergence is uniform, we have

$$\left| \sum_{i=1}^{m} f(x + y + a_i) - f(x)f(y) \right| \leq 0.$$

i.e. f is a solution of the functional equation (1.1). □

Theorem 3.2. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$\left| \sum_{i=1}^{m} [f(x + y + a_i) + f(x - y - a_i)] - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G, \tag{3.4}$$

then either

$$|f(x)| \leq \frac{m + \sqrt{m^2 + 2\delta}}{2}, \; x \in G. \tag{3.5}$$

or

$$\sum_{i=1}^{m} [f(x + y + a_i) + f(x - y - a_i)] = 2f(x)f(y), \quad x, y \in G. \tag{3.6}$$

Proof. By the assertion i) in Proposition 2.2 we get the first case of the theorem. For the second case we have by the inequality (3.4) that

$$\Bigg|\sum_{j=1}^{m}\frac{1}{f(z_n)}\left\{\sum_{i=1}^{m}[f(z_n+x+y+a_j+a_i)+f(z_n-x-y-a_j-a_i)]\right\}$$
$$+\sum_{j=1}^{m}\frac{1}{f(z_n)}\left\{\sum_{i=1}^{m}[f(z_n+x-y-a_j+a_i)+f(z_n-x+y+a_j-a_i)]\right\}$$
$$-2f(x)\frac{1}{f(z_n)}\sum_{j=1}^{m}[f(z_n+y+a_j)+f(z_n-y-a_j)]\Bigg|$$
$$=\Bigg|\sum_{j=1}^{m}\frac{1}{f(z_n)}\Bigg\{\sum_{i=1}^{m}[f(z_n+x+y+a_j+a_i)+f(z_n-x+y+a_j-a_i)]$$
$$-2f(x)f(z_n+y+a_j)\Bigg\}\Bigg|$$
$$+\Bigg|\sum_{j=1}^{m}\frac{1}{f(z_n)}\Bigg\{\sum_{i=1}^{m}[f(z_n+x-y-a_j+a_i)+f(z_n-x-y-a_j-a_i)]$$
$$-2f(x)f(z_n-y-a_j)\Bigg\}\Bigg|$$
$$\leq\sum_{j=1}^{m}\Bigg|\frac{1}{f(z_n)}\Bigg\{\sum_{i=1}^{m}[f(z_n+x+y+a_j+a_i)+f(z_n-x+y+a_j-a_i)]$$
$$-2f(x)f(z_n+y+a_j)\Bigg\}\Bigg|$$

$$+\sum_{j=1}^{m}\left|\frac{1}{f(z_n)}\left\{\sum_{i=1}^{m}[f(z_n+x-y-a_j+a_i)+f(z_n-x-y-a_j-a_i)]\right.\right.$$
$$\left.\left.-2f(x)f(z_n-y-a_j)\right\}\right|$$
$$\leq\frac{2m\delta}{|f(z_n)|},$$

since the convergence is uniform, we have

$$\left|2\sum_{i=1}^{m}[f(x+y+a_i)+f(x-y-a_i)]-4f(x)f(y)\right|\leq 0.$$

i.e. f is a solution of the functional equation (1.2). □

4. Applications

From Theorems 3.1 and 3.2, we easily obtain .

Corollary 4.1. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$|f(x+y+a) - f(x)f(y)| \leq \delta, \quad x, y \in G, \tag{4.1}$$

then either

$$|f(x)| \leq \frac{1+\sqrt{1+4\delta}}{2}, \ x \in G. \tag{4.2}$$

or

$$f(x+y+a) = f(x)f(y) \ \ x, y \in G. \tag{4.3}$$

Remark 4.1. *Taking $a = 0$ in Corollary 4.1, we find the result obtained in [4].*

Corollary 4.2. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$|f(x+y+a) + f(x-y-a) - 2f(x)f(y)| \leq \delta, \quad x, y \in G, \tag{4.4}$$

then either

$$|f(x)| \leq \frac{1+\sqrt{1+2\delta}}{2}, \ x \in G, \tag{4.5}$$

or

$$f(x+y+a) + f(x-y-a) = 2f(x)f(y), \quad x, y \in G. \tag{4.6}$$

Remark 4.2. *Taking $a = 0$ in Corollary 4.2, we find the result obtained in [5].*

Corollary 4.3. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$\left|\sum_{i=1}^{m}[f(x+y+a_i) - f(x-y+a_i)] - 2f(x)f(y)\right| \leq \delta, \quad x, y \in G, \tag{4.7}$$

then either

$$|f(x)| \leq \frac{m+\sqrt{m^2+2\delta}}{2}, \quad x \in G, \tag{4.8}$$

or

$$\sum_{i=1}^{m}[f(x+y+a_i) + f(x-y-a_i)] = 2f(x)f(y), \quad x, y \in G. \tag{4.9}$$

Proof. Let f be a complex-valued function defined on G which satisfies the inequality (4.7), then for all $x, y \in G$ we have

$$\begin{aligned}
2&|f(x)||f(y)+f(-y)| \\
&= |2f(x)f(y) + 2f(x)f(-y)| \\
&= \left|\sum_{i=1}^{m}[f(x+y+a_i) - f(x-y+a_i)]\right. \\
&\qquad \left. - \sum_{i=1}^{m}[f(x+y+a_i) - f(x-y+a_i)] + 2f(x)f(y) + 2f(x)f(-y)\right|
\end{aligned}$$

$$\leq \left| 2f(x)f(y) - \sum_{i=1}^{m}[f(x+y+a_i) - f(x-y+a_i)] \right| + \left| 2f(x)f(-y) - \sum_{i=1}^{m}[f(x-y+a_i) - f(x+y+a_i)] \right| \leq 2\delta.$$

Since f is unbounded it follows that $f(-y) = -f(y)$, for all $y \in G$. Consequently f satisfies the inequality (3.4) and one has the remainder. □

Corollary 4.4. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$\left| \sum_{i=1}^{m}[f(x+y+a_i) + f(x-y+a_i)] - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G, \tag{4.10}$$

then either

$$|f(x)| \leq \frac{m + \sqrt{m^2 + 2\delta}}{2}, \quad x \in G, \tag{4.11}$$

or

$$\sum_{i=1}^{m}[f(x+y+a_i) + f(x-y-a_i)] = 2f(x)f(y) \quad x, y \in G. \tag{4.12}$$

Proof. Let f be a complex-valued function defined on G which satisfies the inequality (4.10), then for all $x, y \in G$ we have

$$\begin{aligned}
2|f(x)||f(y) - f(-y)| \\
&= |2f(x)f(y) - 2f(x)f(-y)| \\
&= \Bigg| \sum_{i=1}^{m} [f(x + y + a_i) + f(x - y + a_i)] \\
&\qquad - \sum_{i=1}^{m} [f(x + y + a_i) + f(x - y + a_i)] \\
&\qquad + 2f(x)f(y) - 2f(x)f(-y) \Bigg| \\
&\le \left| \sum_{i=1}^{m} [f(x - y + a_i) + f(x + y + a_i)] - 2f(x)f(-y) \right| \\
&\qquad + \left| \sum_{i=1}^{m} [f(x + y + a_i) + f(x - y + a_i)] - 2f(x)f(y) \right| \\
&\le 2\delta.
\end{aligned}$$

Since f is unbounded it follows that $f(-y) = f(y)$, for all $y \in G$. Consequently f satisfies the inequality (3.4) and one has the remainder. □

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