

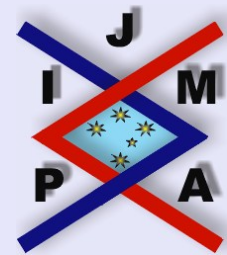
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THE STABILITY OF SOME LINEAR FUNCTIONAL EQUATIONS

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Abstract

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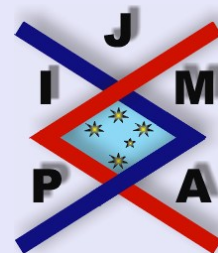


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Abstract

In this note, we deal with the Baker's superstability for the following linear functional equations

$$\sum_{i=1}^m f(x+y+a_i) = f(x)f(y), \quad x, y \in G,$$

$$\sum_{i=1}^m [f(x+y+a_i) + f(x-y-a_i)] = 2f(x)f(y), \quad x, y \in G,$$

where G is an abelian group, a_1, \dots, a_m ($m \in \mathbf{N}$) are arbitrary elements in G and f is a complex-valued function on G .

2000 Mathematics Subject Classification: 39B72.

Key words: Linear functional equations, Stability, Superstability.

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1. Introduction

Let G be an abelian group. The main purpose of this paper is to generalize the results obtained in [4] and [5] for the linear functional equations

$$(1.1) \quad \sum_{i=1}^m f(x + y + a_i) = f(x)f(y), \quad x, y \in G,$$

$$(1.2) \quad \sum_{i=1}^m [f(x + y + a_i) + f(x - y - a_i)] = 2f(x)f(y), \quad x, y \in G,$$

where a_1, \dots, a_m ($m \in \mathbb{N}$), are arbitrary elements in G and f is a complex-valued function on G . In the case where G is a locally compact group, the form of $L^\infty(G)$ solutions of (1.1) (resp. (1.2)) are determined in [2] (resp. [6]). Some particular cases of these linear functional equations are:

- The linear functional equations

$$(1.3) \quad f(x + y + a) = f(x)f(y), \quad x, y \in G,$$

$$(1.4) \quad f(x + y + a) + f(x - y - a) = 2f(x)f(y), \quad x, y \in G,$$

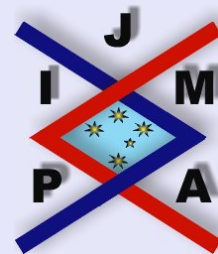
$$(1.5) \quad f(x + y + a) - f(x - y + a) = 2f(x)f(y), \quad x, y \in G,$$

$$(1.6) \quad f(x + y + a) + f(x - y + a) = 2f(x)f(y), \quad x, y \in G,$$

see [1], [2], [6], [7] and [8].

- Cauchy's functional equation

$$(1.7) \quad f(x + y) = f(x)f(y), \quad x, y \in G,$$



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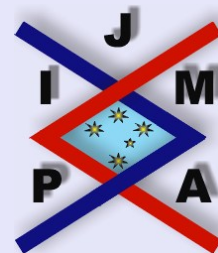
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- D'Alembert's functional equation

$$(1.8) \quad f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in G.$$

To complete our consideration, we give some applications.

We shall need the results below for later use.



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2. General Properties

Proposition 2.1. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$(2.1) \quad \left| \sum_{i=1}^m f(x + y + a_i) - f(x)f(y) \right| \leq \delta, \quad x, y \in G,$$

then one of the assertions is satisfied

i) *If f is bounded, then*

$$(2.2) \quad |f(x)| \leq \frac{m + \sqrt{m^2 + 4\delta}}{2}, \quad x \in G.$$

ii) *If f is unbounded, then there exists a sequence $(z_n)_{n \in \mathbb{N}}$ in G such that $f(z_n) \neq 0$ and $\lim_n |f(z_n)| = +\infty$ and that the convergence of the sequences of functions*

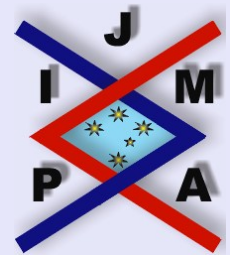
$$(2.3) \quad x \rightarrow \frac{1}{f(z_n)} \sum_{i=1}^m f(z_n + x + a_i), \quad n \in \mathbb{N},$$

to the function

$$x \rightarrow f(x),$$

$$(2.4) \quad x \rightarrow \frac{1}{f(z_n)} \sum_{i=1}^m f(z_n + x + y + a_j + a_i),$$

$$n \in \mathbb{N}, 1 \leq j \leq m, y \in G,$$



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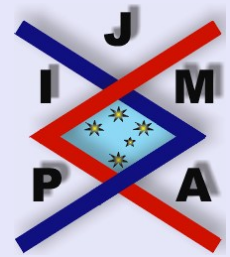


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to the function

$$x \rightarrow f(x + y + a_j),$$

is uniform.

Proof. i) Let $X = \sup |f|$, then for all $x \in G$ we have

$$|f(x)f(x)| \leq mX + \delta$$

from which we obtain that

$$X^2 - mX - \delta \leq 0$$

hence

$$X \leq \frac{m + \sqrt{m^2 + 4\delta}}{2}.$$

ii) Since f is unbounded then there exists a sequence $(z_n)_{n \in \mathbb{N}}$ in G such that $f(z_n) \neq 0$ and $\lim_n |f(z_n)| = +\infty$. Using (2.1) one has

$$\left| \frac{1}{f(z_n)} \sum_{i=1}^m f(z_n + x + a_i) - f(x) \right| \leq \frac{\delta}{|f(z_n)|}, x \in G, n \in \mathbb{N},$$

by letting $n \rightarrow \infty$, we obtain

$$\lim_n \frac{1}{f(z_n)} \sum_{i=1}^m f(z_n + x + a_i) = f(x)$$

and

$$\lim_n \frac{1}{f(z_n)} \sum_{i=1}^m f(z_n + x + y + a_j + a_i) = f(x + y + a_j).$$

□

Proposition 2.2. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$(2.5) \quad \left| \sum_{i=1}^m [f(x + y + a_i) + f(x - y - a_i)] - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G,$$

then one of the assertions is satisfied

i) If f is bounded, then

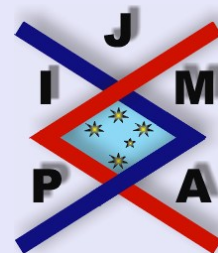
$$(2.6) \quad |f(x)| \leq \frac{m + \sqrt{m^2 + 2\delta}}{2}, \quad x \in G.$$

ii) If f is unbounded, then there exists a sequence $(z_n)_{n \in \mathbb{N}} \in G$ such that $f(z_n) \neq 0$ and $\lim_n |f(z_n)| = +\infty$ and that the convergence of the sequences of functions

$$(2.7) \quad x \rightarrow \frac{1}{f(z_n)} \sum_{i=1}^m [f(z_n + x + a_i) + f(z_n - x - a_i)], \quad n \in \mathbb{N},$$

to the function

$$x \rightarrow 2f(x),$$



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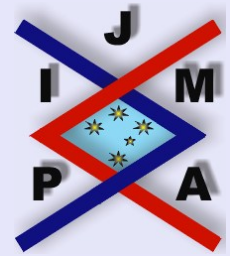


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$$(2.8) \quad x \rightarrow \frac{1}{f(z_n)} \sum_{i=1}^m [f(z_n + x + y + a_j + a_i) + f(z_n - x - y - a_j - a_i)],$$

$$n \in \mathbb{N}, \quad 1 \leq j \leq m, y \in G,$$

to the function

$$x \rightarrow 2f(x + y + a_j),$$

$$(2.9) \quad x \rightarrow \frac{1}{f(z_n)} \sum_{i=1}^m [f(z_n + x - y - a_j + a_i) + f(z_n - x + y + a_j - a_i)],$$

$$n \in \mathbb{N}, \quad 1 \leq j \leq m, y \in G,$$

to the function

$$x \rightarrow 2f(x - y - a_j)$$

is uniform.

Proof. The proof is similar to the proof of Proposition 2.1.

i) Let $X = \sup |f|$, then for all $x \in G$ we have

$$X^2 - mX - \frac{\delta}{2} \leq 0$$

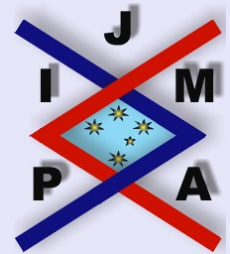
hence

$$X \leq \frac{m + \sqrt{m^2 + 2\delta}}{2}.$$

ii) Follows from the fact that

$$\left| \frac{1}{f(z_n)} \sum_{i=1}^m [f(z_n + x + a_i) + f(z_n - x - a_i)] - 2f(x) \right| \leq \frac{\delta}{|f(z_n)|}, \quad x \in G, \quad n \in \mathbb{N}.$$

□



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3. The Main Results

The main results are the following theorems.

Theorem 3.1. Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that

$$(3.1) \quad \left| \sum_{i=1}^m f(x + y + a_i) - f(x)f(y) \right| \leq \delta, \quad x, y \in G,$$

then either

$$(3.2) \quad |f(x)| \leq \frac{m + \sqrt{m^2 + 4\delta}}{2}, \quad x \in G,$$

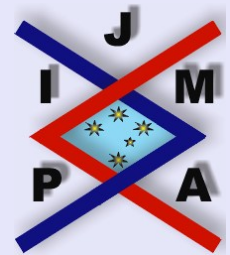
or

$$(3.3) \quad \sum_{i=1}^m f(x + y + a_i) = f(x)f(y), \quad x, y \in G.$$

Proof. The idea is inspired by the paper [3].

If f is bounded, then from (2.2) we obtain the first case of the theorem. For the remainder, we get by using the assertion ii) in Proposition 2.1, for all $x, y \in G$, $n \in \mathbb{N}$

$$\left| \sum_{j=1}^m \frac{1}{f(z_n)} \sum_{i=1}^m f(z_n + x + y + a_j + a_i) - f(x) \frac{1}{f(z_n)} \sum_{j=1}^m f(z_n + y + a_j) \right|$$



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$$\leq \sum_{j=1}^m \left| \frac{1}{f(z_n)} \left\{ \sum_{i=1}^m f(z_n + x + y + a_j + a_i) - f(x)f(z_n + y + a_j) \right\} \right|$$

$$\leq \frac{m\delta}{|f(z_n)|},$$

since the convergence is uniform, we have

$$\left| \sum_{i=1}^m f(x + y + a_i) - f(x)f(y) \right| \leq 0.$$

i.e. f is a solution of the functional equation (1.1). □

Theorem 3.2. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

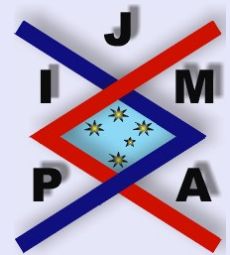
$$(3.4) \quad \left| \sum_{i=1}^m [f(x + y + a_i) + f(x - y - a_i)] - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G,$$

then either

$$(3.5) \quad |f(x)| \leq \frac{m + \sqrt{m^2 + 2\delta}}{2}, \quad x \in G.$$

or

$$(3.6) \quad \sum_{i=1}^m [f(x + y + a_i) + f(x - y - a_i)] = 2f(x)f(y), \quad x, y \in G.$$



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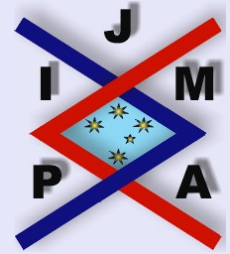
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Proof. By the assertion i) in Proposition 2.2 we get the first case of the theorem. For the second case we have by the inequality (3.4) that

$$\begin{aligned}
 & \left| \sum_{j=1}^m \frac{1}{f(z_n)} \left\{ \sum_{i=1}^m [f(z_n + x + y + a_j + a_i) + f(z_n - x - y - a_j - a_i)] \right\} \right. \\
 & \quad \left. + \sum_{j=1}^m \frac{1}{f(z_n)} \left\{ \sum_{i=1}^m [f(z_n + x - y - a_j + a_i) + f(z_n - x + y + a_j - a_i)] \right\} \right. \\
 & \quad \left. - 2f(x) \frac{1}{f(z_n)} \sum_{j=1}^m [f(z_n + y + a_j) + f(z_n - y - a_j)] \right| \\
 &= \left| \sum_{j=1}^m \frac{1}{f(z_n)} \left\{ \sum_{i=1}^m [f(z_n + x + y + a_j + a_i) + f(z_n - x + y + a_j - a_i)] \right. \right. \\
 & \quad \left. \left. - 2f(x)f(z_n + y + a_j) \right\} \right| \\
 & \quad + \left| \sum_{j=1}^m \frac{1}{f(z_n)} \left\{ \sum_{i=1}^m [f(z_n + x - y - a_j + a_i) + f(z_n - x - y - a_j - a_i)] \right. \right. \\
 & \quad \left. \left. - 2f(x)f(z_n - y - a_j) \right\} \right| \\
 &\leq \sum_{j=1}^m \left| \frac{1}{f(z_n)} \left\{ \sum_{i=1}^m [f(z_n + x + y + a_j + a_i) + f(z_n - x + y + a_j - a_i)] \right. \right. \\
 & \quad \left. \left. - 2f(x)f(z_n + y + a_j) \right\} \right|
 \end{aligned}$$



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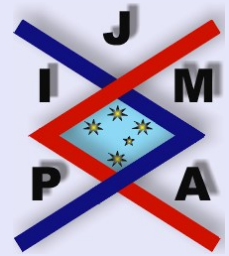


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$$\begin{aligned}
 & + \sum_{j=1}^m \left| \frac{1}{f(z_n)} \left\{ \sum_{i=1}^m [f(z_n + x - y - a_j + a_i) + f(z_n - x - y - a_j - a_i)] \right. \right. \\
 & \quad \left. \left. - 2f(x)f(z_n - y - a_j) \right\} \right| \\
 & \leq \frac{2m\delta}{|f(z_n)|},
 \end{aligned}$$

since the convergence is uniform, we have

$$\left| 2 \sum_{i=1}^m [f(x + y + a_i) + f(x - y - a_i)] - 4f(x)f(y) \right| \leq 0.$$

i.e. f is a solution of the functional equation (1.2). □

4. Applications

From Theorems 3.1 and 3.2, we easily obtain .

Corollary 4.1. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$(4.1) \quad |f(x + y + a) - f(x)f(y)| \leq \delta, \quad x, y \in G,$$

then either

$$(4.2) \quad |f(x)| \leq \frac{1 + \sqrt{1 + 4\delta}}{2}, \quad x \in G.$$

or

$$(4.3) \quad f(x + y + a) = f(x)f(y) \quad x, y \in G.$$

Remark 4.1. *Taking $a = 0$ in Corollary 4.1, we find the result obtained in [4].*

Corollary 4.2. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

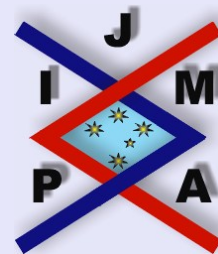
$$(4.4) \quad |f(x + y + a) + f(x - y - a) - 2f(x)f(y)| \leq \delta, \quad x, y \in G,$$

then either

$$(4.5) \quad |f(x)| \leq \frac{1 + \sqrt{1 + 2\delta}}{2}, \quad x \in G,$$

or

$$(4.6) \quad f(x + y + a) + f(x - y - a) = 2f(x)f(y), \quad x, y \in G.$$



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Remark 4.2. Taking $a = 0$ in Corollary 4.2, we find the result obtained in [5].

Corollary 4.3. Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that

$$(4.7) \quad \left| \sum_{i=1}^m [f(x+y+a_i) - f(x-y+a_i)] - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G,$$

then either

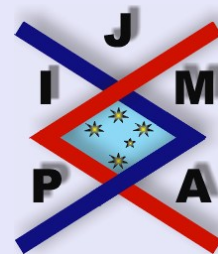
$$(4.8) \quad |f(x)| \leq \frac{m + \sqrt{m^2 + 2\delta}}{2}, \quad x \in G,$$

or

$$(4.9) \quad \sum_{i=1}^m [f(x+y+a_i) + f(x-y-a_i)] = 2f(x)f(y), \quad x, y \in G.$$

Proof. Let f be a complex-valued function defined on G which satisfies the inequality (4.7), then for all $x, y \in G$ we have

$$\begin{aligned} & 2|f(x)||f(y) + f(-y)| \\ &= |2f(x)f(y) + 2f(x)f(-y)| \\ &= \left| \sum_{i=1}^m [f(x+y+a_i) - f(x-y+a_i)] \right. \\ & \quad \left. - \sum_{i=1}^m [f(x+y+a_i) - f(x-y+a_i)] + 2f(x)f(y) + 2f(x)f(-y) \right| \end{aligned}$$



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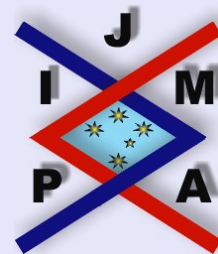


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$$\begin{aligned} &\leq \left| 2f(x)f(y) - \sum_{i=1}^m [f(x+y+a_i) - f(x-y+a_i)] \right| \\ &\quad + \left| 2f(x)f(-y) - \sum_{i=1}^m [f(x-y+a_i) - f(x+y+a_i)] \right| \\ &\leq 2\delta. \end{aligned}$$

Since f is unbounded it follows that $f(-y) = -f(y)$, for all $y \in G$. Consequently f satisfies the inequality (3.4) and one has the remainder. \square

Corollary 4.4. *Let $\delta > 0$. Let G be an abelian group and let f be a complex-valued function defined on G such that*

$$(4.10) \quad \left| \sum_{i=1}^m [f(x+y+a_i) + f(x-y+a_i)] - 2f(x)f(y) \right| \leq \delta, \quad x, y \in G,$$

then either

$$(4.11) \quad |f(x)| \leq \frac{m + \sqrt{m^2 + 2\delta}}{2}, \quad x \in G,$$

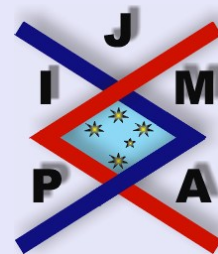
or

$$(4.12) \quad \sum_{i=1}^m [f(x+y+a_i) + f(x-y-a_i)] = 2f(x)f(y) \quad x, y \in G.$$

Proof. Let f be a complex-valued function defined on G which satisfies the inequality (4.10), then for all $x, y \in G$ we have

$$\begin{aligned}
 & 2|f(x)||f(y) - f(-y)| \\
 &= |2f(x)f(y) - 2f(x)f(-y)| \\
 &= \left| \sum_{i=1}^m [f(x+y+a_i) + f(x-y+a_i)] \right. \\
 &\quad \left. - \sum_{i=1}^m [f(x+y+a_i) + f(x-y+a_i)] \right. \\
 &\quad \left. + 2f(x)f(y) - 2f(x)f(-y) \right| \\
 &\leq \left| \sum_{i=1}^m [f(x-y+a_i) + f(x+y+a_i)] - 2f(x)f(-y) \right| \\
 &\quad + \left| \sum_{i=1}^m [f(x+y+a_i) + f(x-y+a_i)] - 2f(x)f(y) \right| \\
 &\leq 2\delta.
 \end{aligned}$$

Since f is unbounded it follows that $f(-y) = f(y)$, for all $y \in G$. Consequently f satisfies the inequality (3.4) and one has the remainder. \square



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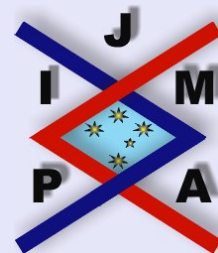
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Belaid Bouikhalene

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