Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 1, Article 24, 2003

SOME VARIANTS OF ANDERSON'S INEQUALITY IN C_1-CLASSES

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Received 15 October, 2002; accepted 19 February, 2003
Communicated by A.G. Babenko

ABSTRACT. The main purpose of this note is to characterize the operators $S \in \ker \Delta_{A,B} \cap C_1$ which are orthogonal to the range of elementary operators, where S is not a smooth point in C_1 by using the $\varphi-$directional derivative.

Key words and phrases: Elementary operators, Schatten $p-$classes, Orthogonality.

2000 *Mathematics Subject Classification.* Primary 47B47, 47A30, 47B20; Secondary 47B10.

1. INTRODUCTION

Let E be a complex Banach space. We first define orthogonality in E. We say that $b \in E$ is orthogonal to $a \in E$ if for all complex λ there holds

$$\|a + \lambda b\| \geq \|a\| . \tag{1.1}$$

This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint with the open ball $K(0, \|a\|)$, i.e, if and only if this complex line is a tangent one.

Note that if b is orthogonal to a, then a need not be orthogonal to b. If E is a Hilbert space, then from (1.1) follows $\langle a, b\rangle = 0$, i.e, orthogonality in the usual sense. This notion and first results concerning the orthogonality in linear metric space was given by G. Birkhoff [2].

Next we define the von Neumann-Schatten classes C_p $(1 \leq p < \infty)$. Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space H and let $T \in B(H)$ be compact, and let $s_1(X) \geq s_2(X) \geq \cdots \geq 0$ denote the singular

ISSN (electronic): 1443-5756

This Work was supported by the research center project No. **Math/1422/10** and for the second author by the research center project No. **Math/1422/22**.
017-03

values of T, i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to belong to the Schatten $p-$classes C_p if

$$\|T\|_p = \left[\sum_{j=1}^{\infty} s_j(T)^p\right]^{\frac{1}{p}} = [tr(T)^p]^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

where tr denotes the trace functional. Hence C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ is the class of compact operators with

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$$

denoting the usual operator norm. For the general theory of the Schatten $p-$classes the reader is referred to [13].

Recall that the norm $\|\cdot\|$ of the B-space V is said to be Gâteaux differentiable at non-zero elements $x \in V$ if

$$\lim_{\mathbb{R}\ni t\to 0} \frac{\|x+ty\| - \|x\|}{t} = \operatorname{Re} D_x(y),$$

for all $y \in V$. Here $\mathbb{R}$ denotes the set of all reals, Re denotes the real part and D_x is the unique support functional (in the dual space V^*) such that $\|D_x\| = 1$ and $D_x(x) = \|x\|$. The Gâteaux differentiability of the norm at x implies that x is a smooth point of the sphere of radius $\|x\|$. It is well known (see [7] and references therein) that for $1 < p < \infty$, C_p is a uniformly convex Banach space. Therefore every non-zero $T \in C_p$ is a smooth point and in this case the support functional of T is given by

$$D_T(X) = tr\left[\frac{|T|^{p-1} UX^*}{\|T\|_p^{p-1}}\right]$$

for all $X \in C_p$ $(1 < p < \infty)$, where $T = U|T|$ is the polar decomposition of T.

In [1] Anderson proved that if A is a normal operator on Hilbert space H, then $AS = SA$ implies that for all bounded linear operator X there holds

$$\|S + AX - XA\| \ge \|S\|. \tag{1.2}$$

This means that the range of the derivation $\delta_A : B(H) \to B(H)$ defined by $\delta_A(X) = AX - XA$ is orthogonal to its kernel. This result has been generalized in two directions, by extending the class of elementary mappings

$$\tilde{E} : B(H) \to B(H); \tilde{E}(X) = \sum_{i=1}^{n} A_i X B_i$$

and

$$E : B(H) \to B(H); E(X) = \sum_{i=1}^{n} A_i X B_i - X,$$

where $(A_1, A_2, \ldots, A_n), (B_1, B_2, \ldots, B_n)$ are $n-$tuples of bounded operators on H and by extending the inequality (1.2) to C_p-classes with $1 < p < \infty$, see [3], [7], [10] and [11].

The Gâteaux derivative concept was used in [4], [5], [6], [8], [9] and [15] and, in order to characterize those operators for which the range of a derivation is orthogonal. In these papers, the attention was directed to C_p-classes for some $p > 1$.

The main purpose of this note is to characterize the operators $S \in C_1$ which are orthogonal to the range of elementary operators, where S is not a smooth point in C_1 by using the φ-directional derivative.

Recall that the operator S is a smooth point of the corresponding sphere in C_1 if and only if either S is injective or S^* is injective.

It is very interesting to point out that this result has been done in C_p-classes with $1 < p < \infty$ but, at least to our acknowledge, it was not given, till now, for C_1-classes.

It is well known see ([6]) that the norm $\|\cdot\|$ of the B-space V is said to be φ-directional differentiable at non-zero elements $x \in V$ if

$$\lim_{\mathbb{R}\ni t\to 0} \frac{\|x + te^{i\varphi}y\| - \|x\|}{t} = D_{x,\varphi}(y),$$

for all $y \in V$. Therefore for every non-zero $T \in C_1$ which is not a smooth point, the support functional of T is given by

$$D_{\varphi,T}(S) = \operatorname{Re}\left\{e^{i\varphi} tr(U^*Y)\right\} + \|QYP\|_{C_1},$$

for all $X \in C_1$, where $S = U|S|$ is the polar decomposition of X, $P = P_{\ker X}$, $Q = Q_{\ker X^*}$.

2. MAIN RESULTS

Let $\phi : B(H) \to B(H)$ be a linear map, that is, $\phi(\alpha X + \beta Y) = \alpha\phi(X) + \beta\phi(Y)$, for all α, β, X, Y, and satisfying the following condition:

$$tr(X\phi(Y)) = tr(\phi(X)Y), \text{ for all } X, Y \in C_1.$$

Let $S \in C_1$ and put

$$\mathcal{U} = \{X \in B(H) : \phi(X) \in C_1\}.$$

Let $\psi : \mathcal{U} \to C_1$ defined by

$$\psi(X) = S + \phi(X).$$

Theorem 2.1. [12] *Let $V \in C_1$. Then,*

$$\|S + \phi(X)\|_{C_1} \geq \|\psi(S)\|_{C_1}, \text{ for all } X \in C_1,$$

if and only if $U^* \in \ker\phi$, where $\psi(V) = U|\psi(V)|$.

As a first consequence of this result we have the following theorem.

Theorem 2.2. *Let $S \in C_1 \cap \ker\phi$. The following assertions are equivalent:*

(1)
$$\|S + \phi(X)\|_{C_1} \geq \|S\|_{C_1}, \text{ for all } X \in C_1,$$

(2) $U^* \in \ker\phi$, *where $S = U|S|$.*

Our main purpose in this paper is to use the general result in Theorem 2.1 in order to characterize all those operators $S \in C_1 \cap \ker\phi$ which are orthogonal to $Ran(\phi \mid C_1)$ (the range of $\phi \mid C_1$) when ϕ is one of the following elementary operators:

(1) $E_{A,B} : B(H) \to B(H)$ defined by

$$E_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X,$$

where $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ are n-tuples of operators in $B(H)$.

(2) $\Delta_{A,B} : B(H) \to B(H)$ defined by

$$\Delta_{A,B}(X) = AXB - X,$$

where A and B are operators in $B(H)$.

(3) $\delta_{A,B} : B(H) \to B(H)$ is defined by

$$\delta_{A,B}(X) = AX - XB,$$

where A and B are operators in $B(H)$.

(4) $\tilde{E}_{A,B} : B(H) \to B(H)$ is defined by

$$\tilde{E}_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i$$

where $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ are n -tuples of operators in $B(H)$.

Note that all the elementary operators recalled above satisfy the assumptions assumed on our abstract general map ϕ.

Let us begin by proving our main results for the elementary operator E.

Theorem 2.3. *Let $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n)$ be n-tuples of operators in $B(H)$ such that*

$$\ker\ E_{A,B}|C_1 \subseteq \ker\ E_{A^*,B^*}|C_1.$$

Assume that

$$\sum_{i=1}^{n} A_i A_i^* \leq 1,\ \sum_{i=1}^{n} A_i^* A_i \leq 1,\ \sum_{i=1}^{n} B_i B_i^* \leq 1 \text{ and } \sum_{i=1}^{n} B_i^* B_i \leq 1 \tag{2.1}$$

and let $S = U\,|S| \in C_1$. Then $S \in \ker\ E_{A,B}$ if, and only if,

$$\|S + E_{A,B}(X)\|_1 \geq \|S\|_1\,,$$

for all $X \in C_1$.

Proof. Let $S \in \ker\ E_{A,B}|C_1$. Then it follows from Theorem 2.1 that

$$\|S + E_{A,B}(X)\|_1 \geq \|S\|_1\,, \tag{2.2}$$

for all $X \in C_1$ if and only if $U^* \in \ker\ E_{A,B}$. The hypothesis $\ker\ E_{A,B} \subseteq \ker\ E_{A^*,B^*}$, implies that $U^* \in \ker\ E_{A^*,B^*}$. Note that $U^* \in \ker\ E_{A,B} \subseteq \ker\ E_{A^*,B^*}$ if and only if

$$tr(U^* E_{A,B}(X)) = 0 = tr(U^* E_{A^*,B^*}(X)). \tag{2.3}$$

Choosing $X \in C_1$ to be the rank one operator $x \otimes y$ it follows from (2.3) that if (2.2) holds then

$$= tr\left(\left(\sum_{i=1}^{n} B_i U^* A_i - U^*\right)(x \otimes y)\right)$$

$$= \left(\sum_{i=1}^{n} B_i U^* A_i x, y\right) - (U^* x, y) = 0$$

and

$$\left(\sum_{i=1}^{n} B_i^* U^* A_i^* x, y\right) - (U^* x, y) = 0$$

for all $x, y \in H$ or

$$E_{A,B}(U) = 0 = E_{A^*,B^*}(U).$$

It is known that if $\sum_{i=1}^{n} B_i B_i^* \leq 1,\ \sum_{i=1}^{n} B_i^* B_i \leq 1$ and $E_{B,B}(S) = 0 = E_{B,B}^*(S)$, then the eigenspaces corresponding to distinct non-zero eigenvalues of the compact positive operator $|S|^2$ reduces each B_i see ([3, Theorem 8], [15, Lemma 2.3]). In particular $|S|$ commutes with each B_i for all $1 \leq i \leq n$. Hence (2.2) holds if and only if,

$$E_{A,B}(S) = 0 = E_{A,B}^*(S).$$

□

Now, we prove a similar result for the operator $\Delta_{A,B}$. Note that in this case we don't need the condition (2.1).

Theorem 2.4. *Let A and B be two operators in $B(H)$ such that*

$$\ker\,\Delta_{A,B}|C_1 \subseteq \ker\,\Delta_{A^*,B^*}|C_1$$

and assume that $S = U\,|S| \in C_1$. Then $S \in \ker\,\Delta_{A,B}|C_1$ if and only if,

$$\|S + \Delta_{A,B}(X)\|_1 \geq \|S\|_1\,, \tag{2.4}$$

for all $X \in C_1$.

Proof. Let $S \in \ker\,\Delta_{A,B}|C_1$. Then it follows from Theorem 2.1 that

$$\|S + \Delta_{A,B}(X)\|_1 \geq \|S\|_1\,,$$

for all $X \in C_1$ if and only if $U^* \in \ker\,\Delta_{A,B}$. By the same arguments as in the proof of the above theorem, it follows that (2.4) holds if and only if

$$AUB = U = A^*UB^* \quad \text{or} \quad B^*U^*A^* = U^* = BU^*A.$$

Multiplying at right by $|S|$ we get

$$AUB|S| = U|S| = A^*UB^*|S|. \tag{2.5}$$

Now as $S \in \ker\,\Delta_{A,B}|C_1 \subseteq \ker\,\Delta_{A^*,B^*}|C_1$, i.e.,

$$ASB = S = A^*SB^*A \quad \text{or} \quad B^*S^*A^* = S^* = BS^*A,$$

then

$$BS^*S = BS^*ASB = S^*SB, \text{ i.e., } B|S| = |S|B.$$

We also get $A|S| = |S|A$, that is, both operators A and B commute with $|S|$. Thus, (2.5) is equivalent to

$$AU|S|B = U|S| = A^*U|S|B^*, \quad \text{i.e., } ASB = S = A^*SB^*.$$

Thus $S \in \ker\,\Delta_{A,B}$. □

Remark 2.5. The above theorem is still true if we consider instead of $\Delta_{A,B}$ the generalized derivation $\delta_{A,B}(X) = AX - XB$. It is still possible to characterize the operators $S \in \ker\phi_{A,B} \cap C_1$ which are orthogonal to $Ran(\phi_{A,B})$, where $\phi_{A,B} = AXB + CXD$. In [13] Shulman stated that there exists a normally represented elementary operator of the form $\sum_{i=1}^{n} A_iXB_i$ with $n > 2$ such that asc$E > 1$, i.e. the range and the kernel have non trival intersection. Hence Theorem 2.1 does not hold in the case where $E_{A,B}$ is replaced by $\phi_{A,B} = \sum_{i=3}^{n} A_iXB_i$

Corollary 2.6. *Let A, B be normal operators in $B(H)$ and let $S = U\,|S| \in C_1$. Then $S \in \ker\Delta_{A,B}$, if and only if,*

$$\|S + \Delta_{A,B}(X)\|_1 \geq \|S\|_1\,,$$

for all $X \in C_1$.

Proof. If A, B are normal operators the Putnam-Fuglede theorem ensures that $\ker\Delta_{A,B} \subseteq \ker\Delta^*_{A,B}$ □

Corollary 2.7. *Let A, B in $B(H)$ be contractions and let $S = U\,|S| \in C_1$. Then $S \in \ker\Delta_{A,B}$, if and only if,*

$$\|S + \Delta_{A,B}(X)\|_1 \geq \|S\|_1\,,$$

for all $X \in C_1$.

Proof. It is known [14, Theorem 2.2] that if A and B are contractions and $S \in C_1$, then $\ker\Delta_{A,B} \subseteq \ker\Delta^*_{A,B}$ and the result holds by the above theorem. □

Remark 2.8. The above corollaries still hold true when we consider $\delta_{A,B}$ instead of $\Delta_{A,B}$.

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