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ON A CONJECTURE OF DE LA GRANDVILLE AND SOLOW CONCERNING POWER MEANS

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ABSTRACT. In a recent paper in this journal De La Grandville and Solow [1] presented a conjecture concerning Power Means. A counterexample to their conjecture is given.

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1. INTRODUCTION

We quote, with minor abbreviations, from De La Grandville and Solow:
"Let x_1, ..., x_n be n positive numbers and

$$M(p) = \left(\sum_{i=1}^{n} f_i x_i^p \right)^{\frac{1}{p}}$$

the mean of order p of the x_i's; $0 < f_i < 1$ and $\sum_{i=1}^{n} f_i = 1$. One of the most important theorems about a general mean is that it is an increasing function of its order. A proof can be found in Hardy, Littlewood and Polya (1952; Theorem 16, pp. 26-27). ... "

... " It is well known that $M(p)$ is increasing with p. It seems that further exact properties of the curve $M(p)$ remain to be discovered."

We now have to distinguish between a bolder conjecture in the *preprint* form of [1] and the published paper which was revised in view of the results we communicated to the authors of [1].

A Conjecture, from the preprint form of [1]**.** *In (M, p) space, the curve $M(p)$ has one and only one inflection point, irrespective of the number and size of the x_i's and the f_i's.* Between

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its limiting values,

$$\lim_{p\to-\infty} M(p) = \min(x_1,\ldots,x_n) \qquad \text{and} \qquad \lim_{p\to\infty} M(p) = \max(x_1,\ldots,x_n),$$

$M(p)$ is in a first phase convex, and then turns concave.

The published form of the conjecture in [1] is for $n = 2$ only. Our present paper serves to show that the restriction to $n = 2$ is necessary. (Whether $n = 2$ is sufficient for the Conjecture to be true is unknown at this stage.)

2. A Counterexample with $n = 3$

As noted in [1], the explicit expressions for the second derivative of $M(p)$ are unpleasant to behold! Computer Algebra packages, however, are less squeamish about messy expressions than are humans. In our counterexample, n is 3. Our counterexample was obtained with Maple (and we omit the plots here and just give relevant numerical values). Maple code, and its output, which provides the counterexample, is given below. For users of Mathematica, the equivalent in Mathematica follows.

```
# maple , x3=1 and f3=(1-f1-f2)
M:= (x1,f1,x2,f2,p) -> (f1*x1^p +f2*x2^p+(1-f1-f2))^(1/p);
M2:=unapply(diff(M(1/9,1/27,2/9,25/27,p),p$2),p);
plot(M2(p),p=-10 .. 10);
map(evalf,[M2(-8),M2(-4),M2(0.1),M2(4)]);
# whose output is
#  [0.001244859453, -0.001233658446, 0.009620297, -0.01197556909]

(* Mathematica x3=1 and f3=(1-f1-f2) *)
M[x1_,f1_,x2_,f2_,p_] := (f1*x1^p +f2*x2^p+(1-f1-f2))^(1/p);
M2[p_]:= Evaluate[D[M[1/9,1/27,2/9,25/27,p],{p,2}]];
Plot[M2[p],{p,-10,10}]
Map[N,{M2[-8],M2[-4],M2[0.1],M2[4]}]
(* whose output is
  [0.00124486, -0.00123366, 0.0096203, -0.0119756] *)
```

In the code, the function `M2` denotes the second derivative of M with respect to p. It is a continuous function of p and has *several* sign changes. For the numeric values of x_i and f_i given in the code, the function `M2` has three zeros, so the function $M(p)$ has three inflection points (in the interval of p studied).

3. Further Results

Hardy, Littlewood and Polya ([2]; Theorem 86, p. 72) give that $p\log(M(p))$ is a convex function of p.

We have some further results related to means, requiring, however, further work. We hope to submit them in a later paper. See also [3].

References

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