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ON INTEGRABILITY OF FUNCTIONS DEFINED BY TRIGONOMETRIC SERIES

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ABSTRACT. The goal of the present paper is to generalize two theorems of R.P. Boas Jr. pertaining to L^p $(p > 1)$ integrability of Fourier series with nonnegative coefficients and weight x^γ. In our improvement the weight x^γ is replaced by a more general one, and the case $p = 1$ is also yielded. We also generalize an equivalence statement of Boas utilizing power-monotone sequences instead of $\{n^\gamma\}$.

Key words and phrases: Sine and cosine series, L^p integrability, equivalence of coefficient conditions, quasi power-monotone sequences.

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1. INTRODUCTION

There are many classical and newer theorems pertaining to the integrability of formal sine and cosine series

$$g(x) := \sum_{n=1}^{\infty} \lambda_n \sin nx,$$

and

$$f(x) := \sum_{n=1}^{\infty} \lambda_n \cos nx.$$

As a nice example, we recall Chen's ([4]) theorem: *If* $\lambda_n \downarrow 0$, *then* $x^{-\gamma}\varphi(x) \in L^p$ *(*φ *means either* f *or* g*),* $p > 1,\ 1/p - 1 < \gamma < 1/p$, *if and only if* $\sum n^{p\gamma+p-2}\lambda_n^p < \infty$.

For notions and notations, please, consult the third section.

We do not recall more theorems because a nice short survey of recent results with references can be found in a recent paper of S. Tikhonov [7], and classical results can be found in the outstanding monograph of R.P. Boas, Jr. [2].

The generalizations of the classical theorems have been obtained in two main directions: to weaken the classical monotonicity condition on the coefficients λ_n; to replace the classical power weight x^γ by a more general one in the integrals. Lately, some authors have used both generalizations simultaneously.

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J. Németh [6] studied the class of $RBVS$ sequences and weight functions more general than the power one in the $L(0,\pi)$ space.

S. Tikhonov [8] also proved two general theorems of this type, but in the L^p-space for $p \geqq 1$; he also used general weights.

Recently D.S. Yu, P. Zhou and S.P. Zhou [9] answered an old problem of Boas ([2], Question 6.12.) in connection with L^p integrability considering weight x^γ, but only under the condition that the sequence $\{\lambda_n\}$ belongs to the class $MVBVS$; their result is the best one among the answers given earlier for special classes of sequences. The original problem concerns nonnegative coefficients.

In the present paper we refer back to an old paper of Boas [3], which was one of the first to study the L^p-integrability with *nonnegative coefficients and weight* x^γ.

We also intend to prove theorems with *nonnegative coefficients,* but with *more general weights than* x^γ.

It can be said that our theorems are the generalizations of Theorems 8 and 9 presented in Boas' paper mentioned above. Boas names these theorems as slight improvements of results of Askey and Wainger [1]. Our theorems jointly generalize these by using more general weights than x^γ, and broaden those to the case $p = 1$, as well.

Comparing our results with those of Tikhonov, as our generalization concerns the coefficients, we omit the condition $\{\lambda_n\} \in RBVS$ and prove the equivalence of (2.2) and (2.3).

In proving our theorems we need to generalize an equivalence statement of Boas [3]. At this step we utilize the quasi β-power-monotone sequences.

2. New Results

We shall prove the following theorems.

Theorem 2.1. *Let* $1 \leqq p < \infty$ *and* $\lambda := \{\lambda_n\}$ *be a nonnegative null-sequence.*

If the sequence $\gamma := \{\gamma_n\}$ *is quasi* β*-power-monotone increasing with a certain* $\beta < p - 1$*, and*

$$\gamma(x)g(x) \in L^p(0,\pi), \tag{2.1}$$

then

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \left(\sum_{k=n}^{\infty} k^{-1}\lambda_k \right)^p < \infty. \tag{2.2}$$

If γ *is also quasi* $\overline{\beta}$*-power-monotone decreasing with a certain* $\overline{\beta} > -1$*, then condition (2.2) is equivalent to*

$$\sum_{n=1}^{\infty} \gamma_n n^{-2} \left(\sum_{k=1}^{n} \lambda_k \right)^p < \infty. \tag{2.3}$$

If the sequence γ *is quasi* β*-power-monotone decreasing with a certain* $\beta > -1 - p$*, and*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \left(\sum_{k=n}^{\infty} |\Delta\lambda_k| \right)^p < \infty, \tag{2.4}$$

then (2.1) holds.

Theorem 2.2. *Let* p *and* λ *be defined as in Theorem 2.1.*

If the sequence γ *is quasi* β*-power-monotone increasing with a certain* $\beta < p - 1$*, and*

$$\gamma(x)f(x) \in L^p(0,\pi), \tag{2.5}$$

then (2.2) holds.

If the sequence γ is quasi β-power-monotone decreasing with a certain $\beta > -1$, then (2.4) implies (2.5).

3. NOTIONS AND NOTATIONS

We shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is *quasi β-power-monotone increasing* (*decreasing*) if there exist a natural number $N := N(\beta, \gamma)$ and a constant $K := K(\beta, \gamma) \geqq 1$ such that

$$Kn^\beta \gamma_n \geqq m^\beta \gamma_m \quad (n^\beta \gamma_n \leqq K m^\beta \gamma_m) \tag{3.1}$$

holds for any $n \geqq m \geqq N$.

If (3.1) holds with $\beta = 0$, then we omit the attribute "β-power" and use the symbols $\uparrow$ ($\downarrow$).

We shall also use the notations $L \ll R$ at inequalities if there exists a positive constant K such that $L \leqq KR$.

A null-sequence $\mathbf{c} := \{c_n\}$ $(c_n \to 0)$ of positive numbers satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta c_n| \leqq K(\mathbf{c}) c_m, \quad (\Delta c_n := c_n - c_{n+1}), \qquad m \in \mathbb{N},$$

with a constant $K(\mathbf{c}) > 0$ is said to be a *sequence of rest bounded variation*, in symbols, $\mathbf{c} \in RBVS$.

A nonnegative sequence $\mathbf{c}$ is said to be a *mean value bounded variation sequence*, in symbols, $\mathbf{c} \in MVBVS$, if there exist a constant $K(\mathbf{c}) > 0$ and a $\lambda \geqq 2$ such that

$$\sum_{k=n}^{2n} |\Delta c_k| \leqq K(\mathbf{c}) n^{-1} \sum_{k=[\lambda^{-1} n]}^{[\lambda n]} c_k, \quad n \in \mathbb{N},$$

where $[\alpha]$ denotes the integral part of α.

In this paper a sequence $\gamma := \{\gamma_n\}$ and a real number $p \geqq 1$ are associated to a function $\gamma(x)$ $(= \gamma_p(x))$, being defined in the following way:

$$\gamma\left(\frac{\pi}{n}\right) := \gamma_n^{1/p}, \ n \in \mathbb{N}; \quad \text{and} \quad K_1(\gamma)\gamma_n \leqq \gamma(x) \leqq K_2(\gamma)\gamma_n$$

holds for all $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right]$.

4. LEMMAS

To prove our theorems we recall one known result and generalize one of Boas' lemmas ([2, Lemma 6.18]).

Lemma 4.1 ([5]). *Let $p \geqq 1$, $\alpha_n \geqq 0$ and $\beta_n > 0$. Then*

$$\sum_{n=1}^{\infty} \beta_n \left(\sum_{k=1}^{n} \alpha_k\right)^p \leqq p^p \sum_{n=1}^{\infty} \beta_n^{1-p} \left(\sum_{k=n}^{\infty} \beta_k\right)^p \alpha_n^p, \tag{4.1}$$

and

$$\sum_{n=1}^{\infty} \beta_n \left(\sum_{k=n}^{\infty} \alpha_k\right)^p \leqq p^p \sum_{n=1}^{\infty} \beta_n^{1-p} \left(\sum_{k=1}^{n} \beta_k\right)^p \alpha_n^p. \tag{4.2}$$

Lemma 4.2. *If* $b_n \geqq 0,\ p \geqq 1,\ s > 0,$ *then*

$$\sum\nolimits_1 := \sum_{n=1}^{\infty} \beta_n \left(\sum_{k=n}^{\infty} b_k \right)^p < \infty \tag{4.3}$$

implies

$$\sum\nolimits_2 := \sum_{n=1}^{\infty} \beta_n n^{-sp} \left(\sum_{k=1}^{n} k^s b_k \right)^p < \infty \tag{4.4}$$

if $n^\delta \beta_n \downarrow$ *with a certain* $\delta > 1 - sp$; *and if* $n^{\overline{\delta}} \beta_n \uparrow$ *with a certain* $\overline{\delta} < 1$, *then (4.4) implies (4.3).*

Thus, if both monotonicity conditions for $\{\beta_n\}$ *hold, then the conditions (4.3) and (4.4) are equivalent.*

Proof of Lemma 4.2. First, suppose (4.3) holds. Write

$$T_n := \sum_{k=n}^{\infty} b_k;$$

then

$$\sum\nolimits_2 = \sum_{n=1}^{\infty} \beta_n n^{-sp} \left(\sum_{k=1}^{n} k^s (T_k - T_{k+1}) \right)^p.$$

By partial summation we obtain

$$\sum\nolimits_2 \ll s^p \sum_{n=1}^{\infty} \beta_n n^{-sp} \left(\sum_{k=1}^{n} k^{s-1} T_k \right)^p =: \sum\nolimits_3.$$

Since $n^\delta \beta_n \downarrow$ with $\delta > 1 - sp$, Lemma 4.1 with (4.1) shows that

$$\begin{aligned} \sum\nolimits_3 &\ll \sum_{n=1}^{\infty} (n^{s-1} T_n)^p (\beta_n n^{-sp})^{1-p} \left(\sum_{k=n}^{\infty} \beta_k k^{-sp} \right)^p \\ &\ll \sum_{n=1}^{\infty} \beta_n T_n^p = \sum\nolimits_1, \end{aligned}$$

this proves that (4.3) $\Rightarrow$ (4.4).

Now suppose that (4.4) holds. First we show that

$$\sum_{n=1}^{\infty} b_n < \infty. \tag{4.5}$$

Denote

$$H_n := \sum_{k=1}^{n} k^s b_k.$$

Then

$$\sum_{k=n}^{N} b_k = \sum_{k=n}^{N} k^{-s} (H_k - H_{k-1}) \leqq s \sum_{k=n}^{N-1} k^{-s-1} H_k + H_N N^{-s}. \tag{4.6}$$

If $p > 1$ then by Hölder's inequality, we obtain

$$\sum_{k=n}^{N-1} k^{-s-1} H_k \beta_k^{\frac{1}{p} - \frac{1}{p}} \leqq \left(\sum_{k=n}^{N-1} H_k^p \beta_k k^{-sp} \right)^{\frac{1}{p}} \left(\sum_{k=n}^{N-1} (k^{-1} \beta_k^{-1/p})^{p/(p-1)} \right)^{\frac{p-1}{p}}. \tag{4.7}$$

Since, $n^{\overline{\delta}}\beta_n \uparrow$ with $\overline{\delta} < 1$, thus

$$\sum_{k=1}^{\infty}(k^{-p}\beta_k^{-1}k^{-\overline{\delta}+\overline{\delta}})^{1/(p-1)} \ll \sum_{k=1}^{\infty} k^{\frac{\overline{\delta}-p}{p-1}} < \infty.$$

This, (4.4) and (4.7) imply that

$$\sum_{k=1}^{\infty} k^{-s-1}H_k < \infty, \tag{4.8}$$

thus $H_N N^{-s}$ tends to zero, herewith, by (4.6), (4.5) is verified, furthermore,

$$\sum_{k=n}^{\infty} b_k \ll \sum_{k=n}^{\infty} k^{-s-1}H_k. \tag{4.9}$$

If $p = 1$, then without Hölder's inequality, the assumption $n^{\overline{\delta}}\beta_n \uparrow$ with a certain $\overline{\delta} < 1$ and (4.4) clearly imply (4.8).

Thus we can apply (4.9) and Lemma 4.1 with (4.2) for any $p \geqq 1$, whence, by $n^{\overline{\delta}}\beta_n \uparrow$ with $\overline{\delta} < 1$, we obtain that

$$\begin{aligned}
\sum_{n=1}^{\infty}\beta_n\left(\sum_{k=n}^{\infty} b_k\right)^p &\ll \sum_{n=1}^{\infty}\beta_n\left(\sum_{k=n}^{\infty} k^{-s-1}H_k\right)^p \\
&\ll \sum_{n=1}^{\infty}(n^{-s-1}H_n)^p\beta_n^{1-p}\left(\sum_{k=1}^{n}\beta_k\right)^p \\
&\ll \sum_{n=1}^{\infty}\beta_n n^{-sp}H_n^p;
\end{aligned}$$

herewith (4.4) $\Rightarrow$ (4.3) is also proved.

The proof of Lemma 4.2 is complete. □

5. PROOF OF THE THEOREMS

Proof of Theorem 2.1. First we prove that (2.1) implies $g(x) \in L(0,\pi)$ and (2.2). If $p > 1$, then, by Hölder's inequality, we get with $p' := p/(p-1)$

$$\int_0^{\pi}|g(x)|dx \leqq \left(\int_0^{\pi}|g(x)\gamma(x)|^p dx\right)^{\frac{1}{p}}\left(\int_0^{\pi}\gamma(x)^{-p'}dx\right)^{\frac{1}{p'}}.$$

Denote $x_n := \frac{\pi}{n},\ n \in \mathbb{N}$. Since $\gamma_n n^{\beta} \uparrow\ (\beta < p-1)$

$$\begin{aligned}
\int_0^{\pi}\gamma(x)^{-p'}dx &\ll \sum_{n=1}^{\infty}\gamma_n^{1/(1-p)}\int_{x_{n+1}}^{x_n}dx \\
&= \sum_{n=1}^{\infty} n^{-2}(\gamma_n n^{\beta})^{1/(1-p)}n^{\beta/(p-1)} \ll 1,
\end{aligned}$$

that is, $g(x) \in L$.

If $p = 1$, then $\gamma_n n^{\beta} \uparrow$ with some $\beta < 0$, thus $\gamma_n \uparrow$, whence

$$\int_0^{\pi}|g(x)|dx \ll \sum_{n=1}^{\infty}\frac{1}{\gamma_n}\int_{x_{n+1}}^{x_n}|g(x)|\gamma(x)dx \ll \frac{1}{\gamma_1}\int_0^{\pi}|g(x)|\gamma(x)dx \ll 1.$$

Integrating $g(x)$, we obtain

$$G(x) := \int_0^x g(t)dt = \sum_{n=1}^{\infty} \frac{\lambda_n}{n}(1-\cos nx) = 2\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin^2 \frac{nx}{2}.$$

Hence

$$G(x_{2k}) \gg \sum_{n=k}^{2k} \frac{\lambda_n}{n}.$$

Denote

$$g_n := \int_{x_{n+1}}^{x_n} |g(x)|dx, \quad n \in \mathbb{N}.$$

Then

$$\begin{aligned}\sum_{k=n}^{\infty} k^{-1}\lambda_k &= \sum_{\nu=0}^{\infty} \sum_{k=2^\nu n}^{2^{\nu+1}n} k^{-1}\lambda_k \\ &\ll \sum_{\nu=0}^{\infty} G(2^{\nu+1}n) \\ &\ll \sum_{\nu=0}^{\infty} \sum_{k=2^{\nu+1}n}^{\infty} g_k \\ &\ll \sum_{\nu=0}^{\infty} \frac{1}{2^{\nu+1}n} \sum_{i=2^\nu n}^{2^{\nu+1}n} \sum_{k=2^{\nu+1}n}^{\infty} g_k \\ &\ll \sum_{\nu=0}^{\infty} \sum_{i=2^\nu n}^{2^{\nu+1}n} \frac{1}{i} \sum_{k=i}^{\infty} g_k \\ &\ll \sum_{i=n}^{\infty} \frac{1}{i} \sum_{k=i}^{\infty} g_k. \qquad (5.1)\end{aligned}$$

Now we have

$$\sum\nolimits_1 := \sum_{n=1}^{\infty} n^{p-2}\gamma_n \left(\sum_{k=n}^{\infty} k^{-1}\lambda_k\right)^p \ll \sum_{n=1}^{\infty} n^{p-2}\gamma_n \left(\sum_{k=n}^{\infty} k^{-1} \sum_{i=k}^{\infty} g_i\right)^p.$$

Applying Lemma 4.1 with (4.2) we obtain that

$$\sum\nolimits_1 \ll \sum_{n=1}^{\infty} \left(n^{-1}\sum_{i=n}^{\infty} g_i\right)^p (n^{p-2}\gamma_n)^{1-p} \left(\sum_{k=1}^{n} k^{p-2}\gamma_k\right)^p.$$

Since $\gamma_n n^\beta \uparrow$ with $\beta < p-1$, we have

$$\sum_{k=1}^{n} \gamma_k k^\beta k^{p-2-\beta} \ll \gamma_n n^\beta \sum_{k=1}^{n} k^{p-2-\beta} \ll \gamma_n n^{p-1}, \qquad (5.2)$$

and thus

$$(n^{p-2}\gamma_n)^{1-p} \left(\sum_{k=1}^{n} k^{p-2}\gamma_k\right)^p \ll \gamma_n n^{2p-2},$$

whence we get

$$\sum\nolimits_1 \ll \sum_{n=1}^{\infty} \gamma_n n^{2p-2} \left(n^{-1}\sum_{i=n}^{\infty} g_i\right)^p.$$

Using again Lemma 4.1 with (4.2) we have

$$\sum\nolimits_1 \ll \sum_{n=1}^{\infty} n^{-p} g_n^p (n^{2p-2}\gamma_n)^{1-p} \left(\sum_{k=1}^{n} k^{2p-2}\gamma_k\right)^p .$$

A similar calculation and consideration as in (5.2) give that

$$\sum_{k=1}^{n} k^{2p-2}\gamma_k \ll \gamma_n n^{2p-1},$$

and

$$(n^{2p-2}\gamma_n)^{1-p} \left(\sum_{k=1}^{n} k^{2p-2}\gamma_k\right)^p \ll \gamma_n n^{3p-2},$$

thus

$$\sum\nolimits_1 \ll \sum_{n=1}^{\infty} \gamma_n n^{2p-2} g_n^p. \tag{5.3}$$

Since

$$\begin{aligned}
\sum_{n=1}^{\infty} \gamma_n n^{2p-2} g_n^p &= \sum_{n=1}^{\infty} \gamma_n n^{2p-2} \left(\int_{x_{n+1}}^{x_n} |g(x)| dx\right)^p \\
&\ll \sum_{n=1}^{\infty} \gamma_n n^{2p-2} \int_{x_{n+1}}^{x_n} |g(x)|^p dx \left(\int_{x_{n+1}}^{x_n} dx\right)^{p-1} \\
&\ll \sum_{n=1}^{\infty} \int_{x_{n+1}}^{x_n} |\gamma(x) g(x)|^p dx \\
&= \int_0^{\pi} |\gamma(x) g(x)|^p dx.
\end{aligned}$$

This and (5.3) prove the implication (2.1) ⇒ (2.2).

Next we verify that (2.4) implies (2.1). Let $x \in (x_{n+1}, x_n]$. Then, using the Abel transformation and the well-known estimation

$$\widetilde{D}_n(x) := \left|\sum_{n=1}^{k} \sin nx\right| \ll x^{-1},$$

we obtain

$$|g(x)| \ll x \sum_{k=1}^{n} k\lambda_k + \left|\sum_{k=n+1}^{\infty} \lambda_k \sin kx\right| \ll x \sum_{k=1}^{n} k\lambda_k + n \sum_{k=n}^{\infty} |\Delta\lambda_k|. \tag{5.4}$$

Denote

$$\Delta_n := \sum_{k=n}^{\infty} |\Delta\lambda_k|.$$

It is easy to see that

$$n\Delta_n \ll n^{-1} \sum_{k=1}^{n} k\Delta_k$$

and, by $\lambda_n \to 0$,

$$\lambda_n \leqq \Delta_n.$$

Thus, by (5.4), we have

$$|g(x)| \ll n^{-1} \sum_{k=1}^{n} k\Delta_k.$$

Hence

$$\int_0^\pi |\gamma(x)g(x)|^p dx = \sum_{n=1}^{\infty} \int_{x_{n+1}}^{x_n} |\gamma(x)g(x)|^p dx \ll \sum_{n=1}^{\infty} \gamma_n n^{-2-p} \left(\sum_{k=1}^{n} k\Delta_k \right)^p .$$

Applying Lemma 4.1 with (4.1), we obtain

$$\int_0^\pi |\gamma(x)g(x)|^p dx \ll \sum_{n=1}^{\infty} (n\Delta_n)^p (\gamma_n n^{-2-p})^{1-p} \left(\sum_{k=n}^{\infty} \gamma_k k^{-2-p} \right)^p .$$

Since $\gamma_n n^\beta \downarrow$ with $\beta > -1-p$, we have

$$\sum_{k=n}^{\infty} \gamma_k k^\beta k^{-2-p-\beta} \ll \gamma_n n^\beta \sum_{k=n}^{\infty} k^{-2-p-\beta} \ll \gamma_n n^{-1-p},$$

and thus

$$(\gamma_n n^{-2-p})^{1-p} \left(\sum_{k=n}^{\infty} \gamma_k k^{-2-p} \right)^p \ll \gamma_n n^{-2}.$$

Collecting these estimations we obtain

$$\int_0^\pi |\gamma(x)g(x)|^p dx \ll \sum_{n=1}^{\infty} \gamma_n n^{p-2} \Delta_n^p = \sum_{n=1}^{\infty} \gamma_n n^{p-2} \left(\sum_{k=n}^{\infty} |\Delta\lambda_k| \right)^p ,$$

herewith the implication (2.4) $\Rightarrow$ (2.1) is also proved.

In order to prove the equivalence of the conditions (2.2) and (2.3), we apply Lemma 4.2 with

$$s = 1, \quad \beta_n = \gamma_n n^{p-2} \quad \text{and} \quad b_k = k^{-1} b_k.$$

Then the assumptions $n^\delta \beta_n \uparrow$ with $\delta < 1$ and $n^{\overline{\delta}} \beta_n \downarrow$ with $\overline{\delta} > 1-p$, determine the following conditions pertaining to γ_n;

$$n^\beta \gamma_n \uparrow \quad \text{with} \quad \beta < p-1 \quad \text{and} \quad n^{\overline{\beta}} \gamma_n \downarrow \quad \text{with} \quad \overline{\beta} > -1. \tag{5.5}$$

The equivalence of (2.2) and (2.3) clearly holds if both monotonicity conditions required in (5.5) hold.

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. As in the proof of Theorem 2.1, first we prove that (2.5) implies (2.2) and $f(x) \in L$. The proof of $f(x) \in L$ runs as that of $g(x) \in L$ in Theorem 2.1.

Integrating $f(x)$, we obtain

$$F(x) := \int_0^x f(t)dt = \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \sin nx,$$

and integrating $F(x)$ we get

$$F_1(x) := \int_0^x F(t)dt = 2 \sum_{n=1}^{\infty} \frac{\lambda_n}{n^2} \sin^2 \frac{nx}{2}.$$

Thus we obtain

$$F_1\left(\frac{\pi}{2k}\right) \gg \sum_{n=k}^{2k} \frac{\lambda_n}{n^2}.$$

Denote

$$f_n := \int_{x_{n+1}}^{x_n} |f(x)|dx, \quad n \in \mathbb{N}, \ \left(x_n = \frac{\pi}{n}\right).$$

Then

$$\begin{aligned} F_1(x_{2n}) &= \int_0^{x_{2n}} F(t)dt \\ &\ll \sum_{k=2n}^{\infty} \int_{x_{k+1}}^{x_k} \left(\int_0^{x_k} |f(t)|dt\right) du \\ &\ll \sum_{k=2n}^{\infty} \frac{1}{k^2} \sum_{\ell=k}^{\infty} \int_{x_{\ell+1}}^{x_\ell} |f(t)|dt = \sum_{k=2n}^{\infty} \frac{1}{k^2} \sum_{\ell=k}^{\infty} f_\ell, \end{aligned}$$

thus

$$\sum_{k=n}^{2n} \frac{\lambda_k}{k} \ll n \sum_{k=2n}^{\infty} \frac{1}{k^2} \sum_{\ell=k}^{\infty} f_\ell.$$

Using the estimation obtained above we have

$$\begin{aligned} \sum_{k=n}^{\infty} k^{-1}\lambda_k &= \sum_{\nu=0}^{\infty} \sum_{k=2^\nu n}^{2^{\nu+1}n} k^{-1}\lambda_k \\ &\ll \sum_{\nu=0}^{\infty} 2^\nu n \sum_{k=2^{\nu+1}n}^{\infty} k^{-2} \sum_{\ell=k}^{\infty} f_\ell \\ &\ll \sum_{\nu=0}^{\infty} 2^\nu n \sum_{i=\nu}^{\infty} \sum_{k=2^{i+1}n}^{2^{i+2}n} k^{-2} \sum_{\ell=2^{i+1}n}^{\infty} f_\ell \\ &\ll \sum_{\nu=0}^{\infty} 2^\nu n \sum_{i=\nu}^{\infty} (2^i_n)^{-1} \sum_{\ell=2^{i+1}n}^{\infty} f_\ell \\ &\ll \sum_{i=0}^{\infty} (2^i n)^{-1} \sum_{\ell=2^{i+1}n}^{\infty} f_\ell \left(\sum_{\nu=0}^{i} 2^\nu n\right) \\ &\ll \sum_{i=0}^{\infty} \sum_{\ell=2^{i+1}n}^{\infty} f_\ell. \end{aligned}$$

Hereafter, as in (5.1), we get that

$$\sum_{k=n}^{\infty} k^{-1}\lambda_k \ll \sum_{i=n}^{\infty} \frac{1}{i} \sum_{\ell=i}^{\infty} f_\ell,$$

and following the method used in the proof of Theorem 2.1 with f_n in place of g_n, the implication (2.5) ⇒ (2.2) can be proved.

The proof of the statement (2.4) ⇒ (2.5) is easier. Namely

$$|f(x)| \leqq \sum_{k=1}^{n} \lambda_k + \left|\sum_{k=n+1}^{\infty} \lambda_k \cos kx\right| \ll \sum_{k=1}^{n} \lambda_k + \frac{1}{x} \sum_{k=n}^{\infty} |\Delta\lambda_k|.$$

Using the notations of Theorem 2.1 and assuming $x \in (x_{n+1}, x_n]$, we obtain

$$\int_{x_{n+1}}^{x_n} |\gamma(x)f(x)|^p dx \ll \gamma_n n^{-2} \left(\sum_{k=1}^{n} \lambda_k\right)^p + \gamma_n n^{-2} \left(n \sum_{k=n}^{\infty} |\Delta\lambda_k|\right)^p$$

and thus, by $\lambda_n \to 0$,

$$\int_0^\pi |\gamma(x)f(x)|^p dx \ll \sum_{n=1}^\infty \gamma_n n^{-2} \left(\sum_{k=1}^n \sum_{m=k}^\infty |\Delta\lambda_m| \right)^p + \sum_{n=1}^\infty \gamma_n n^{p-2} \left(\sum_{k=n}^\infty \Delta\lambda_k \right)^p . \tag{5.6}$$

To estimate the first sum, we again use Lemma 4.1 with (4.1), thus, by $\gamma_n n^\beta \downarrow$ with some $\beta > -1$,

$$\begin{aligned}\sum_{n=1}^\infty \gamma_n n^{-2} \left(\sum_{k=1}^n \Delta_k \right)^p &\ll \sum_{n=1}^\infty \Delta_n^p (\gamma_n n^{-2})^{1-p} \left(\sum_{k=n}^\infty \gamma_k k^{-2} \right)^p \\ &\ll \sum_{n=1}^\infty \gamma_n n^{p-2} \Delta_n^p \equiv \sum_{n=1}^\infty \gamma_n n^{p-2} \left(\sum_{k=n}^\infty |\Delta\lambda_k| \right)^p .\end{aligned}$$

This and (5.6) imply the second assertion of Theorem 2.2, that is, (2.4) ⇒ (2.5).

We have completed our proof. □

REFERENCES

[1] R. ASKEY AND S. WAINGER, Integrability theorems for Fourier series. *Duke Math. J.*, **33** (1966), 223–228.

[2] R.P. BOAS, Jr., *Integrability Theorems for Trigonometric Transforms*, Springer, Berlin, 1967.

[3] R.P. BOAS, Jr., Fourier series with positive coefficients, *J. of Math. Anal. and Applics.*, **17** (1967), 463–483.

[4] Y.-M. CHEN, On the integrability of functions defined by trigonometrical series, *Math. Z.*, **66** (1956), 9–12.

[5] L. LEINDLER, Generalization of inequalities of Hardy and Littlewood, *Acta Sci. Math.*, **31** (1970), 279–285.

[6] J. NÉMETH, Power-monotone-sequences and integrability of trigonometric series, *J. Inequal. Pure and Appl. Math.*, **4**(1) (2003), Art. 3. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=239`].

[7] S. TIKHONOV, On belonging of trigonometric series of Orlicz space, *J. Inequal. Pure and Appl. Math.*, **5**(2) (2004), Art. 22. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=395`].

[8] S. TIKHONOV, On the integrability of trigonometric series, *Math. Notes*, **78**(3) (2005), 437–442.

[9] D.S. YU, P. ZHOU AND S.P. ZHOU, On L^p integrability and convergence of trigonometric series, *Studia Math.*, **182**(3) (2007), 215–226.