Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 2, Issue 3, Article 38, 2001

# SOME PROPERTIES OF THE SERIES OF COMPOSED NUMBERS

LAURENŢIU PANAITOPOL

FACULTY OF MATHEMATICS,
UNIVERSITY OF BUCHAREST,
14 ACADEMIEI ST.,
RO-70109 BUCHAREST, ROMANIA
pan@al.math.unibuc.ro

*Received 27 March, 2001; accepted 11 June, 2001.*
*Communicated by L. Toth*

ABSTRACT. If $c_n$ denotes the $n$-th composed number, one proves inequalities involving $c_n, p_{c_n}, c_{p_n}$, and one shows that the sequences $(p_n)_{n\geq 1}$ and $(c_{p_n})_{n\geq 1}$ are neither convex nor concave.

*Key words and phrases:* Prime Numbers, Composed Numbers, Asymptotic Behaviour, Inequalities, Sums and Series.

2000 *Mathematics Subject Classification.* 11A25, 11N05.

## 1. INTRODUCTION

We are going to use the following notation

$$\begin{aligned} \pi(x) &\text{ the number of prime numbers } \leq x, \\ C(x) &\text{ the number of composed numbers } \leq x, \\ p_n &\text{ the } n\text{-th prime number}, \\ c_n &\text{ the } n\text{-th composed number; } c_1 = 4, c_2 = 6, \dots, \\ \log_2 n &= \log(\log n). \end{aligned}$$

For $x \geq 1$ we have the relation

$$\pi(x) + C(x) + 1 = [x]. \tag{1.1}$$

Bojarincev proved (see [1], [4]) that

$$c_n = n\left(1 + \frac{1}{\log n} + \frac{2}{\log^2 n} + \frac{4}{\log^3 n} + \frac{19}{2}\cdot\frac{1}{\log^4 n} + \frac{181}{6}\cdot\frac{1}{\log^5 n} + o\left(\frac{1}{\log^5 n}\right)\right). \tag{1.2}$$

ISSN (electronic): 1443-5756



The author gratefully acknowledges for partial support from Grant No. 7D/2000 awarded by the Consiliul Naţional al Cercetării Ştiinţifice din Învăţământul Superior, România.

027-01

Let us remark that

$$c_{k+1} - c_k = \begin{cases} 1 & \text{if } c_k + 1 \text{ is composed,} \\ 2 & \text{if } c_k + 1 \text{ is prime.} \end{cases} \tag{1.3}$$

In the proofs from the present paper, we shall need the following facts related to $\pi(x)$ and $p_n$:

$$\text{for } x \geq 67, \quad \pi(x) > \frac{x}{\log x - 0.5} \tag{1.4}$$

(see [7]);

$$\text{for } x \geq 3299, \quad \pi(x) > \frac{x}{\log x - \frac{28}{29}} \tag{1.5}$$

(see [6]);

$$\text{for } x \geq 4, \quad \pi(x) < \frac{x}{\log x - 1.12} \tag{1.6}$$

(see [6]);

$$\text{for } n \geq 1, \quad \pi(x) = \frac{x}{\log x} \sum_{k=0}^{n} \frac{k!}{\log^k x} + O\left(\frac{x}{\log^{n+1} x}\right), \tag{1.7}$$

$$\text{for } n \geq 2, \quad p_n > n(\log n + \log_2 n - 1) \tag{1.8}$$

(see [2] and [3]);

$$\text{for } n \geq 6, \quad p_n < n(\log n + \log_2 n) \tag{1.9}$$

(see [7]).

## 2. INEQUALITIES INVOLVING $c_n$

**Property 1.** We have

$$n\left(1 + \frac{1}{\log n} + \frac{3}{\log^2 n}\right) > c_n > n\left(1 + \frac{1}{\log n} + \frac{1}{\log^2 n}\right) \tag{2.1}$$

whenever $n \geq 4$.

*Proof.* If we take $x = c_n$ in (1.1), then we get

$$\pi(c_n) + n + 1 = c_n. \tag{2.2}$$

Now (1.4) implies that for $n \geq 48$ we have

$$c_n > n + \pi(c_n) > n + \frac{n}{\log n}$$

and then

$$\begin{aligned} c_n > n + \pi(c_n) &> n + \pi\left(n\left(1+\frac{1}{\log n}\right)\right) \\ &> n + \frac{n\left(1+\frac{1}{\log n}\right)}{\log n + \log\left(1+\frac{1}{\log n}\right) - 0.5} \\ &> n + \frac{n\left(1+\frac{1}{\log n}\right)}{\log n} \\ &= n\left(1+\frac{1}{\log n}+\frac{1}{\log^2 n}\right). \end{aligned}$$

By (1.6) and (2.2) it follows that

$$c_n \cdot \frac{\log c_n - 2.12}{\log c_n - 1.12} < n+1.$$

Since $c_n > n$, it follows that $\frac{\log c_n - 2.12}{\log c_n - 1.12} > \frac{\log n - 2.12}{\log n - 1.12}$ hence

$$n+1 > c_n \cdot \frac{\log n - 2.12}{\log n - 1.12}. \tag{2.3}$$

Assume that there would exist $n \geq 1747$ such that

$$c_n \geq n\left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right).$$

Then a direct computation shows that $(12)$ implies

$$\frac{1}{n} \geq \frac{0.88\log n - 6.36}{\log^2 n(\log n - 1.12)}.$$

For $n \geq 1747$, one easily shows that $\frac{0.88\log n - 6.36}{\log n - 1.12} > \frac{1}{31}$, hence $\frac{1}{n} > \frac{1}{31\log^2 n}$. But this is impossible, since for $n \geq 1724$ we have $\frac{1}{n} < \frac{1}{31\log^2 n}$.

Consequently we have $c_n < n\left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right)$. By checking the cases when $n \leq 1746$, one completely proves the stated inequalities. □

**Property 2.** If $n \geq 30,398$, then the inequality

$$p_n > c_n \log c_n$$

holds.

*Proof.* We use (1.8), (2.1) and the inequalities

$$\log\left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right) < \frac{1}{\log n}+\frac{3}{\log^2 n},$$

and

$$n(\log n + \log\log n - 1) > n\left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right)\left(\log n+\frac{1}{\log n}+\frac{3}{\log^2 n}\right),$$

that is $\log\log n > 2 + \frac{4}{\log n} + \frac{4}{\log^2 n} + \frac{6}{\log^3 n} + \frac{9}{\log^4 n}$, which holds if $n \geq 61,800$. Now the proof can be completed by checking the remaining cases. □

**Proposition 2.1.** *We have*

$$\pi(n)p_n > c_n^2$$

*whenever* $n \geq 19,421$.

*Proof.* In view of the inequalities (1.5), (1.8) and (2.1), for $n \geq 3299$ it remains to prove that $\frac{\log n+\log_2 n-1}{\log n-\frac{28}{29}} > \left(1+\frac{1}{\log n}+\frac{3}{\log^2 n}\right)^2$, that is

$$\log\log n > \frac{59}{29} + \frac{5.069}{\log n} - \frac{0.758}{\log^2 n} + \frac{3.207}{\log^3 n} - \frac{8.68}{\log^4 n}.$$

It suffices to show that

$$\log\log n > \frac{59}{29} + \frac{5.069}{\log n}.$$

For $n = 130,000$, one gets $2.466\cdots > 2.4649\ldots$. The checking of the cases when $n < 130,000$ completes the proof. □

## 3. INEQUALITIES INVOLVING $c_{p_n}$ AND $p_{c_n}$

**Proposition 3.1.** *We have*

$$p_n + n < c_{p_n} < p_n + n + \pi(n) \tag{3.1}$$

*for* $n$ *sufficiently large.*

*Proof.* By (1.2) and (1.7) it follows that for $n$ sufficiently large we have $c_n = n+\pi(n)+\frac{n}{\log^2 n}+O\left(\frac{n}{\log^3 n}\right)$, hence

$$c_{p_n} = p_n + n + \frac{p_n}{\log^2 p_n} + O\left(\frac{n}{\log^2 n}\right). \tag{3.2}$$

Thus for $n$ large enough we have $c_{p_n} > p_n + n$.

Since the function $x \mapsto \frac{x}{\log^2 x}$ is increasing, one gets by (1.9)

$$\begin{aligned}\frac{p_n}{\log^2 p_n} &< \frac{n(\log n+\log_2 n)}{(\log n+\log(\log n+\log_2 n))^2}\\ &< \frac{n(\log n+\log_2 n)}{\log n(\log n+2\log_2 n)}\\ &< n\cdot\frac{\log n-\frac{1}{2}\log_2 n}{\log^2 n}\\ &= \pi(n) - \frac{1}{2}\cdot\frac{n\log_2 n}{\log^2 n} + O\left(\frac{n}{\log^2 n}\right).\end{aligned}$$

Both this inequality and (3.2) show that for $n$ sufficiently large we have indeed $c_{p_n} < p_n + n + \pi(n)$. □

**Proposition 3.2.** *If* $n$ *is large enough, then the inequality*

$$p_{c_n} > c_{p_n}$$

*holds.*

*Proof.* By (2.1) it follows that

$$c_{p_n} = \pi(c_{p_n}) + p_n + 1. \tag{3.3}$$

Now (3.1) and (3.3) imply that for $n$ sufficiently large we have $\pi(c_{p_n}) < n + \pi(n)$. But by (2.1) it follows that $c_n > n + \pi(n)$, hence $c_n > \pi(c_{p_n})$. If we assume that $c_{p_n} > p_{c_n}$, then we obtain the contradiction $\pi(c_{p_n}) \geq \pi(p_{c_n}) = c_n$. Consequently we must have $c_{p_n} < p_{c_n}$. □

It is easy to show that the sequence $(c_n)_{n\geq 1}$ is neither convex nor concave. We are lead to the same conclusion by studying the sequences $(c_{p_n})_{n\geq 1}$ and $(p_{c_n})_{n\geq 1}$. Let us say that a sequence $(a_n)_{n\geq 1}$ *has the property* $P$ when the inequality

$$a_{n+1} - 2a_n + a_{n-1} > 0$$

holds for infinitely many indices and the inequality

$$a_{n+1} - 2a_n + a_{n-1} < 0$$

holds also for infinitely many indices. Then we can prove the following fact.

**Proposition 3.3.** *Both sequences* $(c_{p_n})_{n\geq 1}$ *and* $(p_{c_n})_{n\geq 1}$ *have the property* $P$.

In order to prove it we need the following auxiliary result.

**Lemma 3.4.** *If the sequence* $(a_n)_{n\geq n_1}$ *is convex, then for* $m > n \geq n_1$ *we have*

$$\frac{a_m - a_n}{m - n} \geq a_{n+1} - a_n. \tag{3.4}$$

*If the sequence* $(a_n)_{n\geq n_2}$ *is concave, then for* $n > p \geq n_2$ *we have*

$$\frac{a_n - a_p}{n - p} \geq a_{n+1} - a_n \tag{3.5}$$

*whenever* $m > n \geq n_1$.

*Proof.* In the first case, for $i \geq n$ we have $a_{i+1} - a_i \geq a_{n+1} - a_n$, hence $\sum_{i=n}^{m-1}(a_{i+1} - a_i) \geq (m-n)(a_{n+1} - a_n)$, that is (3.4). The inequality (3.5) can be proved similarly. □

*Proof of Proposition 3.3.* Erdös proved in [3] that, with $d_n = p_{n+1} - p_n$, we have $\limsup_{n\to\infty} \frac{\min(d_n, d_{n+1})}{\log n} = \infty$. In particular, the set $M = \{n \mid \min(d_n, d_{n+1}) > 2\log n\}$ is infinite.

For every $n$, at least one of the numbers $n$ and $n+1$ is composed, that is, either $n = c_m$ or $n + 1 = c_m$ for some $m$. Consequently, there exist infinitely many indices $m$ such that $p_{c_m+1} - p_{c_m} > 2\log c_m$. Since $c_{m+1} \geq c_m + 1$ and $c_m > m$, we get infinitely many values of $m$ such that

$$p_{c_{m+1}} - p_{c_m} > 2\log m. \tag{3.6}$$

Let $M'$ be the set of these numbers $m$.

If we assume that the sequence $(p_{c_n})_{n\geq n_1}$ is convex, then (3.4) implies that for $m \in M'$ we have

$$\frac{p_{c_{2m}} - p_{c_m}}{m} \geq p_{c_{m+1}} - p_{c_m} > 2\log m,$$

hence $p_{c_{2m}} > 2m\log m + p_{c_m}$. But this is a contradiction because $c_n \sim n$ and $p_n \sim n\log n$, that is $p_{c_{2m}} \sim 2m\log 2m$ and $p_{c_m} \sim m\log m$.

On the other hand, if we assume that the sequence $(p_{c_n})_{n\geq n_2}$ is concave, then (3.5) implies that for $x \in M'$ we have

$$\frac{p_{c_m} - p_{c[m/2]}}{m - \left[\frac{m}{2}\right]} \geq p_{c_{m+1}} - p_{c_m} > 2\log m,$$

that is

$$1 > \frac{2\left(m - \left[\frac{m}{2}\right]\right)\log m + p_{c[m/2]}}{p_{c_m}}.$$

For $m \to \infty$, $m \in M'$, the last inequality implies the contradiction $1 \geq 1 + \frac{1}{2}$. Consequently the sequence $(p_{c_n})_{n\geq 1}$ has the property $P$.

Now let us assume that the sequence $(c_{p_n})_{n\geq n_1}$ is convex. Then for $n \in M$, $n \geq n_1$, we get by (3.4)

$$\frac{c_{p_{2n}} - c_{p_n}}{n} \geq c_{p_{n+1}} - c_{p_n} \geq p_{n+1} - p_n > 2\log n.$$

If we take $n \to \infty$, $n \in M$, in the inequality $1 > (2n\log n + c_{p_n})/c_{p_{2n}}$, then we obtain the contradiction $1 \geq \frac{3}{2}$.

Finally, if we assume that the sequence $(c_{p_n})_{n\geq n_2}$ is concave, then (3.5) implies that for $n \in M$, $n \geq n_2$, we have

$$\frac{c_{p_n} - c_{p_{[n/2]}}}{n - \left[\frac{n}{2}\right]} \geq c_{p_{n+1}} - c_{p_n} \geq p_{n+1} - p_n > 2\log n,$$

which is again a contradiction. □

## REFERENCES


[1] A.E. BOJARINCEV, Asymptotic expressions for the $n$th composite number, *Ural. Gos. Univ. Mat. Zap.*, **6** (1967), 21–43 (in Russian).

[2] P. DUSART, The $k$th prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$, *Math. Comp.*, **68** (1999), no. 225, 411–415.

[3] P. ERDÖS, Problems and results on the differences of consecutive primes, *Publ. Math. Debrecen*, **1** (1949), 33–37.

[4] J.-P. MASSIAS AND G. ROBIN, Bornes effectives pour certaines fonctions concernant les nombres premiers, *J. Théor. Nombres Bordeaux*, **8** (1996), 215–242.

[5] D.S. MITRINOVIĆ, J. SÁNDOR AND B. CRSTICI, *Handbook of Number Theory*, Kluwer Academic Publishers, Dordrecht - Boston - London, 1996.

[6] L. PANAITOPOL, Several approximations of $\pi(x)$, *Math. Ineq. & Appl.*, **2**(3) (1999), 317–324.

[7] J.B. ROSSER AND L. SCHOENFIELD, Approximate formulas for some functions of prime numbers, *Illinois J. Math.*, **6** (1962), 64–94.