

ON CHEBYSHEV TYPE INEQUALITIES INVOLVING FUNCTIONS WHOSE DERIVATIVES BELONG TO L_p SPACES VIA ISOTONIC FUNCTIONALS

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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper we establish new Chebyshev type inequalities via linear functionals.

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Key words: Chebyshev inequality, Isotonic linear functionals.

Contents

1	Introduction	3
2	Statement of Results	6
3	Proof of Theorem 2.1	9
4	Proof of Theorem 2.2	12
5	Remarks	14
	References	



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 2 of 16

1. Introduction

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two absolutely continuous functions whose derivatives $f', g' \in L_\infty[a, b]$.

The Chebyshev functional is defined by:

$$(1.1) \quad T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right)$$

and the following inequality (see [8]) holds:

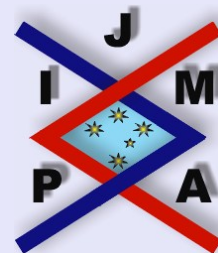
$$(1.2) \quad |F(f, g)| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty.$$

Many researchers have given considerable attention to (1.2) and a number of extensions, generalizations and variants have appeared in the literature, see ([1], [2], [3], [6], [7]) and the references given therein.

In [7] B.G. Pachpatte considered the following functionals:

$$F(f) = \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f \left(\frac{a+b}{2} \right) \right],$$

$$S(f, g) = F(f)F(g) - \frac{1}{b-a} \left[F(f) \int_a^b g(x)dx + F(g) \int_a^b f(x)dx \right] + \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right)$$



On Chebyshev Type Inequalities Involving Functions Whose Derivatives Belong to L_p Spaces via Isotonic Functionals

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 3 of 16

and

$$H(f, g) = \frac{1}{b-a} \int_a^b [F(f)g(x) + F(g)f(x)]dx - 2 \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right).$$

B.G. Pachpatte proved the following results:

Theorem 1.1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$. Then we have the inequalities*

$$(1.3) \quad |T(f, g)| \leq \frac{1}{(b-a)^3} \|f'\|_p \|g'\|_p \int_a^b [B(x)]^{2/q} dx,$$

$$(1.4) \quad |T(f, g)| \leq \frac{1}{2(b-a)^2} \int_a^b [|g(x)| \|f'\|_p + |f(x)| \|g'\|_p] [B(x)]^{1/q} dx,$$

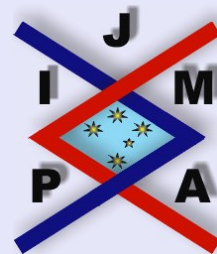
where

$$(1.5) \quad B(x) = \frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1}$$

for $x \in [a, b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.2. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$. Then we have the inequalities:*

$$(1.6) \quad |S(f, g)| \leq \frac{1}{(b-a)^2} M^{2/q} \|f'\|_p \|g'\|_p$$



On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 4 of 16

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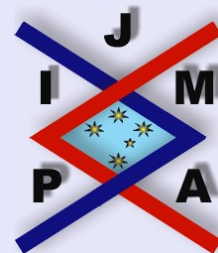
$$(1.7) \quad |H(f, g)| \leq \frac{1}{(b-a)^2} M^{1/q} \int_a^b [|g(x)| \|f'\|_p + |f(x)| \|g'\|_p] dx,$$

where

$$M = \frac{(2^{q+1} + 1)(b-a)^{q+1}}{3(q+1)6^q}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

The main purpose of the present note is to establish inequalities similar to the inequalities (1.3) – (1.6) involving isotonic functionals.



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 5 of 16

2. Statement of Results

Let $I = [a, b]$ a fixed interval. For every $t \in I$ we consider the function $u_t : [a, b] \rightarrow \mathbb{R}$ defined by

$$u_t(x) = \begin{cases} 0, & x \in [a, t), \\ 1, & x \in [t, b]. \end{cases}$$

Let L be a linear class of real valued functions $f : I \rightarrow \mathbb{R}$ having the properties:

$$L_1 : f, g \in L \Rightarrow \alpha f + \beta g \in L, \text{ for all } \alpha, \beta \in \mathbb{R}$$

$$L_2 : u_t \in L \text{ for all } t \in [a, b].$$

An isotonic linear functional is a functional $A : L \rightarrow \mathbb{R}$ having the following properties:

$$A_1 : A(\alpha f + \beta g) = \alpha A(f) + \beta A(g) \text{ for } f, g \in L, \alpha, \beta \in \mathbb{R}$$

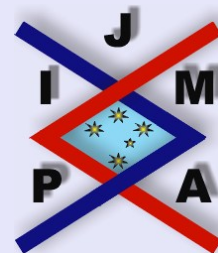
$$A_2 : f \in L, f(t) \geq 0 \text{ on } I \text{ then } A(f) \geq 0.$$

In what follows we denote by \mathcal{M} the set of all isotonic functionals having the properties:

$$M_1 : A \in \mathcal{M} \text{ then } A(u_t) \in L_p(\mathbb{R}) \text{ for all } p \geq 1$$

$$M_2 : A \in \mathcal{M} \text{ then } A(1) = 1.$$

Now, we state our main results as follows.



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 6 of 16

Theorem 2.1. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$ and A, B, C isotonic functionals belong to \mathcal{M} . Then we have the following inequalities:

$$(2.1) \quad |C(fg) - C(f)B(g) - C(g)A(f) + A(f)B(g)| \leq C[K(A, B)] \|f'\|_p \|g'\|_p$$

and

$$(2.2) \quad |2C(fg) - C(f)B(g) - C(g)A(f)| \leq C[H_{f,g}],$$

where

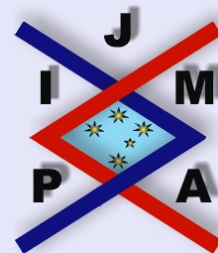
$$K(A, B)(x) = \left(\int_a^b |u_t(x) - A(u_t)|^q dt \right)^{\frac{1}{q}} \left(\int_a^b |u_t(x) - B(u_t)|^q dt \right)^{\frac{1}{q}}$$

and

$$H_{f,g}(x) = |g(x)| \left(\int_a^b |u_t(x) - A(u_t)|^q dt \right)^{\frac{1}{q}} \|f'\|_p + |f(x)| \left(\int_a^b |u_t(x) - B(u_t)|^q dt \right)^{\frac{1}{q}} \|g'\|_p.$$

Theorem 2.2. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$ and A, B two isotonic functionals belong to \mathcal{M} . Then we have the inequality:

$$(2.3) \quad |A(f)A(g) - A(f)C(g) - C(f)A(g) + C(f)C(g)| \leq M^{2/q} \|f'\|_p \|g'\|_p,$$



On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals

I. Gavrea

Title Page

Contents



Go Back

Close

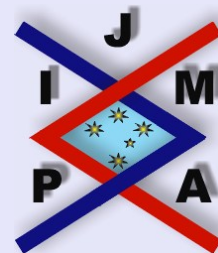
Quit

Page 7 of 16

where

$$M = \int_a^b |A(u_t) - C(u_t)|^q dt$$

and $\frac{1}{p} + \frac{1}{q} = 1$.



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 8 of 16

3. Proof of Theorem 2.1

From the identity:

$$f(x) = f(a) + \int_a^x f'(t)dt$$

and using the definition of the function u_t we obtain the following equality

$$(3.1) \quad f(x) = f(a) + \int_a^b u_t(x)f'(t)dt.$$

Functional A being an isotonic functional from (3.1) we get

$$(3.2) \quad A(f) = f(a) + \int_a^b A(u_t)f'(t)dt.$$

From (3.1) and (3.2) we obtain

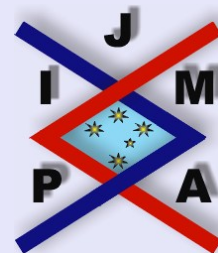
$$(3.3) \quad f(x) - A(f) = \int_a^b [u_t(x) - A(u_t)]f'(t)dt.$$

Similarly we obtain:

$$(3.4) \quad g(x) - B(g) = \int_a^b [u_t(x) - B(u_t)]g'(t)dt.$$

Multiplying the left sides and right sides of (3.3) and (3.4) we have:

$$(3.5) \quad f(x)g(x) - f(x)B(g) - g(x)A(f) + A(f)B(g) \\ = \int_a^b [u_t(x) - A(u_t)]f'(t)dt \int_a^b [u_t(x) - B(u_t)]g'(t)dt.$$



On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 9 of 16

From (3.5) we obtain:

$$(3.6) \quad |f(x)g(x) - f(x)B(g) - g(x)A(f) + A(f)B(g)| \\ \leq \int_a^b |u_t(x) - A(u_t)|f'(t)dt \int_a^b |u_t(x) - B(u_t)||g'(t)|dt.$$

Using Hölder's integral inequality from (3.6) we get:

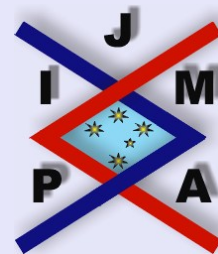
$$(3.7) \quad |f(x)g(x) - f(x)B(g) - g(x)A(f) + A(f)B(g)| \\ \leq \left(\int_a^b |u_t(x) - A(u_t)|^q dt \right)^{\frac{1}{q}} \left(\int_a^b |u_t(x) - B(u_t)|^q dt \right)^{\frac{1}{q}} \|f'\|_p \|g'\|_p.$$

From (3.7) applying the functional C and using the fact that C is an isotonic linear functional we obtain inequality (2.1).

Multiplying both sides of (3.3) and (3.4) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we get:

$$(3.8) \quad 2f(x)g(x) - g(x)A(f) - f(x)B(g) \\ = \int_a^b g(x)[u_t(x) - A(u_t)]f'(t)dt + \int_a^b f(x)[u_t(x) - B(u_t)]g'(t)dt.$$

From (3.8), using the properties of modulus, Hölder's integral inequality we



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 10 of 16

have:

$$\begin{aligned}
 (3.9) \quad & |2f(x)g(x) - g(x)A(f) - f(x)B(g)| \\
 & \leq |g(x)| \left(\int_a^b |u_t(x) - A(u_t)|^q dt \right)^{\frac{1}{q}} \|f'\|_p \\
 & \quad + |f(x)| \left(\int_a^b |u_t(x) - B(u_t)|^q dt \right)^{\frac{1}{q}} \|g'\|_p
 \end{aligned}$$

or

$$(3.10) \quad |2f(x)g(x) - g(x)A(f) - f(x)B(g)| \leq H_{f,g}(x).$$

The functional C being an isotonic linear functional we have:

$$\begin{aligned}
 (3.11) \quad & C(|2f(x)g(x) - g(x)A(f) - f(x)B(g)|) \\
 & \geq |2C(fg) - C(g)A(f) - C(f)B(g)|.
 \end{aligned}$$

From (3.10) applying the functional C and using (3.11) we obtain inequality (2.2).

The proof of Theorem 2.1 is complete.



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 11 of 16

4. Proof of Theorem 2.2

From (3.1) we have:

$$(4.1) \quad f(x) - f(y) = \int_a^b [u_t(x) - u_t(y)]f'(t)dt$$

and

$$(4.2) \quad g(x) - g(y) = \int_a^b [u_t(x) - u_t(y)]g'(t)dt.$$

Applying the functionals A and C in (4.1) and (4.2) we obtain

$$(4.3) \quad A(f) - C(f) = \int_a^b [A(u_t) - C(u_t)]f'(t)dt$$

and

$$(4.4) \quad A(g) - C(g) = \int_a^b [A(u_t) - C(u_t)]g'(t)dt.$$

Multiplying the left sides and right sides of (4.3) and (4.4) we have

$$(4.5) \quad \begin{aligned} A(f)A(g) - A(f)C(g) - A(g)C(f) + C(f)C(g) \\ = \int_a^b [A(u_t) - C(u_t)]f'(t)dt \int_a^b [A(u_t) - C(u_t)]g'(t)dt. \end{aligned}$$



On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals

I. Gavrea

Title Page

Contents



Go Back

Close

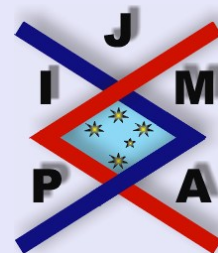
Quit

Page 12 of 16

Using Hölder's integral inequality from (4.5) we obtain

$$|A(f)A(g) - A(f)C(g) - A(g)C(f) + C(f)C(g)| \leq \left(\int_a^b |A(u_t) - C(u_t)|^q dt \right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p.$$

The last inequality proves the theorem.



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 13 of 16

5. Remarks

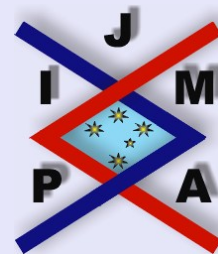
a) For

$$A(f) = B(f) = C(f) = \frac{1}{b-a} \int_a^b f(x) dx$$

then from Theorem 2.1 we obtain the results from Theorem 1.1.

b) Inequality (1.6) is a particular case of the inequality (2.3) when $A = F$,

$$C(f) = \frac{1}{b-a} \int_a^b f(x) dx.$$



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 14 of 16

References

- [1] S.S. DRAGOMIR, On Simpson's quadrature formula for differentiable mappings whose derivatives belong to L_p spaces and applications, *J. KSIAM*, **2**(2) (1998), 57–65.
- [2] S.S. DRAGOMIR AND S. WANG, A new inequality of Ostrowski type in L_p norm, *Indian J. Math.*, **40**(3) (1998), 299–304.
- [3] H.P. HEINIG AND L. MALIGRANDA, Chebyshev inequality in function spaces, *Real Analysis and Exchange*, **17** (1991-92), 211–217.
- [4] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
- [5] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.
- [6] B.G. PACHPATTE, On Ostrowski-Grüss-Chebyshev type inequalities for functions whose modulus of derivatives are convex, *J. Inequal. Pure and Appl. Math.*, **6**(4) (2005), Art. 128. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=602>].
- [7] B.G. PACHPATTE, On Chebyshev type inequalities involving functions whose derivatives belong to L_p spaces, *J. Inequal. Pure and Appl. Math.*, **7**(2) (2006), Art. 58. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=675>].



On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals

I. Gavrea

Title Page

Contents



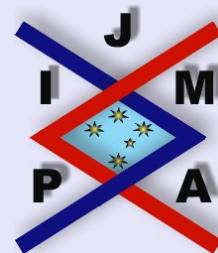
Go Back

Close

Quit

Page 15 of 16

- [8] J.E. PEČARIĆ, F. PORCHAN AND Y. TANG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, San Diego, 1992.



**On Chebyshev Type Inequalities
Involving Functions Whose
Derivatives Belong to L_p
Spaces via Isotonic Functionals**

I. Gavrea

Title Page

Contents



Go Back

Close

Quit

Page 16 of 16