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ON CHEBYSHEV TYPE INEQUALITIES INVOLVING FUNCTIONS WHOSE DERIVATIVES BELONG TO L_p SPACES VIA ISOTONIC FUNCTIONALS

I. GAVREA

Technical University of Cluj-Napoca
Department of Mathematics
Str. C. Daicoviciu 15
3400 Cluj-Napoca, Romania

EMail: Ioan.Gavrea@math.utcluj.ro

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Abstract

In this paper we establish new Chebyshev type inequalities via linear functionals.

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Contents

1. Introduction

Let $f, g : [a, b] \to \mathbb{R}$ be two absolutely continuous functions whose derivatives $f', g' \in L_\infty[a, b]$.

The Chebyshev functional is defined by:

$$(1.1)\quad T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

and the following inequality (see [8]) holds:

$$(1.2)\quad |F(f,g)| \le \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty.$$

Many researchers have given considerable attention to (1.2) and a number of extensions, generalizations and variants have appeared in the literature, see ([1], [2], [3], [6], [7]) and the references given therein.

In [7] B.G. Pachpatte considered the following functionals:

$$F(f) = \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right],$$

$$S(f,g) = F(f)F(g) - \frac{1}{b-a}\left[F(f)\int_a^b g(x)dx + F(g)\int_a^b f(x)dx\right] + \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

and

$$H(f,g)=\frac{1}{b-a}\int_a^b [F(f)g(x)+F(g)f(x)]dx \\ -2\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right).$$

B.G. Pachpatte proved the following results:

Theorem 1.1. *Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions whose derivatives $f',g'\in L_p[a,b]$, $p>1$. Then we have the inequalities*

$$|T(f,g)|\le\frac{1}{(b-a)^3}\|f'\|_p\|g'\|_p\int_a^b [B(x)]^{2/q}dx, \tag{1.3}$$

$$|T(f,g)|\le\frac{1}{2(b-a)^2}\int_a^b [|g(x)|\|f'\|_p+|f(x)|\|g'\|_p][B(x)]^{1/q}dx, \tag{1.4}$$

where

$$B(x)=\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1} \tag{1.5}$$

for $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.2. *Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous functions whose derivatives $f',g'\in L_p[a,b]$, $p>1$. Then we have the inequalities:*

$$|S(f,g)|\le\frac{1}{(b-a)^2}M^{2/q}\|f'\|_p\|g'\|_p \tag{1.6}$$

and

$$|H(f,g)| \le \frac{1}{(b-a)^2} M^{1/q} \int_a^b [|g(x)| \|f'\|_p + |f(x)| \|g'\|_p] dx, \tag{1.7}$$

where

$$M = \frac{(2^{q+1}+1)(b-a)^{q+1}}{3(q+1)6^q}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

The main purpose of the present note is to establish inequalities similar to the inequalities (1.3) – (1.6) involving isotonic functionals.

2. Statement of Results

Let $I = [a, b]$ a fixed interval. For every $t \in I$ we consider the function $u_t : [a, b] \to \mathbb{R}$ defined by

$$u_t(x) = \begin{cases} 0, & x \in [a, t), \\ 1, & x \in [t, b]. \end{cases}$$

Let L be a linear class of real valued functions $f : I \to \mathbb{R}$ having the properties:

$$L_1 : \quad f, g \in L \Rightarrow \alpha f + \beta g \in L, \text{ for all } \alpha, \beta \in \mathbb{R}$$
$$L_2 : \quad u_t \in L \text{ for all } t \in [a, b].$$

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:

$$A_1 : \quad A(\alpha f + \beta g) = \alpha A(f) + \beta A(g) \text{ for } f, g \in L, \ \alpha, \beta \in \mathbb{R}$$
$$A_2 : \quad f \in L, \ f(t) \geq 0 \text{ on } I \text{ then } A(f) \geq 0.$$

In what follows we denote by $\mathcal{M}$ the set of all isotonic functionals having the properties:

$$M_1 : \quad A \in \mathcal{M} \text{ then } A(u_t) \in L_p(\mathbb{R}) \text{ for all } p \geq 1$$
$$M_2 : \quad A \in \mathcal{M} \text{ then } A(1) = 1.$$

Now, we state our main results as follows.

Theorem 2.1. *Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$ and A, B, C isotonic functionals belong to $\mathcal{M}$. Then we have the following inequalities:*

$$|C(fg) - C(f)B(g) - C(g)A(f) + A(f)B(g)| \leq C[K(A, B)]\|f'\|_p\|g'\|_p \tag{2.1}$$

and

$$|2C(fg) - C(f)B(g) - C(g)A(f)| \leq C[H_{f,g}], \tag{2.2}$$

where

$$K(A, B)(x) = \left(\int_a^b |u_t(x) - A(u_t)|^q dt\right)^{\frac{1}{q}} \left(\int_a^b |u_t(x) - B(u_t)|^q\right)^{\frac{1}{q}}$$

and

$$H_{f,g}(x) = |g(x)| \left(\int_a^b |u_t(x) - A(u_t)|^q dt\right)^{\frac{1}{q}} \|f'\|_p + |f(x)| \left(\int_a^b |u_t(x) - B(u_t)|^q dt\right)^{\frac{1}{q}} \|g'\|_p.$$

Theorem 2.2. *Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$ and A, B two isotonic functionals belong to $\mathcal{M}$. Then we have the inequality:*

$$|A(f)A(g) - A(f)C(g) - C(f)A(g) + C(f)C(g)| \leq M^{2/q}\|f'\|_p\|g'\|_p, \tag{2.3}$$

where

$$M=\int_a^b |A(u_t)-C(u_t)|^q dt$$

and $\frac{1}{p}+\frac{1}{q}=1.$

3. Proof of Theorem 2.1

From the identity:

$$f(x) = f(a) + \int_a^x f'(t)dt$$

and using the definition of the function u_t we obtain the following equality

$$f(x) = f(a) + \int_a^b u_t(x) f'(t)dt. \tag{3.1}$$

Functional A being an isotonic functional from (3.1) we get

$$A(f) = f(a) + \int_a^b A(u_t) f'(t)dt. \tag{3.2}$$

From (3.1) and (3.2) we obtain

$$f(x) - A(f) = \int_a^b [u_t(x) - A(u_t)] f'(t)dt. \tag{3.3}$$

Similarly we obtain:

$$g(x) - B(g) = \int_a^b [u_t(x) - B(u_t)] g'(t)dt. \tag{3.4}$$

Multiplying the left sides and right sides of (3.3) and (3.4) we have:

$$\begin{aligned} f(x)g(x) - f(x)B(g) - g(x)A(f) + A(f)B(g) \\ = \int_a^b [u_t(x) - A(u_t)] f'(t)dt \int_a^b [u_t(x) - B(u_t)] g'(t)dt. \end{aligned} \tag{3.5}$$

From (3.5) we obtain:

$$|f(x)g(x) - f(x)B(g) - g(x)A(f) + A(f)B(g)| \leq \int_a^b |u_t(x) - A(u_t)|f'(t)dt \int_a^b |u_t(x) - B(u_t)||g'(t)|dt. \tag{3.6}$$

Using Hölder's integral inequality from (3.6) we get:

$$|f(x)g(x) - f(x)B(g) - g(x)A(f) + A(f)B(g)| \leq \left(\int_a^b |u_t(x) - A(u_t)|^q dt\right)^{\frac{1}{q}} \left(\int_a^b |u_t(x) - B(u_t)|^q\right)^{\frac{1}{q}} \|f'\|_p \|g'\|_p. \tag{3.7}$$

From (3.7) applying the functional C and using the fact that C is an isotonic linear functional we obtain inequality (2.1).

Multiplying both sides of (3.3) and (3.4) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we get:

$$2f(x)g(x) - g(x)A(f) - f(x)B(g) = \int_a^b g(x)[u_t(x) - A(u_t)]f'(t)dt + \int_a^b f(x)[u_t(x) - B(u_t)]g'(t)dt. \tag{3.8}$$

From (3.8), using the properties of modulus, Hölder's integral inequality we have:

$$|2f(x)g(x) - g(x)A(f) - f(x)B(g)| \le |g(x)| \left(\int_a^b |u_t(x) - A(u_t)|^q dt \right)^{\frac{1}{q}} \|f'\|_p + |f(x)| \left(\int_a^b |u_t(x) - B(u_t)|^q dt \right)^{\frac{1}{q}} \|g'\|_p \tag{3.9}$$

or

$$|2f(x)g(x) - g(x)A(f) - f(x)B(g)| \le H_{f,g}(x). \tag{3.10}$$

The functional C being an isotonic linear functional we have:

$$C(|2f(x)g(x) - g(x)A(f) - f(x)B(g)|) \ge |2C(fg) - C(g)A(f) - C(f)B(g)|. \tag{3.11}$$

From (3.10) applying the functional C and using (3.11) we obtain inequality (2.2).

The proof of Theorem 2.1 is complete.

4. Proof of Theorem 2.2

From (3.1) we have:

$$f(x) - f(y) = \int_a^b [u_t(x) - u_t(y)] f'(t) dt \tag{4.1}$$

and

$$g(x) - g(y) = \int_a^b [u_t(x) - u_t(y)] g'(t) dt. \tag{4.2}$$

Applying the functionals A and C in (4.1) and (4.2) we obtain

$$A(f) - C(f) = \int_a^b [A(u_t) - C(u_t)] f'(t) dt \tag{4.3}$$

and

$$A(g) - C(g) = \int_a^b [A(u_t) - C(u_t)] g'(t) dt. \tag{4.4}$$

Multiplying the left sides and right sides of (4.3) and (4.4) we have

$$\begin{aligned} &A(f)A(g) - A(f)C(g) - A(g)C(f) + C(f)C(g) \\ &\qquad = \int_a^b [A(u_t) - C(u_t)] f'(t) dt \int_a^b [A(u_t) - C(u_t)] g'(t) dt. \end{aligned} \tag{4.5}$$

Using Hölder's integral inequality from (4.5) we obtain

$$|A(f)A(g) - A(f)C(g) - A(g)C(f) + C(f)C(g)| \leq \left(\int_a^b |A(u_t) - C(u_t)|^q dt \right)^{\frac{2}{q}} \|f'\|_p \|g'\|_p.$$

The last inequality proves the theorem.

5. Remarks

a) For

$$A(f) = B(f) = C(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

then from Theorem 2.1 we obtain the results from Theorem 1.1.

b) Inequality (1.6) is a particular case of the inequality (2.3) when $A = F$,

$$C(f) = \frac{1}{b-a}\int_a^b f(x)dx.$$

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