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SOME REMARKS ON A PAPER BY A. MCD. MERCER

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Abstract

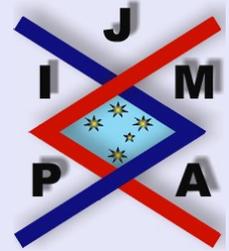
In this note we give a necessary and sufficient condition in order that an inequality established by A. McD. Mercer to be true for every convex sequence.

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1. Introduction

In [1] A. McD. Mercer proved the following result:

If the sequence $\{u_k\}$ is convex then

$$(1.1) \quad \sum_{k=0}^n \left[\frac{1}{n+1} - \frac{1}{2^n} \binom{n}{k} \right] u_k \geq 0.$$

In [2] this inequality was generalized to the following:

Suppose that the polynomial

$$(1.2) \quad \sum_{k=0}^n a_k x^k$$

has $x = 1$ as a double root and the coefficients c_k , $k = 0, 1, \dots, n - 2$ of the polynomial

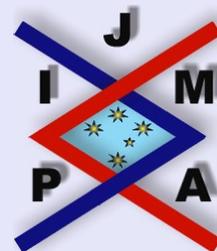
$$(1.3) \quad \frac{\sum_{k=0}^n a_k x^k}{(x-1)^2} = \sum_{k=0}^{n-2} c_k x^k$$

are positive. Then

$$(1.4) \quad \sum_{k=0}^n a_k u_k \geq 0$$

if the sequence $\{u_k\}$ is convex.

The aim of this note is to show that the inequality (1.4) holds for every convex sequence $\{u_k\}$ if and only if the polynomial given by (1.2) has $x = 1$ as a double root and the coefficients c_k ($k = 0, 1, \dots, n - 2$) of the polynomial given by (1.3) are positive.



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2. A Result of Tiberiu Popoviciu

Let n be a fixed natural number and

$$(2.1) \quad x_0 < x_1 < \cdots < x_n$$

$n + 1$ distinct points on the real axis. We denote by S the linear subspace of the real functions defined on the set of the points (2.1). If a_0, a_1, \dots, a_n are $n + 1$ fixed real numbers we define the linear functional $A, A : S \rightarrow \mathbb{R}$ by

$$(2.2) \quad A(f) = \sum_{k=0}^n a_k f(x_k).$$

T. Popoviciu ([3]) proved the following results:

Theorem 2.1.

(a) *The functional A is zero for every polynomial of degree at the most one if and only if there exist the constants $\alpha_0, \alpha_1, \dots, \alpha_{n-2}$ independent of the function f , such that the following equality holds:*

$$(2.3) \quad A(f) = \sum_{k=0}^{n-2} \alpha_k [x_k, x_{k+1}, x_{k+2}; f],$$

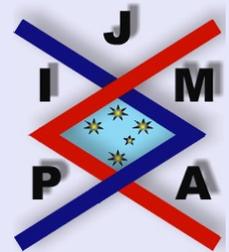
where $[x_k, x_{k+1}, x_{k+2}; f]$ is divided difference of the function f .

(b) *If there exists an index k ($0 \leq k \leq n - 2$) such that $\alpha_k \neq 0$, then*

$$(2.4) \quad A(f) \geq 0,$$

for every convex function f if and only if

$$(2.5) \quad \alpha_i \geq 0, \quad i = 0, 1, \dots, n - 2.$$



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3. Main Result

Theorem 3.1. Let a_0, a_1, \dots, a_n be $n+1$ fixed real numbers such that $\sum_{k=0}^n a_k^2 > 0$. The inequality

$$(3.1) \quad \sum_{k=0}^n a_k u_k \geq 0$$

holds for every convex sequence $\{u_k\}$ if and only if the polynomial given by (1.2) has $x = 1$ as a double root and all coefficients c_k of the polynomial given by (1.3) are positive.

Proof. The sufficiency of the theorem was proved by A. Mc D. Mercer in [2].

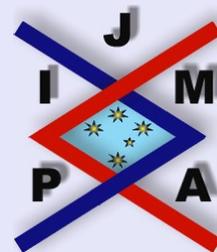
We suppose that the inequality (3.1) is valid for every convex sequence. The sequences $\{1\}$, $\{-1\}$, $\{k\}$ and $\{-k\}$ are convex sequences. By (3.1) we get

$$(3.2) \quad \sum_{k=0}^n a_k = 0$$

$$\sum_{k=1}^n k a_k = 0.$$

We denote by $f, f : [0, 1] \rightarrow \mathbb{R}$, the polygonal line having its vertices $(\frac{k}{n}, u_k)$, $k = 0, 1, \dots, n$.

The sequence $\{u_k\}$ is convex if and only if the function f is convex.



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Let us denote by

$$A(f) = \sum_{k=0}^n a_k f\left(\frac{k}{n}\right).$$

The inequality (3.1) holds for every convex sequence $\{u_k\}$ if and only if

$$(3.3) \quad A(f) \geq 0$$

for every function f which is convex on the set $\{0, \frac{1}{n}, \dots, \frac{n}{n}\}$.

By (3.2) we have

$$A(P) = 0$$

for every polynomial P having its degree at the most one. Using Popoviciu's Theorem 2.1, it follows that there exist the constants $\alpha_0, \alpha_1, \dots, \alpha_{n-2}$, independent of the function f such that

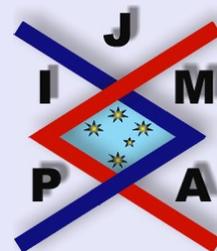
$$(3.4) \quad A(f) = \sum_{k=0}^{n-2} \alpha_k \left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; f \right],$$

for every function f defined of the set $\{0, \frac{1}{n}, \dots, \frac{n}{n}\}$.

By the equality

$$\sum_{k=0}^n \alpha_k \left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; f \right] = \sum_{k=0}^n a_k f\left(\frac{k}{n}\right),$$

we get $\alpha_k = \frac{2}{n^2} c_k, k = 0, 1, \dots, n - 2$.



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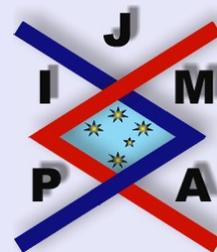
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Because $x = 1$ is a double root for the polynomial given by (1.2) we have

$$\sum_{k=0}^n c_k \neq 0.$$

Using again Popoviciu's Theorem (b), $A(f) \geq 0$ if and only if $c_k \geq 0$, $k = 0, \dots, n - 2$, and our theorem is proved. \square



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4. Another Proof of (1.1)

Let us consider the Bernstein operator B_n ,

$$(4.1) \quad B_n(f)(x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k = 0, 1, \dots, n$.

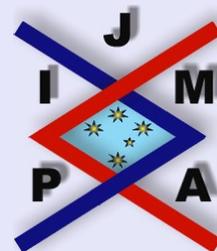
It is well known that for every convex function f , B_n is a convex function too. For such a function, we have, by Jensen's inequality,

$$(4.2) \quad \int_0^1 B_n(f)(x) dx \geq B_n(f)\left(\frac{1}{2}\right).$$

On the other hand we have

$$(4.3) \quad \int_0^1 p_{n,k}(x) dx = \frac{1}{n+1},$$
$$p_{n,k}\left(\frac{1}{2}\right) = \binom{n}{k} \frac{1}{2^n}, \quad k = 0, 1, \dots, n.$$

Now, the inequality (1.1) follows by (4.2) and (4.3).



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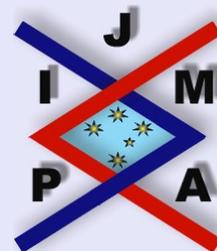
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