Journal of Inequalities in Pure and
Applied Mathematics

ANOTHER VERSION OF ANDERSON'S INEQUALITY IN THE IDEAL OF ALL COMPACT OPERATORS

SALAH MECHERI

King Saud University, College of Science
Department of Mathematics
P.O. Pox 2455, Riyah 11451
Saudi Arabia

EMail: mecherisalah@hotmail.com

volume 6, issue 3, article 90,
2005.

*Received 02 April, 2003;
accepted 16 June, 2005.*

Communicated by: *A. Babenko*

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ISSN (electronic): 1443-5756
041-03

Abstract

This note studies how certain problems in quantum theory have motivated some recent research in pure Mathematics in matrix and operator theory. The mathematical key is that of a commutator. We introduce the notion of the pair (A, B) of operators having the Fuglede-Putnam's property in the ideal of all compact operators. The characterization of this class leads us to generalize some recent results. We also give some applications of these results.

2000 Mathematics Subject Classification: Primary 47B47, 47A30, 47B20; Secondary 47B10.
Key words: Generalized derivation, Orthogonality, Compact operators.

I would like to thank the referee for his careful reading of the paper. His valuable suggestions, critical remarks, and pertinent comments made numerous improvements throughout. This research was supported by the K.S.U. research center project no. Math/2005/04.

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1. Introduction

Let H denote a separable infinite-dimensional complex Hilbert space. Let

$$\mathcal{L}(H) \supset \mathcal{K}(H) \supset C_p \supset \mathcal{F}(H)$$

$(0 < p < \infty)$ denote, respectively, the class of all bounded linear operators, the class of compact operators, the Schatten p-class, and the class of finite rank operators on H. All operators herein are assumed to be linear and bounded. Let $\|\cdot\|_p$, $\|\cdot\|_\infty$ denote, respectively, the C_p- norm and the $\mathcal{K}(H)$-norm. Let $\mathcal{I}$ be a proper bilateral ideal of $\mathcal{L}(H)$. It is well known that if $\mathcal{I} \neq \{0\}$, then $\mathcal{K}(H) \supset \mathcal{I} \supset \mathcal{F}(H)$. For $A, B \in \mathcal{L}(H)$ we define the generalized derivation $\delta_{A,B}$ as follows

$$\delta_{A,B}(X) = AX - XB$$

for $X \in \mathcal{L}(H)$ (so that $\delta_{A,A} = \delta_A$). In [1, Theorem 1.7], J. Anderson shows that if A is normal and commutes with T then,

$$\|T - (AX - XA)\| \geq \|T\|, \tag{1.1}$$

for all $X \in \mathcal{L}(H)$. In [11] we generalized this inequality, showing that if the pair (A, B) has the Fuglede-Putnam's property (in particular if A and B are normal operators) and $AT = TB$, then for all $X \in \mathcal{L}(H)$,

$$\|T - (AX - XB)\| \geq \|T\|.$$

The related inequality (1.1) was obtained by P.J. Maher [13, Theorem 3.2] showing that if A is normal and $AT = TA$, where $T \in C_p$, then

$$\|T-(AX-XA)\|_p \geq \|T\|_p$$

for all $X \in \mathcal{L}(H)$, where C_p is the von Neumann-Schatten class,

$1 \leq p < \infty$ and $\|\cdot\|_p$ its norm. In [12] we generalized P.J. Maher's result, proving that if the pair (A, B) has the Fuglede-Putnam's property $(FP)_{C_p}$, then

$$\|T-(AX-XB)\|_p \geq \|T\|_p$$

for all $X \in \mathcal{L}(H)$, and for all $T \in C_p \cap \ker \delta_{A,B}$. In [9] F. Kittaneh shows that if the pair (A, B) has the Fuglede-Putnam's property in $\mathcal{L}(H)$ then

$$\|T-(AX-XB)\|_I \geq \|T\|_I$$

for all $X \in \mathcal{L}(H)$, and for all $T \in I \cap \ker \delta_{A,B}$. In order to generalize these results, we prove that if the pair (A, B) has the $(FP)_{\mathcal{K}(H)}$ property (the Fuglede-Putnam's property in $\mathcal{K}(H)$), then

$$\|T-(AX-XB)\|_\infty \geq \|T\|_\infty$$

for all $X \in \mathcal{K}(H)$ and for all $T \in \mathcal{K}(H) \cap \ker \delta_{A,B}$. That is, the zero generalized commutator is the generalized commutator in $\mathcal{K}(H)$ of T.

A.H. Almoadjil [2] shows that if A is normal and for every $X \in \mathcal{L}(H)$, $A^2X = XA^2$ and $A^3X = XA^3$, then $AX = XA$. However F. Kittaneh [7] generalizes the Almoadjil's theorem by choosing A and B^* subnormal. There are of course other co-prime pairs of powers of A and B, such as 2 and $2n+1$ or 3 and $2n+1$ (with 3 and $2n+1$ co-prime), for which a similar result can be proved. Notice here that for such co-prime powers of A and B, the hypothesis that the pair (A, B) has the $(FP)_{\mathcal{K}(H)}$ property implies that $\delta^m_{A,B}(X) = 0$ for some integer $m > 1$, and the conclusion $X \in \ker \delta_{A,B}$ is a consequence of the following general result: Let $\delta^m_{A,B}$ denote an $m-$times application of $\delta_{A,B}$. If the pair (A, B) has the $(FP)_{\mathcal{K}(H)}$ property and $\delta^m_{A,B}(X) = 0$ for some integer $m > 1$, then $\delta_{A,B}(X) = 0$.

2. Orthogonality

We begin by the following definition of the orthogonality in the sense of G. Birkhoff [3] which generalizes the idea of orthogonality in Hilbert space.

Definition 2.1. *Let* $\mathbb{C}$ *be the field of complex numbers and let* E *be a normed linear space. Let* $x, y \in E$*. If* $\|x - \lambda y\| \geq \|\lambda y\|$ *for all* $\lambda \in \mathbb{C}$*, then* x *is said to be orthogonal to* y*. Let* F *and* G *be two subspaces in* E*. If* $\|x + y\| \geq \|y\|$*, for all* $x \in F$ *and for all* $y \in G$*, then* F *is said to be orthogonal to* G*.*

Definition 2.2. *Let* $A, B \in \mathcal{L}(H)$*. We say that the pair* (A, B) *satisfies* $(FP)_{\mathcal{K}(H)}$*, if* $AC = CB$ *where* $C \in \mathcal{K}(H)$ *implies* $A^*C = CB^*$*.*

Theorem 2.1. *Let* $A, B \in \mathcal{L}(H)$*. If* A *and* B *are normal operators, then*

$$\|S - (AX - XB)\|_\infty \geq \|S\|_\infty$$

for all $X \in \mathcal{L}(H)$ *and for all* $S \in \ker \delta_{A,B} \cap \mathcal{K}(H)$*.*

Proof. Let $S = U|S|$ be the polar decomposition of S, where U is an isometry such that $\ker U = \ker |S|$. Since

$$\|U^*S\|_\infty \leq \|U^*\|_\infty \|S\|_\infty = \|S\|_\infty$$

for all $S \in \mathcal{K}(H)$,

$$\begin{aligned} \|S - (AX - XB)\|_\infty &\geq \sup_n |(U^*[S - (AX - XB)]\varphi_n, \varphi_n)| \\ &= \sup_n([|S| - U^*(AX - XB)]\varphi_n, \varphi_n) \end{aligned} \tag{2.1}$$

for any orthonormal basis $\{\varphi_n\}_{n\geq 1}$ of H. Since $AS = SB$ and A, B are normal operators, then it follows from the Fuglede-Putnam's theorem that $S^*A = BS^*$; consequently $S^*AS = BS^*S$ or $S^*SB = BS^*S$, i.e, $B\,|S| = |S|\,B$. Since $|S|$ is a compact normal operator and commutes with B, there exists an orthonormal basis $\{f_k\} \cup \{g_m\}$ of H such that $\{f_k\}$ consists of common eigenvectors of B and $|S|$, and $\{g_m\}$ is an orthonormal basis of $\ker|S|$. Since $\{f_k\}$ is an orthonormal basis of the normal operator B, then there exists a scalar α_k such that $f_k = \alpha_k f_k$ and $B^* f_k = \overline{\alpha}_k f_k$; consequently

$$\begin{aligned}
\langle U^*(AX - XB)f_k, |S|\,f_k\rangle &= \langle S^*(AX - XB)f_k, f_k\rangle \\
&= \langle (B(S^*X) - (S^*X)B)f_k,\ f_k\rangle = 0.
\end{aligned}$$

That is, $\langle U^*(AX - XB)f_k, f_k\rangle = 0$. In (2.1) take $\{\varphi_n\} = \{f_k\} \cup \{g_m\}$ as an orthonormal basis of H. Then

$$\begin{aligned}
\|S - (AX - XB)\|_\infty &\geq \sup_n([|S| - U^*(AX - XB)]\varphi_n, \varphi_n) \\
&= \sup_{k,m}[|S|\,f_k,\ f_k) + (U^*(AX - XB)g_m, g_m)] \\
&\geq \sup_k(|S|\,f_k,\ f_k) \\
&= \||S|\| = \|S\|_\infty\,.
\end{aligned}$$

□

Theorem 2.2. *Let* $A, B \in \mathcal{L}(H)$. *If the pair* (A, B) *satisfies the* $(FP)_{\mathcal{K}(H)}$ *property, then*

(2.2) $$\|\delta_{A,B}(X) + S\|_\infty \geq \|S\|_\infty\,,$$

for all $X \in \mathcal{K}(H)$, *and for all* $S \in \mathcal{K}(H) \cap \ker(\delta_{A,B})$. *In particular we have*

$$R(\delta_{A,B}\,|_{\mathcal{K}(H)}) \cap \ker(\delta_{A,B}\,|_{\mathcal{K}(H)}) = \{0\}\,, \tag{2.3}$$

where $R(\delta_{A,B})$ *and* $\ker(\delta_{A,B})$ *denote the range and the kernel of* $\delta_{A,B}$.

Proof. It is well known that if the pair (A, B) satisfies the $(FP)_{\mathcal{K}(H)}$ property, then $\overline{R(S)}$ reduces A, $\ker^{\perp} S$ reduces B and $A\,|_{\overline{R(S)}}$, $B\,|_{\ker^{\perp} S}$ are normal operators. Letting $S_0 : \ker^{\perp} S \to \overline{R(S)}$ be the quasi-affinity defined by setting $S_0 x = Sx$ for each $x \in \ker^{\perp} S$, then it results that $\delta_{A_1,B_1}(S_0) = \delta_{A_1^*,B_1^*}(S_0) = 0$. Let $A = A_1 \oplus A_2$, with respect to $H = \overline{R(S)} \oplus \overline{R(S)}^{\perp}$, $B = B_1 \oplus B_2$, with respect to $H = \ker(S)^{\perp} \oplus \ker S$ and $X : \overline{R(S)} \oplus \overline{R(S)}^{\perp} \to \ker(S)^{\perp} \oplus \ker S$ have the matrix representation

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Then we have

$$\|S - (AX - XB)\|_\infty = \left\| \begin{bmatrix} S_1 - (A_1X_1 - X_1B_1) & * \\ * & * \end{bmatrix} \right\|_\infty .$$

The result of I.C. Gohberg and M.G. Krein [6] guarantees that

$$\|S - (AX - XB)\|_\infty \geq \|S_1 - (A_1X_1 - X_1B_1)\|_\infty .$$

Since A_1 and B_1 are two normal operators, it results from Theorem 2.2 that

$$\|S_1 - (A_1X_1 - X_1B_1)\|_\infty \geq \|S_1\|_\infty = \|S\|_\infty$$

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and

$$\|S-(AX-XB)\|_{\infty} \geq \|S_1-(A_1X_1-X_1B_1)\|_{\infty} \geq \|S_1\|_{\infty} = \|S\|_{\infty}.$$

$\square$

We can ask "Is the sufficient condition in Theorem 2.2 necessary?"

3. Examples and Applications

The related topic of approximation by commutators $AX - XA$ or by generalized commutator $AX - XB$, which has attracted much interest, has its roots in quantum theory. The Heinsnberg Uncertainly principle may be mathematically formulated as saying that there exists a pair A, X of linear transformations and a non-zero scalar α for which

$$AX - XA = \alpha I. \tag{3.1}$$

Clearly, (3.1) cannot hold for square matrices A and X and for bounded linear operators A and X. This prompts the question:

How close can $AX - XA$ be the identity?

Williams [17] proved that if A is normal, then, for all X in $B(H)$,

$$\|I - (AX - XA)\| \geq \|I\|. \tag{3.2}$$

Mecheri [14] generalized Williams inequality (3.2): he proved that if A, B are normal, then for all $X \in B(H)$

$$\|I - (AX - XB)\| \geq \|I\|. \tag{3.3}$$

Anderson [1] generalized Williams inequality (3.2): he proved that if A is normal and commutes with B then, for all $X \in B(H)$

$$\|B - (AX - XA)\| \geq \|B\|. \tag{3.4}$$

Maher [13] obtained the C_p variants of Anderson's result. Mecheri [14] studied approximation by generalized commutators $AX - XC$: he showed that the following inequality holds

$$||B - (AX - XC)||_p \geq ||B||_p, \tag{3.5}$$

for all $X \in C_p$ if and only if $B \in \ker \delta_{A,B}$. In Theorem 2.2 we obtained the $\mathcal{K}(H)$ of Maher and Mecheri's results.

In the previous inequality (3.5) the zero generalized commutator is a generalized commutator approximant in C_P of B.

Now we are ready to give some operators for which the inequality (2.2) holds.

Corollary 3.1. *Let* $A, B \in L(H)$. *Then the pair* (A, B) *has the* $(FP)_{\mathcal{K}(H)}$ *property in each of the following cases:*

1. *If* $A, B \in \mathcal{L}(H)$ *such that* $\|Ax\| \geq \|x\| \geq \|Bx\|$ *for all* $x \in H$.
2. *If* A *is invertible and* B *such that* $\|A^{-1}\| \, \|B\| \leq 1$.
3. *If* $A = B$ *is a cyclic subnormal operator.*

Proof. The result of Y. Tong [16, Lemma 1] guarantees that the above condition implies that for all $T \in \ker(\delta_{A,B} \mid \mathcal{K}(H))$, $\overline{R(T)}$ reduces A, $\ker(T)^{\perp}$ reduces B, and $A \mid_{\overline{R(T)}}$ and $B \mid_{\ker(T)^{\perp}}$ are unitary operators. Hence it results from Theorem 2.2 that the pair (A, B) has the property $(FP)_{\mathcal{K}(H)}$ and the result holds by the above theorem. The above inequality holds in particular if $A = B$ is isometric, in other words $\|Ax\| = \|x\|$ for all $x \in H$.

(2) In this case it suffices to take $A_1 = \|B\|^{-1} A$ and $B_1 = \|B\|^{-1} B$, then $\|A_1 x\| \geq \|x\| \geq \|B_1 x\|$ and the result holds by (1) for all $x \in H$.
(3) Since T commutes with A, it follows that T is subnormal [18]. But any compact subnormal operator is normal. Hence T is normal. Now $AT = TA$ implies $A^*T = TA^*$, i.e, the pair (A, A) has the $(FP)_{\mathcal{K}(H)}$ property. ☐

Theorem 3.2. *Let $A, B \in \mathcal{L}(H)$ such that the pairs (A, A) and (B, B) have the $(FP)_{\mathcal{K}(H)}$ property. If $\sigma(A) \cap \sigma(B) = \phi$, then*

$$\|T - \delta_{A\oplus B, A\oplus B}(X)\|_\infty \geq \|T\|_\infty$$

for all $X \in \mathcal{K}(H)$, and for all $T \in \mathcal{K}(H) \cap \ker(\delta_{A,B})$.

Proof. It suffices to show that the pair $(A\oplus B, A\oplus B)$ has the $(FP)_{\mathcal{K}(H)}$ property. Let

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

be in $\mathcal{K}(H \oplus H)$. If $(A \oplus B)T = T(A \oplus B)$, then $AT_1 = T_1A$, $BT_4 = T_4B$, $AT_2 = T_2B$ and $BT_3 = T_3A$. Since $\sigma(A) \cap \sigma(B) = \phi$, then $\delta_{A,B}$, $\delta_{B,A}$ are invertible [12]. Consequently $T_2 = T_3 = 0$ and since (A, A) and (B, B) have the $(FP)_{\mathcal{K}(H)}$ property, $AT_1^* = T_1^*A$ and $BT_4^* = T_4^*B$, that is, $(A \oplus B)T^* = T^*(A \oplus B)$. ☐

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4. On the Commutant of A and its Powers

In this section we will be interested on the investigation of the relation between the commutant of a bounded linear operator A and its powers.

Lemma 4.1. *Let* $A, B \in \mathcal{L}(H)$. *Then*

$$R(\delta_{A,B}) \cap \ker \delta_{A,B} = \{0\} \Leftrightarrow \ker \delta^m_{A,B} = \ker \delta_{A,B},$$

for all $m \geq 1$.

Proof. Suppose that $R(\delta_{A,B}) \cap \ker \delta_{A,B} = \{0\}$. It suffices to prove that

$$\ker \delta^2_{A,B} \subset \ker \delta_{A,B}.$$

If $X \in \ker \delta^2_{A,B}$, then $\delta_{A,B}(X) \in R(\delta_{A,B}) \cap \ker \delta_{A,B} = \{0\}$, i.e. $X \in \ker \delta_{A,B}$. Conversely if $Y \in R(\delta_{A,B}) \cap \ker \delta_{A,B}$, then $Y = \delta_{A,B}(X)$ for some $X \in \mathcal{L}(H)$ and $\delta_{A,B}(Y) = 0$. Consequently we have $\delta^2_{A,B}(X) = 0$, i.e. $X \in \ker \delta^2_{A,B} = \ker \delta_{A,B}$. Then we obtain $\delta_{A,B}(X) = 0$, i.e. $Y = 0$. □

Lemma 4.2. *If* $R(\delta_{A,B}) \cap \ker \delta_{A,B} = \{0\}$, *then*

$$\ker \delta_{A,B} = \bigcap_{i=2}^{\infty} \ker \delta_{A^i,B^i}.$$

Proof. Note that $\ker \delta_{A,B} \subset \bigcap_{i=2}^{\infty} \ker \delta_{A^i,B^i}$. Hence it suffices to prove the opposite inclusion. If $X \in \bigcap_{i=2}^{\infty} \ker \delta_{A^i,B^i}$, then $A^2X = XB^2$ and $A^3X = XB^3$. Hence $A^2XB = XB^3$ and $AXB^2 = A^3X$. Let $C = AX - XB$. Then,

$$A^2C = A^3X - A^2XB = XB^3 - XB^3 = 0;$$

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$$CB^2 = AXB^2 - XB^3 = A^3X - A^3X = 0;$$
$$ACB = A^2XB - AXB^2 = XB^3 - XB^3 = 0;$$

hence

$$A(AC - CB) = A^2C - ACB = 0; \tag{4.1}$$

$$(AC - CB)B = ACB - CB^2 = 0. \tag{4.2}$$

Thus (4.1) and (4.2) imply that

$$AC - CB \in R(\delta_{A,B}) \cap \ker \delta_{A,B} = \{0\},$$

from which it results that $AC = CB$. Hence

$$C \in R(\delta_{A,B}) \cap \ker \delta_{A,B},$$

that is, $C = 0$ and thus $AX = XB$, i.e, $X \in \ker \delta_{A,B}$. □

Theorem 4.3. *If (A, B) has the $(FP)_{\mathcal{K}(H)}$ property, then*

$$\ker \delta^m_{A,B} = \ker \delta_{A,B} = \bigcap_{i=2}^{\infty} \ker \delta_{A^i,B^i},\ m \geq 1.$$

In particular if $A^2X = XB^2$ and $A^3X = XB^3$ for some $X \in \mathcal{K}(H)$, then $AX = XB$.

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Proof. This is an immediate consequence of Lemma 4.1, Lemma 4.2 and Theorem 2.2. ☐

Remark 1. *The above theorem generalizes the results of F. Kittaneh [9] and Almoadjil [2]. In [8] F. Kittaneh shows that if the pair* (A, B) *has the* $(FP)_{\mathcal{L}(H)}$ *property, then for all* $T \in \ker(\delta_{A,B} \mid_{\mathcal{I}})$ *and for all* $X \in \mathcal{I}$,

$$\left\| \delta_{A,B}(X) + S \right\|_{\mathcal{I}} \geq \left\| S \right\|_{\mathcal{I}} .$$

In Theorem 2.2 we show that it suffices that the pair (A, B) *has the* $(FP)_{\mathcal{K}(H)}$ *property for which* $R(\delta_{A,B} \mid_{\mathcal{K}(H)})$ *is orthogonal to* $\ker(\delta_{A,B} \mid_{\mathcal{K}(H)})$. *The results of this paper are also true in the case where* $\mathcal{K}(H)$ *is replaced by a two sided ideal of* $\mathcal{L}(H)$. *Hence Theorem 2.2 generalizes the results of F. Kittaneh [8], [9] and of S. Mecheri [12].*

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