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ON SOME FUNDAMENTAL INTEGRAL INEQUALITIES AND THEIR DISCRETE ANALOGUES

B.G. PACHPATTE

DEPARTMENT OF MATHEMATICS, MARATHWADA UNIVERSITY
AURANGABAD 431 004 (MAHARASHTRA) INDIA.
bgpachpatte@hotmail.com

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ABSTRACT. The aim of the present paper is to establish some new integral inequalities in two independent variables and their discrete analogues which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in the qualitative theory of certain partial differential and finite difference equations.

Key words and phrases: Integral inequalities, discrete analogues, two independent variables, partial differential and difference equations, nondecreasing, nonincreasing, terminal value, non-self-adjoint hyperbolic partial differential equation.

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1. INTRODUCTION

The integral and finite difference inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential and finite difference equations. During the past few years, many such new inequalities have been discovered, which are motivated by certain applications (see [1]–[10]). In the qualitative analysis of some classes of partial differential and finite difference equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new inequalities in order to achieve a diversity of desired goals. Our main objective here is to establish some useful integral inequalities involving functions of two independent variables and their discrete analogues which can be used as ready and powerful tools in the analysis of certain classes of partial differential and finite difference equations.

2. STATEMENT OF RESULTS

In what follows, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, $N_0 = \{0, 1, 2, \dots\}$ are the given subsets of $\mathbb{R}$. The first order partial derivatives of a function $z(x, y)$ defined for $x, y \in \mathbb{R}$ with respect to x and y are denoted by $z_x(x, y)$ and $z_y(x, y)$ respectively. We

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use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. Throughout the paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals, sums and products involved exist on the respective domains of their definitions.

We need the inequalities in the following lemma, which are the slight variants of the inequalities given in [5, pp. 12, 28].

Lemma 2.1. *Let $u(t)$, $a(t)$, $b(t)$ be nonnegative and continuous functions defined for $t \in \mathbb{R}_+$.*

(α_1) *Assume that $a(t)$ is nondecreasing for $t \in \mathbb{R}_+$. If*

$$u(t) \le a(t) + \int_0^t b(s)\, u(s)\, ds,$$

for $t \in \mathbb{R}_+$, then

$$u(t) \le a(t) \exp\left(\int_0^t b(s)\, ds \right),$$

for $t \in \mathbb{R}_+$.

(α_2) *Assume that $a(t)$ is nonincreasing for $t \in \mathbb{R}_+$. If*

$$u(t) \le a(t) + \int_t^\infty b(s)\, u(s)\, ds,$$

for $t \in \mathbb{R}_+$, then

$$u(t) \le a(t) \exp\left(\int_t^\infty b(s)\, ds \right),$$

for $t \in \mathbb{R}_+$.

The proofs of the inequalities in (α_1), (α_2) can be completed as in [5, pp. 12, 28, 325-326] (see also [4]). Here we omit the details.

Our main results on integral inequalities are established in the following theorems.

Theorem 2.2. *Let $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$ be nonnegative continuous functions defined for $x, y \in \mathbb{R}_+$.*

(a_1) *If*

$$u(x,y) \le a(x,y) + b(x,y) \int_0^x \int_y^\infty c(s,t)\, u(s,t)\, dtds, \tag{2.1}$$

for $x, y \in \mathbb{R}_+$, then

$$u(x,y) \le a(x,y) + b(x,y)\, e(x,y) \exp\left(\int_0^x \int_y^\infty c(s,t)\, b(s,t)\, dtds \right), \tag{2.2}$$

for $x, y \in \mathbb{R}_+$, where

$$e(x,y) = \int_0^x \int_y^\infty c(s,t)\, a(s,t)\, dtds, \tag{2.3}$$

for $x, y \in \mathbb{R}_+$.

(a_2) *If*

$$u(x,y) \le a(x,y) + b(x,y) \int_x^\infty \int_y^\infty c(s,t)\, u(s,t)\, dtds, \tag{2.4}$$

for $x, y \in \mathbb{R}_+$, then

$$u(x,y) \le a(x,y) + b(x,y)\, \bar{e}(x,y) \exp\left(\int_x^\infty \int_y^\infty c(s,t)\, b(s,t)\, dtds \right), \tag{2.5}$$

for $x, y \in \mathbb{R}_+$, *where*

$$\bar{e}(x,y) = \int_x^\infty \int_y^\infty c(s,t)\, a(s,t)\, dtds, \tag{2.6}$$

for $x, y \in \mathbb{R}_+$.

Theorem 2.3. *Let* $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$ *be nonnegative continuous functions defined for* $x, y \in \mathbb{R}_+$.

(b_1) *Assume that* $a(x,y)$ *is nondecreasing in* $x \in \mathbb{R}_+$. *If*

$$u(x,y) \le a(x,y) + \int_0^x b(s,y)\, u(s,y)\, ds + \int_0^x \int_y^\infty c(s,t)\, u(s,t)\, dtds, \tag{2.7}$$

for $x, y \in \mathbb{R}_+$, *then*

$$u(x,y) \le p(x,y)\left[a(x,y) + A(x,y)\exp\left(\int_0^x \int_y^\infty c(s,t)\, p(s,t)\, dtds\right)\right], \tag{2.8}$$

for $x, y \in \mathbb{R}_+$, *where*

$$p(x,y) = \exp\left(\int_0^x b(s,y)\, ds\right), \tag{2.9}$$

$$A(x,y) = \int_0^x \int_y^\infty c(s,t)\, p(s,t)\, a(s,t)\, dtds, \tag{2.10}$$

for $x, y \in \mathbb{R}_+$.

(b_2) *Assume that* $a(x,y)$ *is nonincreasing in* $x \in \mathbb{R}_+$. *If*

$$u(x,y) \le a(x,y) + \int_x^\infty b(s,y)\, u(s,y)\, ds + \int_x^\infty \int_y^\infty c(s,t)\, u(s,t)\, dtds, \tag{2.11}$$

for $x, y \in \mathbb{R}_+$, *then*

$$u(x,y) \le \bar{p}(x,y)\left[a(x,y) + \bar{A}(x,y)\exp\left(\int_x^\infty \int_y^\infty c(s,t)\, \bar{p}(s,t)\, dtds\right)\right], \tag{2.12}$$

for $x, y \in \mathbb{R}_+$, *where*

$$\bar{p}(x,y) = \exp\left(\int_x^\infty b(s,y)\, ds\right), \tag{2.13}$$

$$\bar{A}(x,y) = \int_x^\infty \int_y^\infty c(s,t)\, \bar{p}(s,t)\, a(s,t)\, dtds, \tag{2.14}$$

for $x, y \in \mathbb{R}_+$.

Theorem 2.4. *Let* $u(x,y)$, $a(x,y)$, $b(x,y)$ *be nonnegative continuous functions defined for* $x, y \in \mathbb{R}_+$ *and* $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ *be a continuous function which satisfies the condition*

$$0 \le F(x,y,u) - F(x,y,v) \le K(x,y,v)(u - v),$$

for $u \ge v \ge 0$, *where* $K(x,y,v)$ *is a nonnegative continuous function defined for* $x, y, v \in \mathbb{R}_+$.

(c_1) *Assume that* $a(x,y)$ *is nondecreasing in* $x \in \mathbb{R}_+$. *If*

$$u(x,y) \le a(x,y) + \int_0^x b(s,y)\, u(s,y)\, ds + \int_0^x \int_y^\infty F(s,t,u(s,t))\, dtds, \tag{2.15}$$

for $x, y \in \mathbb{R}_+$*, then*

$$u(x,y) \le p(x,y)\left[a(x,y) + B(x,y) \times \exp\left(\int_0^x \int_y^\infty K(s,t,p(s,t)\,a(s,t))\,p(s,t)\,dtds\right)\right], \tag{2.16}$$

for $x, y \in \mathbb{R}_+$*, where*

$$B(x,y) = \int_0^x \int_y^\infty F(s,t,p(s,t)\,a(s,t))\,dtds, \tag{2.17}$$

for $x, y \in \mathbb{R}_+$ *and* $p(x,y)$ *is defined by (2.9).*

(c_2) *Assume that* $a(x,y)$ *is nonincreasing in* $x \in \mathbb{R}_+$*. If*

$$u(x,y) \le a(s,y) + \int_x^\infty b(s,y)\,u(s,y)\,ds + \int_x^\infty \int_y^\infty F(s,t,u(s,t))\,dtds, \tag{2.18}$$

for $x, y \in \mathbb{R}_+$*, then*

$$u(x,y) \le \bar{p}(x,y)\left[a(x,y) + \bar{B}(x,y) \times \exp\left(\int_x^\infty \int_y^\infty K(s,t,\bar{p}(s,t)\,a(s,t))\,\bar{p}(s,t)\,dtds\right)\right], \tag{2.19}$$

for $x, y \in \mathbb{R}_+$*, where*

$$\bar{B}(x,y) = \int_x^\infty \int_y^\infty F(s,t,\bar{p}(s,t)\,a(s,t))\,dtds, \tag{2.20}$$

for $x, y \in \mathbb{R}_+$ *and* $\bar{p}(x,y)$ *is defined by (2.13).*

We require the following discrete version of Lemma 2.1 to establish the discrete analogues of Theorems 2.3 and 2.4.

Lemma 2.5. *Let* $u(n)$*,* $a(n)$*,* $b(n)$ *be nonnegative functions defined for* $n \in N_0$*.*

(β_1) *Assume that* $a(n)$ *is nondecreasing for* $n \in N_0$*. If*

$$u(n) \le a(n) + \sum_{s=0}^{n-1} b(s)\,u(s),$$

for $n \in N_0$*, then*

$$u(n) \le a(n) \prod_{s=0}^{n-1} [1 + b(s)],$$

for $n \in N_0$*.*

(β_2) *Assume that* $a(n)$ *is nonincreasing for* $n \in N_0$*. If*

$$u(n) \le a(n) + \sum_{s=n+1}^{\infty} b(s)\,u(s),$$

for $n \in N_0$*, then*

$$u(n) \le a(n) \prod_{s=n+1}^{\infty} [1 + b(s)],$$

for $n \in N_0$*.*

The proof of (β_1) can be completed by following the proof of Theorem 1° in [6, p. 256] and closely looking at the proof of Theorem 4.2.2 in [5, p. 326]. For the proof of (β_2), see [10] and also [4].

The discrete analogues of Theorems 2.2 – 2.4 are given in the following theorems.

Theorem 2.6. *Let* $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$ *be nonnegative functions defined for* $m,n \in N_0$.

(p_1) *If*

$$u(m,n) \le a(m,n) + b(m,n) \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} c(s,t)\, u(s,t), \tag{2.21}$$

for $m,n \in N_0$, *then*

$$u(m,n) \le a(m,n) + b(m,n)\, f(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{\infty} c(s,t)\, b(s,t)\right], \tag{2.22}$$

for $m,n \in N_0$, *where*

$$f(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} c(s,t)\, a(s,t), \tag{2.23}$$

for $m,n \in N_0$.

(p_2) *If*

$$u(m,n) \le a(m,n) + b(m,n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} c(s,t)\, u(s,t), \tag{2.24}$$

for $m,n \in N_0$, *then*

$$u(m,n) \le a(m,n) + b(m,n)\, \bar{f}(m,n) \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} c(s,t)\, b(s,t)\right], \tag{2.25}$$

for $m,n \in N_0$, *where*

$$\bar{f}(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} c(s,t)\, a(s,t), \tag{2.26}$$

for $m,n \in N_0$.

Theorem 2.7. *Let* $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$ *be nonnegative functions defined for* $m,n \in N_0$.

(q_1) *Assume that* $a(m,n)$ *is nondecreasing in* $m \in N_0$. *If*

$$u(m,n) \le a(m,n) + \sum_{s=0}^{m-1} b(s,n)\, u(s,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} c(s,t)\, u(s,t), \tag{2.27}$$

for $m,n \in N_0$, *then*

$$u(m,n) \le q(m,n) \left[a(m,n) + G(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{\infty} c(s,t)\, q(s,t)\right]\right], \tag{2.28}$$

for $m, n \in N_0$*, where*

$$q(m,n) = \prod_{s=0}^{m-1} [1 + b(s,n)], \tag{2.29}$$

$$G(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} c(s,t)\, q(s,t)\, a(s,t), \tag{2.30}$$

for $m, n \in N_0$.

(q_2) *Assume that* $a(m,n)$ *is nonincreasing in* $m \in N_0$*. If*

$$u(m,n) \le a(m,n) + \sum_{s=m+1}^{\infty} b(s,n)\, u(s,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} c(s,t)\, u(s,t), \tag{2.31}$$

for $m, n \in N_0$*, then*

$$u(m,n) \le \bar{q}(m,n) \left[a(m,n) + \bar{G}(m,n) \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} c(s,t)\, \bar{q}(s,t) \right] \right], \tag{2.32}$$

for $m, n \in N_0$*, where*

$$\bar{q}(m,n) = \prod_{s=m+1}^{\infty} [1 + b(s,n)], \tag{2.33}$$

$$\bar{G}(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} c(s,t)\, \bar{q}(s,t)\, a(s,t), \tag{2.34}$$

for $m, n \in N_0$.

Theorem 2.8. *Let* $u(m,n)$, $a(m,n)$, $b(m,n)$ *be nonnegative functions defined for* $m, n \in N_0$ *and* $L : N_0^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a function which satisfies the condition*

$$0 \le L(m,n,u) - L(m,n,v) \le M(m,n,v)(u - v),$$

for $u \ge v \ge 0$*, where* $M(m,n,v)$ *is a nonnegative function for* $m, n \in N_0$*,* $v \in \mathbb{R}_+$.

(r_1) *Assume that* $a(m,n)$ *is nondecreasing in* $m \in N_0$*. If*

$$u(m,n) \le a(m,n) + \sum_{s=0}^{m-1} b(s,n)\, u(s,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} L(s,t,u(s,t)), \tag{2.35}$$

for $m, n \in N_0$*, then*

$$u(m,n) \le q(m,n) \Bigg[a(m,n) + H(m,n) \times \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{\infty} M(s,t,q(s,t)\, a(s,t))\, q(s,t) \right] \Bigg], \tag{2.36}$$

for $m, n \in N_0$*, where*

$$H(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} L(s,t,q(s,t)\, a(s,t)), \tag{2.37}$$

for $m, n \in N_0$ *and* $q(m,n)$ *is defined by (2.29).*

(q_2) *Assume that* $a(m,n)$ *is nonincreasing in* $m \in N_0$. *If*

$$u(m,n) \leq a(m,n) + \sum_{s=m+1}^{\infty} b(s,n)\, u(s,n) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} L(s,t,u(s,t)), \tag{2.38}$$

for $m,n \in N_0$, *then*

$$u(m,n) \leq \bar{q}(m,n)\Bigg[a(m,n) + \bar{H}(m,n) \times \prod_{s=m+1}^{\infty}\left[1 + \sum_{t=n+1}^{\infty} M(s,t,\bar{q}(s,t)\, a(s,t))\, \bar{q}(s,t)\right]\Bigg], \tag{2.39}$$

for $m,n \in N_0$, *where*

$$\bar{H}(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} L(s,t,\bar{q}(s,t)\, a(s,t)), \tag{2.40}$$

for $m,n \in N_0$ *and* $\bar{q}(m,n)$ *is defined by (2.33).*

3. PROOFS OF THEOREMS 2.2 – 2.4

Since the proofs resemble one another, we give the details for (a_1), (b_1) and (c_1); the proofs of (a_2), (b_2) and (c_2) can be completed by following the proofs of the above mentioned results with suitable changes.

(a_1) Define a function $z(x,y)$ by

$$z(x,y) = \int_0^x \int_y^\infty c(s,t)\, u(s,t)\, dt ds. \tag{3.1}$$

Then (2.1) can be restated as

$$u(x,y) \leq a(x,y) + b(x,y)\, z(x,y). \tag{3.2}$$

From (3.1) and (3.2) we have

$$\begin{aligned} z(x,y) &\leq \int_0^x \int_y^\infty c(s,t)\left[a(s,t) + b(s,t)\, z(s,t)\right] dt ds \\ &= e(x,y) + \int_0^x \int_y^\infty c(s,t)\, b(s,t)\, z(s,t)\, dt ds, \end{aligned} \tag{3.3}$$

where $e(x,y)$ is defined by (2.3). Clearly, $e(x,y)$ is nonnegative, continuous, nondecreasing in x and nonincreasing in y for $x,y \in \mathbb{R}_+$. First we assume that $e(x,y) > 0$ for $x,y \in \mathbb{R}_+$. From (3.3) it is easy to observe that

$$\frac{z(x,y)}{e(x,y)} \leq 1 + \int_0^x \int_y^\infty c(s,t)\, b(s,t)\, \frac{z(s,t)}{e(s,t)} dt ds. \tag{3.4}$$

Define a function $v(x,y)$ by the right hand side of (3.4), then $v(0,y)=v(x,\infty)=1$, $\frac{z(x,y)}{e(x,y)}\leq v(x,y)$, $v(x,y)$ is nonincreasing in y, $y\in\mathbb{R}_+$ and

$$v_x(x,y) = \int_y^\infty c(x,t)\,b(x,t)\,\frac{z(x,t)}{e(x,t)}dt \leq \int_y^\infty c(x,t)\,b(x,t)\,v(x,t)\,dt \leq v(x,y)\int_y^\infty c(x,t)\,b(x,t)\,dt. \tag{3.5}$$

Treating y, $y\in\mathbb{R}_+$ fixed in (3.5), dividing both sides of (3.5) by $v(x,y)$, setting $x=s$ and integrating the resulting inequality from 0 to x, $x\in\mathbb{R}_+$ we get

$$v(x,y)\leq\exp\left(\int_0^x\int_y^\infty c(s,t)\,b(s,t)\,dtds\right). \tag{3.6}$$

Using (3.6) in $\frac{z(x,y)}{e(x,y)}\leq v(x,y)$, we have

$$z(x,y)\leq e(x,y)\exp\left(\int_0^x\int_y^\infty c(s,t)\,b(s,t)\,dtds\right). \tag{3.7}$$

The desired inequality (2.2) follows from (3.2) and (3.7).

If $e(x,y)$ is nonnegative, we carry out the above procedure with $e(x,y)+\varepsilon$ instead of $e(x,y)$, where $\varepsilon>0$ is an arbitrary small constant, and then subsequently pass to the limit as $\varepsilon\to 0$ to obtain (2.2).

(b_1) Define a function $z(x,y)$ by

$$z(x,y)=\int_0^x\int_y^\infty c(s,t)\,u(s,t)\,dtds. \tag{3.8}$$

Then (2.7) can be restated as

$$u(x,y)\leq z(x,y)+\int_0^x b(s,y)\,u(s,y)\,ds. \tag{3.9}$$

Clearly $z(x,y)$ is a nonnegative, continuous and nondecreasing function in x, $x\in\mathbb{R}_+$. Treating y, $y\in\mathbb{R}_+$ fixed in (3.9) and using part (α_1) of Lemma 2.1 to (3.9), we get

$$u(x,y)\leq z(x,y)\,p(x,y), \tag{3.10}$$

where $p(x,y)$ is defined by (2.9). From (3.10) and (3.8) we have

$$u(x,y)\leq p(x,y)\left[a(x,y)+v(x,y)\right], \tag{3.11}$$

where

$$v(x,y)=\int_0^x\int_y^\infty c(s,t)\,u(s,t)\,dtds. \tag{3.12}$$

From (3.11) and (3.12) we get

$$v(x,y) \leq \int_0^x\int_y^\infty c(s,t)\,p(s,t)\left[a(s,t)+v(s,t)\right]dtds = A(x,y)+\int_0^x\int_y^\infty c(s,t)\,p(s,t)\,v(s,t)\,dtds,$$

where $A(x,y)$ is defined by (2.10). Clearly, $A(x,y)$ is nonnegative, continuous, nondecreasing in x, $x \in \mathbb{R}_+$ and nonincreasing in y, $y \in \mathbb{R}_+$. Now, by following the proof of (a_1), we obtain

$$v(x,y) \le A(x,y) \exp\left(\int_0^x \int_y^\infty c(s,t)\, p(s,t)\, dtds \right). \tag{3.13}$$

Using (3.13) in (3.11) we get the required inequality in (2.8).

(c_1) Define a function $z(x,y)$ by

$$z(x,y) = a(x,y) + \int_0^x \int_y^\infty F(s,t,u(s,t))\, dtds. \tag{3.14}$$

Then (2.15) can be restated as

$$u(x,y) \le z(x,y) + \int_0^x b(s,y)\, u(s,y)\, ds. \tag{3.15}$$

Clearly, $z(x,y)$ is a nonnegative, continuous and nondecreasing function in x, $x \in \mathbb{R}_+$. Treating y, $y \in \mathbb{R}_+$ fixed in (3.15) and using part (α_1) of Lemma 2.1 to (3.15), we obtain

$$u(x,y) \le z(x,y)\, p(x,y), \tag{3.16}$$

where $p(x,y)$ is defined by (2.9). From (3.16) and (3.15) we have

$$u(x,y) \le p(x,y)\left[a(x,y) + v(x,y)\right], \tag{3.17}$$

where

$$v(x,y) = \int_0^x \int_y^\infty F(s,t,u(s,t))\, dtds. \tag{3.18}$$

From (3.18), (3.17) and the hypotheses on F it follows that

$$\begin{aligned} v(x,y) \le \int_0^x \int_y^\infty & [F(s,t,p(s,t)(a(s,t)+v(s,t))) \\ & - F(s,t,p(s,t)\,a(s,t)) + F(s,t,p(s,t)\,a(s,t))]dtds \\ & \le B(x,y) + \int_0^x \int_y^\infty K(s,t,p(s,t)\,a(s,t))\, p(s,t)\, v(s,t)\, dtds. \end{aligned} \tag{3.19}$$

Clearly, $B(x,y)$ is nonnegative, continuous and nondecreasing in x and nonincreasing in y for $x, y \in \mathbb{R}_+$. By following the proof of (a_1), we get

$$v(x,y) \le B(x,y) \exp\left(\int_0^x \int_y^\infty K(s,\ t,\ p(s,t)\ a(s,t))\ p(s,t)\, dtds \right). \tag{3.20}$$

The required inequality (2.16) follows from (3.17) and (3.20).

4. PROOFS OF THEOREMS 2.6 – 2.8

We give the proofs of (p_1), (q_1), (r_1) only; the proofs of (p_2), (q_2), (r_2) can be completed by following the proofs of the above mentioned inequalities.

(p_1) Define a function $z(m,n)$ by

$$z(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} c(s,t)\, u(s,t). \tag{4.1}$$

Then (2.21) can be stated as

$$u(m,n) \le a(m,n) + b(m,n)\, z(m,n). \tag{4.2}$$

From (4.1) and (4.2) we have

$$\begin{aligned}(4.3)\qquad z(m,n) &\le \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\left[a(s,t)+b(s,t)\,z(s,t)\right]\\ &= f(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\,b(s,t)\,z(s,t)\,,\end{aligned}$$

where $f(m,n)$ is defined by (2.23). Clearly, $f(m,n)$ is nonnegative, nondecreasing in m and nonincreasing in n for $m,n\in N_0$. First, we assume that $f(m,n)>0$ for $m,n\in N_0$. From (4.3) we observe that

$$\frac{z(m,n)}{f(m,n)} \le 1+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\,b(s,t)\,\frac{z(s,t)}{f(s,t)}.$$

Define a function $v(m,n)$ by

$$(4.4)\qquad v(m,n) = 1+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} c(s,t)\,b(s,t)\,\frac{z(s,t)}{f(s,t)},$$

then $\frac{z(m,n)}{f(m,n)} \le v(m,n)$ and

$$\begin{aligned}(4.5)\qquad &[v(m+1,n)-v(m,n)]-[v(m+1,n+1)-v(m,n+1)]\\ &\qquad = c(m,n+1)\,b(m,n+1)\,\frac{z(m,n+1)}{f(m,n+1)}\\ &\qquad \le c(m,n+1)\,b(m,n+1)\,v(m,n+1)\,.\end{aligned}$$

From (4.5) and using the facts that $v(m,n)>0$, $v(m,n+1)\le v(m,n)$ for $m,n\in N_0$, we observe that

$$\begin{aligned}(4.6)\qquad &\frac{[v(m+1,n)-v(m,n)]}{v(m,n)}-\frac{[v(m+1,n+1)-v(m,n+1)]}{v(m,n+1)}\\ &\qquad \le c(m,n+1)\,b(m,n+1)\,.\end{aligned}$$

Keeping m fixed in (4.6), set $n=t$ and sum over $t=n,n+1,\dots,r-1$ ($r\ge n+1$ is arbitrary in N_0) to obtain

$$(4.7)\qquad \frac{[v(m+1,n)-v(m,n)]}{v(m,n)}-\frac{[v(m+1,r)-v(m,r)]}{v(m,r)} \le \sum_{t=n+1}^{r} c(m,t)\,b(m,t)\,,$$

Noting that $\lim_{r\to\infty} v(m,r)=\lim_{r\to\infty} v(m+1,r)=1$ and by letting $r\to\infty$ in (4.7) we get

$$\frac{[v(m+1,n)-v(m,n)]}{v(m,n)} \le \sum_{t=n+1}^{\infty} c(m,t)\,b(m,t)\,,$$

i.e.,

$$(4.8)\qquad v(m+1,n) \le \left[1+\sum_{t=n+1}^{\infty} c(m,t)\,b(m,t)\right] v(m,n)\,.$$

Now, by keeping n fixed in (4.8) and setting $m = s$ and substituting $s = 0, 1, 2, \ldots, m-1$ successively and using the fact that $v(0, n) = 1$, we get

$$v(m,n) \leq \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{\infty} c(m,t)\, b(m,t)\right]. \tag{4.9}$$

Using (4.9) in $\frac{z(m,n)}{f(m,n)} \leq v(m,n)$ we have

$$z(m,n) \leq f(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{\infty} c(s,t)\, b(s,t)\right]. \tag{4.10}$$

The required inequality in (2.22) follows from (4.2) and (4.10).

If $f(m,n)$ is nonnegative, then we carry out the above procedure with $f(m,n) + \varepsilon$ instead of $f(m,n)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.22).

(q_1) Define a function $z(m,n)$ by

$$z(m,n) = a(m,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} c(s,t)\, u(s,t), \tag{4.11}$$

then (2.27) can be restated as

$$u(m,n) \leq z(m,n) + \sum_{s=0}^{m-1} b(s,n)\, u(s,n). \tag{4.12}$$

Clearly, $z(m,n)$ is nonnegative and nondecreasing in m, $m \in N_0$. Treating n, $n \in N_0$ fixed in (4.12) and using part (β_1) of Lemma 2.5 to (4.12), we obtain

$$u(m,n) \leq z(m,n)\, q(m,n), \tag{4.13}$$

where $q(m,n)$ is defined by (2.29). From (4.13) and (4.11) we have

$$u(m,n) \leq q(m,n)\left[a(m,n) + v(m,n)\right], \tag{4.14}$$

where

$$v(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} c(s,t)\, u(s,t). \tag{4.15}$$

From (4.14) and (4.15), it is easy to see that

$$v(m,n) \leq G(m,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} c(s,t)\, q(s,t)\, v(s,t),$$

where $G(m,n)$ is as defined by (2.30). The rest of the proof of (2.28) can be completed by following the proof of (p_1) given above, and we omit the further details.

(r_1) The proof follows by closely looking at the proofs of (p_1), (q_1) and (c_1) given above. Here we leave the details to the reader.

5. Some Applications

In this section we present some immediate applications of part (a_2) of Theorem 2.2 to study certain properties of solutions of the following terminal value problem for the hyperbolic partial differential equation

$$u_{xy}(x,y) = h(x,y,u(x,y)) + r(x,y), \tag{5.1}$$

$$u(x,\infty) = \sigma_\infty(x),\ u(\infty,y) = \tau_\infty(y),\ \ u(\infty,\infty) = d, \tag{5.2}$$

where $h : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$, $r : \mathbb{R}_+^2 \to \mathbb{R}$, σ_∞, $\tau_\infty : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and d is a real constant.

The following theorem deals with the estimate on the solution of (5.1) – (5.2)

Theorem 5.1. *Suppose that*

$$|h(x,y,u)| \le c(x,y)|u|, \tag{5.3}$$

$$\left|\sigma_\infty(x) + \tau_\infty(y) - d + \int_x^\infty \int_y^\infty r(s,t)\,dtds\right| \le a(x,y), \tag{5.4}$$

where $a(x,y)$, $c(x,y)$ are as defined in part (a_2) of Theorem 2.2. Let $u(x,y)$ be a solution of (5.1) – (5.2) for $x, y \in \mathbb{R}_+$, then

$$|u(x,y)| \le a(x,y) + \bar{e}(x,y)\exp\left(\int_x^\infty \int_y^\infty c(s,t)\,dtds\right), \tag{5.5}$$

for $x, y \in \mathbb{R}_+$, where $\bar{e}(x,y)$ is defined by (2.6).

Proof. If $u(x,y)$ is a solution of (5.1) – (5.2), then it can be written as (see [1, p. 80])

$$u(x,y) = \sigma_\infty(x) + \tau_\infty(y) - d + \int_x^\infty \int_y^\infty [h(s,t,u(s,t)) + r(s,t)]\,dtds, \tag{5.6}$$

for $x, y \in \mathbb{R}_+$. From (5.6), (5.3), (5.4) we get

$$|u(x,y)| \le a(x,y) + \int_x^\infty \int_y^\infty c(s,t)|u(s,t)|\,dtds. \tag{5.7}$$

Now, a suitable application of part (a_2) of Theorem 2.2 to (5.7) yields the required estimate in (5.5). □

Our next result deals with the uniqueness of the solutions of (5.1) – (5.2).

Theorem 5.2. *Suppose that the function h in (5.1) satisfies the condition*

$$|h(x,y,u) - h(x,y,v)| \le c(x,y)|u - v|, \tag{5.8}$$

where $c(x,y)$ is as defined in Theorem 2.2. Then the problem (5.1) – (5.2) has at most one solution on $\mathbb{R}_+^2$.

Proof. The problem (5.1) – (5.2) is equivalent to the integral equation (5.6). Let $u(x,y)$, $v(x,y)$ be two solutions of (5.1) – (5.2). From (5.6), (5.8) we have

$$|u(x,y) - v(x,y)| \le \int_x^\infty \int_y^\infty c(s,t)|u(s,t) - v(s,t)|\,dtds. \tag{5.9}$$

Now a suitable application of part (a_2) of Theorem 2.2 yields $u(x,y) = v(x,y)$, i.e., there is at most one solution to the problem (5.1) – (5.2). □

We note that the inequality given in part (b_2) of Theorem 2.3 can be used to obtain the bound and uniqueness of the solutions of the following non-self-adjoint hyperbolic partial differential equation

$$u_{xy}(x,y) = (r(x,y)u(x,y))_x + h(x,y,u(x,y)), \tag{5.10}$$

with the given terminal value conditions in (5.2), under some suitable conditions on the functions involved in (5.10) – (5.2). We also note that the inequalities given here have many applications, however, various applications of other inequalities is left for another time.

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