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A CLASS OF INEQUALITIES RELATED TO THE ANGLE BISECTORS AND THE SIDES OF A TRIANGLE

SHAN-HE WU AND ZHI-HUA ZHANG

DEPARTMENT OF MATHEMATICS
LONGYAN COLLEGE
LONGYAN FUJIAN 364012
PEOPLE'S REPUBLIC OF CHINA
wushanhe@yahoo.com.cn

ZIXING EDUCATIONAL RESEARCH SECTION
CHENZHOU, HUNAN 423400
PEOPLE'S REPUBLIC OF CHINA
zxzh1234@163.com

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ABSTRACT. In the present paper a class of geometric inequalities concerning the angle bisectors and the sides of a triangle are established. Moreover an interesting open problem is proposed.

Key words and phrases: Inequality; Triangle; Angle bisector; Cyclic sum; Best coefficient.

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1. INTRODUCTION AND MAIN RESULTS

For a given triangle ABC we assume that $A,\ B,\ C$ denote its angles, a, b, c denote the lengths of its corresponding sides, w_a, w_b, w_c denote respectively the bisectors of angles $A,\ B,\ C$. Let R, r and s be the circumradius, the inradius and the semi-perimeter of a triangle respectively. In addition we will customarily use the symbols $\sum$ (cyclic sum) and $\prod$ (cyclic product), such as

$$\sum f(a) = f(a) + f(b) + f(c), \quad \prod f(a) = f(a)f(b)f(c).$$

The angle bisectors of triangles have many interesting properties. In particular, inequalities for angle bisectors is a very attractive subject and plays an important role in the study of geometry. A large number of related results can be found in the well-known monographs [1] – [3]. In recent years, we have given considerable attention to these inequalities (see [4] – [8]). Recently,

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the following interesting double inequality concerning the angle bisectors and the sides, which was presented by X.-Zh. Yang, T.-Y. Ma and W.-L. Dong in [9, 10], has come to our attention:

$$\frac{3\sqrt{3}}{2}+\left(\frac{8}{3}-\frac{3\sqrt{3}}{2}\right)\left(1-\frac{2r}{R}\right)\le\sum\frac{w_a}{a}\le\frac{3\sqrt{3}}{2}+2\sqrt{3}\left(\frac{R}{2r}-1\right). \tag{1.1}$$

The above result also motivates us to investigate some similar inequalities. We give here sharp lower and upper bounds for the sum $\sum\frac{a}{w_a}$. Moreover, in Section 3 the obtained result will be used for establishing an analogue of inequality (1.1).

Theorem 1.1. *In any triangle ABC the following double inequalities hold*

$$\frac{1}{2}\left(\frac{s}{r}+\sqrt{3}\right)\le\sum\frac{a}{w_a}\le\frac{\sqrt{2}}{2}\left(\frac{s}{r}+2\sqrt{6}-3\sqrt{3}\right), \tag{1.2}$$

with equality if and only if the triangle is an equilateral. Furthermore, $\frac{1}{2}$ and $\frac{\sqrt{2}}{2}$ are the best coefficients in (1.2).

2. PROOF OF THEOREM 1.1

To prove Theorem 1.1, we shall use the following known results [2, p. 3, p. 241] (see also [1])

Lemma 2.1. *In any triangle ABC we have the following inequalities*

$$s^4\le s^2(4R^2+20Rr-2r^2)-r(4R+r)^3, \tag{2.1}$$

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\le s^2\le 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}, \tag{2.2}$$

with equality if and only if the triangle is isosceles.

$$s\le\frac{1}{\sqrt{3}}(4R+r), \tag{2.3}$$

with equality if and only if the triangle is equilateral.

In any acute triangle ABC we have

$$s^2\ge 4R^2+4Rr+r^2, \tag{2.4}$$

with equality if and only if the triangle is equilateral.

Proof of Theorem 1.1. By the formula for angle bisector of triangle ABC $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, we have

$$\begin{aligned}\sum\frac{a}{w_a}&=\sum(\csc B+\csc C)\sin\frac{A}{2}\\&=\left(\sum\csc A\right)\left(\sum\sin\frac{A}{2}\right)-\frac{1}{2}\sum\sec\frac{A}{2}.\end{aligned}$$

Based on the above result, it follows from the identity $\prod\cot\frac{A}{2}=\frac{s}{r}$ that the inequality (1.2) is equivalent to the following inequality

$$\begin{aligned}\frac{1}{2}\prod\cot\frac{A}{2}+\frac{\sqrt{3}}{2}&\le\left(\sum\csc A\right)\left(\sum\sin\frac{A}{2}\right)-\frac{1}{2}\sum\sec\frac{A}{2}\\&\le\frac{\sqrt{2}}{2}\prod\cot\frac{A}{2}+\frac{4\sqrt{3}-3\sqrt{6}}{2}.\end{aligned} \tag{2.5}$$

Using a substitution $A \to \pi - 2A$, $B \to \pi - 2B$, $C \to \pi - 2C$ in (2.5), then the inequality (2.5) can be translated to

$$\frac{1}{2}\prod \tan A + \frac{\sqrt{3}}{2} \le \left(\sum \csc 2A\right)\left(\sum \cos A\right) - \frac{1}{2}\sum \csc A \le \frac{\sqrt{2}}{2}\prod \tan A + \frac{4\sqrt{3}-3\sqrt{6}}{2}. \tag{2.6}$$

Now, in order to prove the inequality (2.5), it is enough to prove that the inequality (2.6) to be valid for any acute triangle.

Note that the following known identities for a triangle [2, p. 55-60]:

$$\sum \csc 2A = \frac{s^4 + s^2(2r^2 - 8Rr - 4R^2) + 16R^3r + 20R^2r^2 + 8Rr^3 + r^4}{4rs\left(s^2 - 4R^2 - 4Rr - r^2\right)},$$

$$\prod \tan A = \frac{2rs}{s^2 - 4R^2 - 4Rr - r^2},$$

$$\sum \csc A = \frac{s^2 + 4Rr + r^2}{2rs},$$

$$\sum \cos A = \frac{R + r}{R},$$

taking these identities into (2.6), we find that the inequality (2.6) is equivalent to

$$4Rr^2s^2 + 2\sqrt{3}\left(s^2 - 4R^2 - 4Rr - r^2\right)srR \le H \le 4\sqrt{2}Rr^2s^2 + (8\sqrt{3} - 6\sqrt{6})\left(s^2 - 4R^2 - 4Rr - r^2\right)srR, \tag{2.7}$$

where

$$H = (R + r)\left[s^4 + s^2(2r^2 - 8Rr - 4R^2) + 16R^3r + 20R^2r^2 + 8Rr^3 + r^4\right] - R\left(s^2 + 4Rr + r^2\right)\left(s^2 - 4R^2 - 4Rr - r^2\right).$$

Let us now prove the inequality (2.7) to be valid for any acute triangle.

Using the inequalities (2.1), (2.3) and (2.4), we have

$$\begin{aligned} & H - 4\sqrt{2}Rr^2s^2 - \left(8\sqrt{3} - 6\sqrt{6}\right)\left(s^2 - 4R^2 - 4Rr - r^2\right)srR \\ & \le (R + r)\left[s^2(4R^2 + 20Rr - 2r^2) - r(4R + r)^3 \right.\\ & \qquad \left. + s^2(2r^2 - 8Rr - 4R^2) + 16R^3r + 20R^2r^2 + 8Rr^3 + r^4\right] \\ & \qquad - R\left(s^2 + 4Rr + r^2\right)\left(s^2 - 4R^2 - 4Rr - r^2\right) - 4\sqrt{2}Rr^2s^2 \\ & \qquad - (8 - 6\sqrt{2})\left(s^2 - 4R^2 - 4Rr - r^2\right)(4R + r)rR \\ & = s^2\left[\left(-40 + 24\sqrt{2}\right)R + \left(6 + 2\sqrt{2}\right)r\right]Rr + \left(160 - 96\sqrt{2}\right)R^4r \\ & \qquad + \left(152 - 120\sqrt{2}\right)R^3r^2 + \left(52 - 48\sqrt{2}\right)R^2r^3 + \left(6 - 6\sqrt{2}\right)Rr^4 \\ & = F(s, R, r). \end{aligned} \tag{2.8}$$

From Euler's inequality $R \ge 2r$, we observe that $(-40 + 24\sqrt{2})R + (6 + 2\sqrt{2})r < 0$.

Case 1. When $R > (\sqrt{2}+1)r$, by inequality (2.4) we have

$$\begin{aligned} F(s,R,r) &\le (4R^2+4Rr+r^2)\left[\left(-40+24\sqrt{2}\right)R+\left(6+2\sqrt{2}\right)r\right]Rr+\left(160-96\sqrt{2}\right)R^4r \\ &\quad + (152-120\sqrt{2})R^3r^2+\left(52-48\sqrt{2}\right)R^2r^3+\left(6-6\sqrt{2}\right)Rr^4 \\ &= \left(16-16\sqrt{2}\right)R^3r^2+\left(36-16\sqrt{2}\right)R^2r^3+\left(12-4\sqrt{2}\right)Rr^4 \\ &= -4Rr^2\left[R-\left(\sqrt{2}+1\right)r\right]\left[\left(4\sqrt{2}-4\right)R+\left(4\sqrt{2}-5\right)r\right] \\ &< 0. \end{aligned}$$

Case 2. When $2r \le R \le (\sqrt{2}+1)r$, by inequality (2.2) we get

$$\begin{aligned} F(s,R,r) &\le \left[2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\right] \\ &\quad \times \left[\left(-40+24\sqrt{2}\right)R+\left(6+2\sqrt{2}\right)r\right]Rr+\left(160-96\sqrt{2}\right)R^4r \\ &\quad + \left(152-120\sqrt{2}\right)R^3r^2+\left(52-48\sqrt{2}\right)R^2r^3+\left(6-6\sqrt{2}\right)Rr^4 \\ &= 4Rr^3(R-2r)[F_1(R,r)+F_2(R,r)], \end{aligned}$$

where

$$F_1(R,r) = \left[\left(20-12\sqrt{2}\right)\left(\frac{R}{r}\right)-3-\sqrt{2}\right]\sqrt{\left(\frac{R}{r}\right)\left(\frac{R}{r}-2\right)},$$

$$F_2(R,r) = \left(20-12\sqrt{2}\right)\left(\frac{R}{r}\right)^2+\left(-19+7\sqrt{2}\right)\left(\frac{R}{r}\right)+\sqrt{2}.$$

We deduce from $2 \le R/r \le \sqrt{2}+1$ that

$$\begin{aligned} F_1(R,r) &\le \left[\left(20-12\sqrt{2}\right)\left(\sqrt{2}+1\right)-3-\sqrt{2}\right]\sqrt{\left(\sqrt{2}+1\right)\left(\sqrt{2}+1-2\right)} \\ &= 7\sqrt{2}-7, \end{aligned}$$

$$\begin{aligned} F_2(R,r) &\le \left(20-12\sqrt{2}\right)\left(\sqrt{2}+1\right)^2+\left(-19+7\sqrt{2}\right)\left(\sqrt{2}+1\right)+\sqrt{2} \\ &= 7-7\sqrt{2}, \end{aligned}$$

which leads to $F(s,R,r) \le 0$.

Consequently

$$H-4\sqrt{2}Rr^2s^2-\left(8\sqrt{3}-6\sqrt{6}\right)\left(s^2-4R^2-4Rr-r^2\right)srR \le 0. \tag{2.9}$$

On the other hand, utilizing the inequalities (2.3) and (2.4), we have

$$\begin{aligned} &H-4Rr^2s^2-2\sqrt{3}\left(s^2-4R^2-4Rr-r^2\right)srR \\ &\ge (R+r)\left[s^4+s^2(2r^2-8Rr-4R^2)+16R^3r+20R^2r^2+8Rr^3+r^4\right] \\ &\qquad - R\left(s^2+4Rr+r^2\right)\left(s^2-4R^2-4Rr-r^2\right) \\ &\qquad - 4\sqrt{2}Rr^2s^2-2\left(s^2-4R^2-4Rr-r^2\right)(4R+r)R \\ &= r\left(s^2-4R^2-4Rr-3r^2\right)^2+4r(R+r)F(s,R,r), \end{aligned}$$

where

$$F(s,R,r) = -s^2(3R-2r)+12R^3+4R^2r-Rr^2-2r^3.$$

By Euler's inequality $R \geq 2r$, we conclude that $3R - 2r > 0$. Using the inequality (2.2) yields

$$\begin{aligned} F(s,R,r) &\geq -(3R-2r)\left[2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\right] \\ &\qquad + 12R^3+4R^2r-Rr^2-2r^3 \\ &= 2(R-2r)\left[R(3R-5r)+r^2-(3R-2r)\sqrt{R^2-2Rr}\right] \\ &= 2(R-2r)\frac{[Rr^2(3R-2r)+r^4]}{R(3R-5r)+r^2+(3R-2r)\sqrt{R^2-2Rr}} \\ &\geq 0. \end{aligned}$$

Consequently

$$H - 4Rr^2s^2 - 2\sqrt{3}\left(s^2-4R^2-4Rr-r^2\right)srR \geq 0. \tag{2.10}$$

Combining (2.9) and (2.10) yields the inequality (2.7), then from (2.7), the inequality (1.2) follows immediately. Moreove, from the process of proving inequality (1.2), it is easy to observe that the equalities hold in (1.2) if and only if the triangle is equilateral.

Next, we need to show that the coefficients $\frac{1}{2}$ and $\frac{\sqrt{2}}{2}$ in (1.2) are best possible in the strong sense.

Consider the inequality (1.2) in a general form as

$$\lambda\left(\frac{s}{r}+\frac{2\sqrt{3}}{\lambda}-3\sqrt{3}\right) \leq \sum\frac{a}{w_a} \leq k\left(\frac{s}{r}+\frac{2\sqrt{3}}{k}-3\sqrt{3}\right). \tag{2.11}$$

Putting $a = 1, b = 1, c = 2t$ in (2.11) yields that

$$\begin{aligned} \lambda(1+t)+\left(2\sqrt{3}-3\sqrt{3}\lambda\right)t\sqrt{\frac{1-t}{1+t}} &\leq \frac{4t^2+(1+2t)\sqrt{2-2t}}{2+2t} \\ &\leq k(1+t)+\left(2\sqrt{3}-3\sqrt{3}k\right)t\sqrt{\frac{1-t}{1+t}}. \end{aligned} \tag{2.12}$$

In (2.12), passing the limit as $t \to 0$ and $t \to 1$ respectively, we find that $\lambda \leq \frac{1}{2}$ and $k \geq \frac{\sqrt{2}}{2}$. Thus the best possible values for λ and k in (2.11) is that $\lambda_{\max} = \frac{1}{2}$, $k_{\min} = \frac{\sqrt{2}}{2}$. This completes the proof of Theorem 1.1. □

3. An Application

As an application of Theorem 1.1, we establish an analogue of the inequality (1.1), as follows.

Theorem 3.1. *In any triangle ABC the following double inequalities hold*

$$2\sqrt{3}+\frac{3}{2}\left(1-\frac{2r}{R}\right) \leq \sum\frac{a}{w_a} \leq 2\sqrt{3}+2\sqrt{2}\left(\frac{R}{2r}-1\right), \tag{3.1}$$

with equality if and only if the triangle is equilateral. Furthermore, $2\sqrt{2}$ is the best coefficient in the right-hand side of inequality (3.1).

Proof. Applying Theorem 1.1 and Blundon's inequality [11] $s \leq 2R + (3\sqrt{3}-4)r$, it follows that

$$\sum\frac{a}{w_a} \leq \frac{\sqrt{2}}{2}\left(\frac{s}{r}+2\sqrt{6}-3\sqrt{3}\right) \leq 2\sqrt{3}+2\sqrt{2}\left(\frac{R}{2r}-1\right).$$

On the other hand, by using Theorem 1.1 and Gerretsen's inequality [12] $s^2 \geq 16Rr - 5r^2$, we have

$$\begin{aligned}
&\sum \frac{a}{w_a} - 2\sqrt{3} - \frac{3}{2}\left(1 - \frac{2r}{R}\right) \\
&\quad \geq \frac{1}{2}\left(\frac{s}{r} + \sqrt{3}\right) - 2\sqrt{3} - \frac{3}{2}\left(1 - \frac{2r}{R}\right) \\
&\quad = \frac{1}{2Rr}[sR - 3(R-2r)r - 6\sqrt{3}Rr] \\
&\quad \geq \frac{1}{2Rr}\left[R\sqrt{16Rr - 5r^2} - 3(R-2r)r - 3\sqrt{3}Rr\right] \\
&\quad = \frac{R^2(16Rr - 5r^2) - \left[3(R-2r)r + 3\sqrt{3}Rr\right]^2}{2Rr\left[R\sqrt{16Rr - 5r^2} + 3(R-2r)r + 3\sqrt{3}Rr\right]} \\
&\quad = \frac{16(R-2r)^3 + (55 - 18\sqrt{3})(R-2r)^2 + (64 - 36\sqrt{3})(R-2r)}{2R\left[R\sqrt{16Rr - 5r^2} + 3(R-2r)r + 3\sqrt{3}Rr\right]} \\
&\quad \geq 0,
\end{aligned}$$

so that

$$\sum \frac{a}{w_a} \geq 2\sqrt{3} + \frac{3}{2}\left(1 - \frac{2r}{R}\right).$$

The inequality (3.1) is proved. It follows directly from Theorem 1.1 that the equalities hold in (3.1) if and only if the triangle is equilateral.

Let us now show that the coefficient $2\sqrt{2}$ in the right-hand side of inequality (3.1) is best possible.

Consider the inequality (3.1) in a general form as

$$\sum \frac{a}{w_a} \leq 2\sqrt{3} + \mu\left(\frac{R}{2r} - 1\right). \tag{3.2}$$

Putting $a = 1$, $b = 1$, $c = 2t$ in (3.2) yields that

$$\frac{4t^2 + (1+2t)\sqrt{2-2t}}{2+2t} \leq \frac{\mu}{4\sqrt{1-t^2}} + \left(2\sqrt{3} - \mu\right) t\sqrt{\frac{1-t}{1+t}}. \tag{3.3}$$

Passing the limit as $t \to 0$ in (3.3), we get $\mu \geq 2\sqrt{2}$. Thus the best possible value for μ in (3.2) should be $\mu_{\min} = 2\sqrt{2}$. The proof of Theorem 3.1 is complete. □

It is worth noticing that the coefficient $\frac{3}{2}$ is not best possible for the left-hand side of inequality (3.1), this may lead us to further discussion of the following significant problem.

Open Problem. *Determine the best coefficient μ for which the inequality below holds*

$$\sum \frac{a}{w_a} \geq 2\sqrt{3} + \mu\left(1 - \frac{2r}{R}\right). \tag{3.4}$$

It seems that the problem is complicated and difficult. Indeed, it is unable to be solved in a same way as the foregoing technique.

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