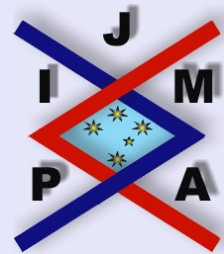


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Abstract

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Abstract

Classical inequalities and convex functions are used to get cyclical inequalities involving the elements of a triangle.

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1. Introduction

In what follows we are concerned with inequalities involving the elements of a triangle. Many of these inequalities have been documented in an extensive lists that appear in the work of Botema [2] and Mitrinović [5]. In this paper, using classical inequalities and convex functions some new inequalities for a triangle are obtained.



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2. The Inequalities

In the sequel we present some cyclical inequalities for the triangle. We begin with:

Theorem 2.1. *Let a, b, c be the sides of triangle ABC and let s be its semiperimeter. Then,*

$$(2.1) \quad \frac{1}{18} \sum_{\text{cyclic}} \left\{ \frac{1}{(s-a)(s-b)} \right\}^{\frac{1}{2}} \geq \left\{ \sum_{\text{cyclic}} \frac{a^2 + bc}{b+c} \right\}^{-1}.$$

Proof. First, we will prove that

$$(2.2) \quad \sqrt{\frac{1}{(s-a)(s-b)}} + \sqrt{\frac{1}{(s-b)(s-c)}} + \sqrt{\frac{1}{(s-c)(s-a)}} \geq \frac{9}{s}.$$

In fact, taking into account the AM-GM inequality, we get

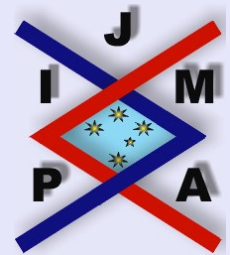
$$(2.3) \quad \frac{s}{3} = \frac{(s-a) + (s-b) + (s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)},$$

and

$$(2.4) \quad \frac{\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c}}{3} \geq \sqrt[6]{(s-a)(s-b)(s-c)}.$$

Multiplying up (2.3) and (2.4) yields

$$\frac{s(\sqrt{s-a} + \sqrt{s-b} + \sqrt{s-c})}{9} \geq \sqrt{(s-a)(s-b)(s-c)}$$



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or equivalently,

$$\frac{s}{9} \left(\sqrt{\frac{1}{(s-a)(s-b)}} + \sqrt{\frac{1}{(s-b)(s-c)}} + \sqrt{\frac{1}{(s-c)(s-a)}} \right) \geq 1$$

and (2.2) is proved.

Now we will see that

$$(2.5) \quad s \leq \frac{1}{2} \sum_{cyclic} \frac{a^2 + bc}{b + c}$$

or equivalently,

$$(2.6) \quad \frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} - (a + b + c) \geq 0$$

holds. In fact, (2.6) is a consequence of the well known inequality

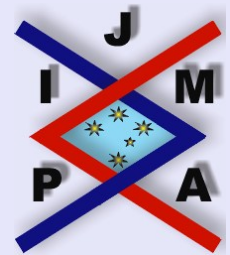
$$(2.7) \quad X^2 + Y^2 + Z^2 \geq XY + XZ + YZ \quad X, Y, Z \in \mathbb{R}$$

that can be obtained by rewriting the inequality

$$(X - Y)^2 + (X - Z)^2 + (Y - Z)^2 \geq 0.$$

After reducing (2.6) to a common denominator and some straightforward algebra, we get

$$\frac{a^2 + bc}{b + c} + \frac{b^2 + ca}{c + a} + \frac{c^2 + ab}{a + b} - (a + b + c) = \frac{a^4 + b^4 + c^4 - a^2c^2 - a^2b^2 - b^2c^2}{(a + b)(a + c)(b + c)}.$$



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Setting $X = a^2, Y = b^2$ and $Z = c^2$ into (2.7), we have

$$a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2 \geq 0$$

and (2.5) is proved. Note that equality holds when $a = b = c$. That is, when $\triangle ABC$ is equilateral.

Finally, (2.1) immediately follows from (2.2) and (2.5) and the theorem is proved. \square

Next we state and prove a key result to generate cyclical inequalities.

Theorem 2.2. *Let a_1, a_2, \dots, a_n be positive real numbers and let $s_k = S - (n-1)a_k, k = 1, 2, \dots, n$ where $S = a_1 + a_2 + \dots + a_n$. If $a_k, s_k, k = 1, 2, \dots, n$ lie in the domain of a convex function f , then*

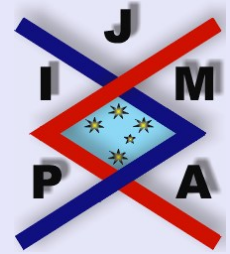
$$(2.8) \quad \sum_{k=1}^n f(s_k) \geq \sum_{k=1}^n f(a_k).$$

Proof. Without loss of generality, we can assume that $a_1 \geq a_2 \geq \dots \geq a_n$. Now it is easy to see that the vector

$$\begin{aligned} & (S - (n-1)a_n, S - (n-1)a_{n-1}, \dots, S - (n-1)a_1) \\ &= (a_1 + \dots + a_{n-1} - na_n, a_1 + \dots - na_{n-1} + a_n, \dots, -na_1 + \dots + a_{n-1} + a_n) \end{aligned}$$

majorizes [7] the vector (a_1, a_2, \dots, a_n) . Namely,

$$s_n + s_{n-1} + \dots + s_{n-\ell+1} \geq a_1 + a_2 + \dots + a_\ell$$



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for $\ell = 1, 2, \dots, n-1$, and equality for $\ell = n$. Taking into account Karamata's inequality [6] we have

$$\sum_{k=1}^n f(S - (n-1)a_k) \geq \sum_{k=1}^n f(a_k)$$

and the proof is complete. \square

Theorem 2.3. *In any $\triangle ABC$ the following inequality holds:*

$$(2.9) \quad \prod_{cyclic} (a+b-c)^{a+b-c} \geq a^b b^c c^a,$$

where a, b, c are the sides of the triangle.

Proof. Applying Theorem 2.2 to the function $f(x) = x \ln x$ that is convex for $x > 0$, we get

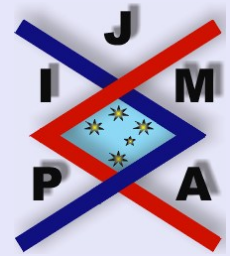
$$(2.10) \quad (a+b-c)^{a+b-c} (b+c-a)^{b+c-a} (c+a-b)^{c+a-b} \geq a^a b^b c^c.$$

Now we claim that

$$(2.11) \quad a^a b^b c^c \geq \left(\frac{a+b+c}{3} \right)^{a+b+c} \geq a^b b^c c^a$$

and the statement immediately follows from (2.10) and (2.11).

Inequalities in (2.11) have been proved in [3] using the weighted AM-GM-HM inequality [4]. Note that equality holds when $a = b = c$. Namely, when $\triangle ABC$ is equilateral. This completes the proof. \square



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Emil Artin in [1] proved that $f(x) = \ln \Gamma(x)$ is convex for $x > 0$ where $\Gamma(x)$ is the Euler Gamma Function. Then, applying Theorem 2.2 to $f(x)$, we have

Theorem 2.4. *In any triangle ABC , we have*

$$(2.12) \quad \prod_{cyclic} \Gamma(a + b - c) \geq \prod_{cyclic} \Gamma(a).$$

Using other convex functions and carrying out this procedure we get the following new inequalities:

Theorem 2.5. *Let a, b and c be the sides of triangle ABC . Then*

$$(2.13) \quad \prod_{cyclic} (a + b - c)^{a+b} \geq a^{s+a/2} b^{s+b/2} c^{s+c/2}$$

holds.

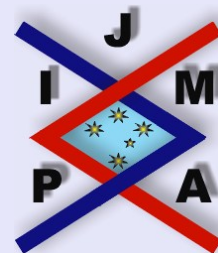
Proof. Applying Theorem 2.2 to the function $f(x) = (x + a + b + c) \ln x$ that is convex for $x > 0$, we get from

$$f(a + b - c) + f(b + c - a) + f(c + a - b) \geq f(a) + f(b) + f(c)$$

that

$$\begin{aligned} 2(a + b) \ln(a + b - c) + 2(b + c) \ln(b + c - a) + 2(c + a) \ln(c + a - b) \\ \geq (2a + b + c) \ln a + (a + 2b + c) \ln b + (a + b + 2c) \ln c \end{aligned}$$

and we are done. □



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The function $f(x) = \frac{x^3}{1+x}$ is convex for $x > 0$. In fact, $f'(x) = \frac{x^2(3+2x)}{(1+x)^2} > 0$ and $f''(x) = \frac{2x(3+3x+x^2)}{(1+x)^3} > 0$. Hence, f is increasing and convex. Applying again Theorem 2.2 to $f(x)$, we have

Theorem 2.6. *In any triangle ABC the following inequality*

$$(2.14) \quad \sum_{cyclic} \frac{(a+b-c)^3}{1+a+b-c} \geq \sum_{cyclic} \frac{a^3}{1+a}$$

holds.

Observe that the preceding procedure can be used to generate many triangle inequalities.

Before stating our next result we give a lemma that we will use later on.

Lemma 2.7. *Let x, y, z and a, b, c be strictly positive real numbers. Then, we have*

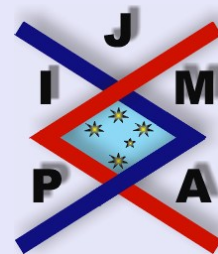
$$(2.15) \quad 3(yza^2 + zxb^2 + xyc^2) \geq (a\sqrt{yz} + b\sqrt{zx} + c\sqrt{xy})^2.$$

Proof. Let $\vec{u} = (\sqrt{yz}, \sqrt{zx}, \sqrt{xy})$ and $\vec{v} = (a, b, c)$. By applying Cauchy-Buniakovski-Schwarz's inequality, we get

$$[(\sqrt{yz}, \sqrt{zx}, \sqrt{xy}) \cdot (a, b, c)]^2 \leq \|(\sqrt{yz}, \sqrt{zx}, \sqrt{xy})\|^2 \|(a, b, c)\|^2$$

or equivalently,

$$(2.16) \quad (a\sqrt{yz} + b\sqrt{zx} + c\sqrt{xy})^2 \leq (yz + zx + xy)(a^2 + b^2 + c^2).$$



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On the other hand, applying the rearrangement inequality yields

$$a^2yz + b^2zx + c^2xy \geq b^2yz + c^2zx + a^2xy,$$

$$a^2yz + b^2zx + c^2xy \geq b^2xy + a^2zx + c^2yz.$$

Hence, the right hand side of (2.15) becomes

$$(yz + zx + xy)(a^2 + b^2 + c^2) \leq 3(yza^2 + zxb^2 + xyc^2)$$

and the proof is complete. \square

In particular, setting $x = a + b - c$, $y = c + a - b$ and $z = b + c - a$ into the preceding lemma, we get the following

Theorem 2.8. *If a, b and c are the sides of triangle ABC , then*

$$(2.17) \quad \sum_{cyclic} a^3b \sin^2 \frac{C}{2} \geq \frac{1}{3} \left\{ \sum_{cyclic} a\sqrt{(s-a)(s-b)} \right\}^2.$$

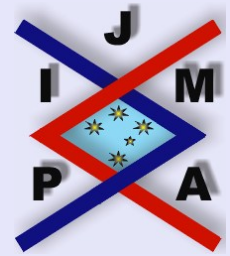
Proof. Taking into account the Law of Cosines, we have

$$\sum_{cyclic} a^3b \sin^2 \frac{C}{2} = \frac{1}{2} \sum_{cyclic} a^3b(1 - \cos C) = \frac{1}{2} \sum_{cyclic} a^2[c^2 - (a-b)^2].$$

On the other hand,

$$\left\{ \sum_{cyclic} a\sqrt{(s-a)(s-b)} \right\}^2 = \frac{1}{2} \left\{ \sum_{cyclic} a\sqrt{c^2 - (a-b)^2} \right\}^2.$$

Now, inequality (2.17) immediately follows from (2.15) and the proof is completed. \square



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