Journal of Inequalities in Pure and Applied Mathematics

SOME CYCLICAL INEQUALITIES FOR THE TRIANGLE

JOSÉ LUIS DÍAZ-BARRERO

Applied Mathematics III
Universitat Politècnica de Catalunya
Jordi Girona 1-3, C2
08034 Barcelona, Spain

EMail: jose.luis.diaz@upc.edu

ISSN (electronic): 1443-5756
047-04

volume 6, issue 1, article 20, 2005.

Received 03 March, 2004; accepted 1 February, 2005.

Communicated by: A. Lupaş

Abstract

Classical inequalities and convex functions are used to get cyclical inequalities involving the elements of a triangle.

2000 Mathematics Subject Classification: 26D15.
Key words: Inequalities, Geometric inequalities, Triangle inequalities, Circular inequalities

Contents

1. Introduction

In what follows we are concerned with inequalities involving the elements of a triangle. Many of these inequalities have been documented in an extensive lists that appear in the work of Botema [2] and Mitrinović [5]. In this paper, using classical inequalities and convex functions some new inequalities for a triangle are obtained.

2. The Inequalities

In the sequel we present some cyclical inequalities for the triangle. We begin with:

Theorem 2.1. *Let* a, b, c *be the sides of triangle* ABC *and let* s *be its semiperimeter. Then,*

$$\frac{1}{18}\sum_{cyclic}\left\{\frac{1}{(s-a)(s-b)}\right\}^{\frac{1}{2}} \geq \left\{\sum_{cyclic}\frac{a^2+bc}{b+c}\right\}^{-1}. \tag{2.1}$$

Proof. First, we will prove that

$$\sqrt{\frac{1}{(s-a)(s-b)}}+\sqrt{\frac{1}{(s-b)(s-c)}}+\sqrt{\frac{1}{(s-c)(s-a)}} \geq \frac{9}{s}. \tag{2.2}$$

In fact, taking into account the AM-GM inequality, we get

$$\frac{s}{3} = \frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}, \tag{2.3}$$

and

$$\frac{\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c}}{3} \geq \sqrt[6]{(s-a)(s-b)(s-c)}. \tag{2.4}$$

Multiplying up (2.3) and (2.4) yields

$$\frac{s(\sqrt{s-a}+\sqrt{s-b}+\sqrt{s-c})}{9} \geq \sqrt{(s-a)(s-b)(s-c)}$$

or equivalently,

$$\frac{s}{9}\left(\sqrt{\frac{1}{(s-a)(s-b)}}+\sqrt{\frac{1}{(s-b)(s-c)}}+\sqrt{\frac{1}{(s-c)(s-a)}}\right)\geq 1$$

and (2.2) is proved.

Now we will see that

$$s\leq\frac{1}{2}\sum_{cyclic}\frac{a^2+bc}{b+c} \tag{2.5}$$

or equivalently,

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}-(a+b+c)\geq 0 \tag{2.6}$$

holds. In fact, (2.6) is a consequence of the well known inequality

$$X^2+Y^2+Z^2\geq XY+XZ+YZ \quad X,Y,Z\in\mathbb{R} \tag{2.7}$$

that can be obtained by rewriting the inequality

$$(X-Y)^2+(X-Z)^2+(Y-Z)^2\geq 0.$$

After reducing (2.6) to a common denominator and some straightforward algebra, we get

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}-(a+b+c)=\frac{a^4+b^4+c^4-a^2c^2-a^2b^2-b^2c^2}{(a+b)(a+c)(b+c)}.$$

Setting $X = a^2, Y = b^2$ and $Z = c^2$ into (2.7), we have

$$a^4 + b^4 + c^4 - a^2b^2 - a^2c^2 - b^2c^2 \geq 0$$

and (2.5) is proved. Note that equality holds when $a = b = c$. That is, when $\triangle ABC$ is equilateral.

Finally, (2.1) immediately follows from (2.2) and (2.5) and the theorem is proved. □

Next we state and prove a key result to generate cyclical inequalities.

Theorem 2.2. *Let $a_1, a_2, \dots, a_n$ be positive real numbers and let $s_k = S - (n-1)a_k$, $k = 1, 2, \dots, n$ where $S = a_1 + a_2 + \cdots + a_n$. If a_k, s_k, $k = 1, 2, \dots, n$ lie in the domain of a convex function f, then*

$$\sum_{k=1}^{n} f(s_k) \geq \sum_{k=1}^{n} f(a_k). \tag{2.8}$$

Proof. Without loss of generality, we can assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. Now it is easy to see that the vector

$$(S - (n-1)a_n, S - (n-1)a_{n-1}, \dots, S - (n-1)a_1)$$
$$= (a_1 + \cdots + a_{n-1} - na_n, a_1 + \cdots - na_{n-1} + a_n, \dots, -na_1 + \cdots + a_{n-1} + a_n)$$

majorizes [7] the vector $(a_1, a_2, \dots, a_n)$. Namely,

$$s_n + s_{n-1} + \cdots + s_{n-\ell+1} \geq a_1 + a_2 + \cdots + a_\ell$$

for $\ell = 1, 2, \dots, n-1$, and equality for $\ell = n$. Taking into account Karamata's inequality [6] we have

$$\sum_{k=1}^{n} f(S-(n-1)a_k) \ge \sum_{k=1}^{n} f(a_k)$$

and the proof is complete. □

Theorem 2.3. *In any $\triangle ABC$ the following inequality holds:*

$$\prod_{cyclic} (a+b-c)^{a+b-c} \ge a^b b^c c^a, \tag{2.9}$$

where a, b, c are the sides of the triangle.

Proof. Applying Theorem 2.2 to the function $f(x) = x \ln x$ that is convex for $x > 0$, we get

$$(a+b-c)^{a+b-c}(b+c-a)^{b+c-a}(c+a-b)^{c+a-b} \ge a^a b^b c^c. \tag{2.10}$$

Now we claim that

$$a^a b^b c^c \ge \left(\frac{a+b+c}{3}\right)^{a+b+c} \ge a^b b^c c^a \tag{2.11}$$

and the statement immediately follows from (2.10) and (2.11).

Inequalities in (2.11) have been proved in [3] using the weighted AM-GM-HM inequality [4]. Note that equality holds when $a = b = c$. Namely, when $\triangle ABC$ is equilateral. This completes the proof. □

Emil Artin in [1] proved that $f(x) = \ln \Gamma(x)$ is convex for $x > 0$ where $\Gamma(x)$ is the Euler Gamma Function. Then, applying Theorem 2.2 to $f(x)$, we have

Theorem 2.4. *In any triangle ABC, we have*

$$\prod_{\text{cyclic}} \Gamma(a+b-c) \geq \prod_{\text{cyclic}} \Gamma(a). \tag{2.12}$$

Using other convex functions and carrying out this procedure we get the following new inequalities:

Theorem 2.5. *Let a, b and c be the sides of triangle ABC. Then*

$$\prod_{\text{cyclic}} (a+b-c)^{a+b} \geq a^{s+a/2}\, b^{s+b/2}\, c^{s+c/2} \tag{2.13}$$

holds.

Proof. Applying Theorem 2.2 to the function $f(x) = (x+a+b+c)\ln x$ that is convex for $x > 0$, we get from

$$f(a+b-c) + f(b+c-a) + f(c+a-b) \geq f(a) + f(b) + f(c)$$

that

$$\begin{aligned} 2(a+b)\ln(a+b-c) + 2(b+c)\ln(b+c-a) + 2(c+a)\ln(c+a-b) \\ \geq (2a+b+c)\ln a + (a+2b+c)\ln b + (a+b+2c)\ln c \end{aligned}$$

and we are done. □

The function $f(x) = \frac{x^3}{1+x}$ is convex for $x > 0$. In fact, $f'(x) = \frac{x^2(3+2x)}{(1+x)^2} > 0$ and $f''(x) = \frac{2x(3+3x+x^2)}{(1+x)^3} > 0$. Hence, f is increasing and convex. Applying again Theorem 2.2 to $f(x)$, we have

Theorem 2.6. *In any triangle ABC the following inequality*

$$\sum_{\text{cyclic}} \frac{(a+b-c)^3}{1+a+b-c} \geq \sum_{\text{cyclic}} \frac{a^3}{1+a} \tag{2.14}$$

holds.

Observe that the preceding procedure can be used to generate many triangle inequalities.

Before stating our next result we give a lemma that we will use later on.

Lemma 2.7. *Let x, y, z and a, b, c be strictly positive real numbers. Then, we have*

$$3\left(yza^2 + zxb^2 + xyc^2\right) \geq \left(a\sqrt{yz} + b\sqrt{zx} + c\sqrt{xy}\right)^2. \tag{2.15}$$

Proof. Let $\overrightarrow{u} = \left(\sqrt{yz}, \sqrt{zx}, \sqrt{xy}\right)$ and $\overrightarrow{v} = (a, b, c)$. By applying Cauchy-Buniakovski-Schwarz's inequality, we get

$$\left[\left(\sqrt{yz}, \sqrt{zx}, \sqrt{xy}\right) \cdot (a, b, c)\right]^2 \leq \left\|\left(\sqrt{yz}, \sqrt{zx}, \sqrt{xy}\right)\right\|^2 \|(a, b, c)\|^2$$

or equivalently,

$$\left(a\sqrt{yz} + b\sqrt{zx} + c\sqrt{xy}\right)^2 \leq (yz + zx + xy)(a^2 + b^2 + c^2). \tag{2.16}$$

On the other hand, applying the rearrangement inequality yields

$$a^2yz + b^2zx + c^2xy \geq b^2yz + c^2zx + a^2xy,$$
$$a^2yz + b^2zx + c^2xy \geq b^2xy + a^2zx + c^2yz.$$

Hence, the right hand side of (2.15) becomes

$$(yz + zx + xy)(a^2 + b^2 + c^2) \leq 3(yza^2 + zxb^2 + xyc^2)$$

and the proof is complete. □

In particular, setting $x = a + b - c$, $y = c + a - b$ and $z = b + c - a$ into the preceding lemma, we get the following

Theorem 2.8. *If a, b and c are the sides of triangle ABC, then*

$$\sum_{\textit{cyclic}} a^3 b \sin^2 \frac{C}{2} \geq \frac{1}{3} \left\{ \sum_{\textit{cyclic}} a\sqrt{(s-a)(s-b)} \right\}^2 . \tag{2.17}$$

Proof. Taking into account the Law of Cosines, we have

$$\sum_{\text{cyclic}} a^3 b \sin^2 \frac{C}{2} = \frac{1}{2} \sum_{\text{cyclic}} a^3 b(1 - \cos C) = \frac{1}{2} \sum_{\text{cyclic}} a^2[c^2 - (a-b)^2].$$

On the other hand,

$$\left\{ \sum_{\text{cyclic}} a\sqrt{(s-a)(s-b)} \right\}^2 = \frac{1}{2} \left\{ \sum_{\text{cyclic}} a\sqrt{c^2 - (a-b)^2} \right\}^2 .$$

Now, inequality (2.17) immediately follows from (2.15) and the proof is completed. □

References

[1] E. ARTIN, *The Gamma Function*, Holt, Reinhart and Winston, Inc., New York, 1964.

[2] O. BOTEMA et al., *Geometric Inequalities*, Wolters Noordhoff Publishing, Groningen, 1969.

[3] J.L. DÍAZ-BARRERO, Rational identities and inequalities involving Fibonacci and Lucas numbers, *J. Ineq. Pure and Appl. Math.*, **4**(5) (2003), Art. 83. [ONLINE: http://jipam.vu.edu.au/article.php?sid=324]

[4] G. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge, 1997.

[5] D.S. MITIRINOVIĆ, J.E. PEČARIĆ AND V. VOLENEC, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

[6] M. ONUCU, *Inegalităţi*, GIL, Zalău, România, 2003.

[7] E.W. WEISSTEIN et al., Majorization, *MathWorld– A Wolfram Web Resource*, (2004). [ONLINE: http://matworld.wolfram.com/Majorization.html]