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AN INTEGRAL INEQUALITY BOUNDING THE AUTOCORRELATION OF A PULSE OR SEQUENCE AT A KNOWN LAG

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ABSTRACT. This paper gives best bounds for the ratio $\int_a^{b-t} f(x)\, f(x+t)\, dx / \int_a^b f^2(x)\, dx$ for any square-summable real function $f(x)$ on the interval $(a, b]$. Similarly, bounds are established for the autocorrelation of any pulse or finite-length sequence at any known lag, and the family of pulses and sequences attaining these bounds is identified. The form of this family is related to a half-cycle of a sinusoid. Stronger bounds are suggested for pulses known to be non-negative and unimodal or concave.

Key words and phrases: Inequalities, Auto-correlation, Bounds.

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1. INTRODUCTION

This paper presents a derivation of the inequality

$$\left| \frac{\int_a^{b-t} f(x)\, f(x+t)\, dx}{\int_a^b f^2(x)\, dx} \right| \le \cos\left(\frac{\pi}{\lceil \frac{b-a}{t} \rceil + 1} \right) \qquad 0 < t \le b-a \tag{1.1}$$

for any square-summable real function $f(x)$ on the interval $(a, b]$, and demonstrates that the bound is the best possible. The notation $\lceil \cdot \rceil$ denotes the 'lowest integer not less than' function. The result is obtained by a shift in origin after derivation of the inequality

$$|A(t)| \le \cos\left(\frac{\pi}{\lceil T/t \rceil + 1} \right) \qquad 0 < t \le T, \tag{1.2}$$

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where

$$A(t) \equiv \frac{\int_0^{T-t} f(x)\, f(x+t)\, dx}{\int_0^T f^2(x)\, dx} \tag{1.3}$$

is the autocorrelation function of a pulse of duration T. As the notation suggests, the autocorrelation function is principally thought of as a function of the lag t for known $f(x)$ and T. In that context familiar results are $A(0) = 1$ and $|A(t)| \leq 1$. Here the lag (as a proportion of the pulse duration) is deemed to be known and the bounds, equation (1.2), are obtained for any square-summable pulse $f(x)$. If T is unknown then the parameter of interest is the lag proportion t/T, and equation (1.2) can be written accordingly. Also if the pulse duration is known only to be less than or equal to some figure T then the same bounds hold. Equation (1.2) is obtained by first developing a discrete analogy which bounds the autocorrelation of a real sequence.

The motivating example for this work was the placement of bounds on a correlation arising in medical Doppler ultrasound, where the goal is the description of blood-flow. Scatterers of ultrasound (groupings of red cells) within the blood can be regarded as being distributed uniformly and randomly within an insonated volume. Therefore the power of the signal measured on reception at the transmitter-receiver is the sum of many contributions with uniform random phases. A short time later the scatterers have moved with the rest of the blood but with little change in their relative positions. Some scatterers have entered the insonated volume and some have left. The correlation between the powers received at these two times is related to the velocity of blood v, the time interval τ, and the intensity profile of the ultrasound beam. The 'pulse' $f(x)$ is this intensity profile as a one-dimensional function of space in the direction of the blood velocity. So the product $|v\tau|$ is t, and the spatial extent of the intensity function is T. If sides lobes are ignored this extent is the width of the central lobe of the intensity function. Even if the *shape* of the intensity function is unknown the correlation between the powers at the two times is bounded according to equation (1.2) and more strongly according to the results for unimodal and concave functions obtained in Sections 3 and 4. If the correlation is determined experimentally this in turn bounds v.

There appears to be little published regarding such bounds on an autocorrelation. Communications engineers are more interested in the design of sequences with desirable (small) autocorrelation over a range of lags. Upper bounds have been given for the autocorrelation of maximal-length pseudo-random sequences [1], while lower bounds for the maximum magnitude of cross-correlation functions and autocorrelation functions for sets of complex-valued sequences have been considered [2].

2. ANALYSIS

Equation (1.2) is derived by considering the definite integral to be a sum of infinitesimal terms. So we first study the autocorrelation of a finite real sequence of P values $\{f_n\}$, $n = 1, 2, \ldots, P$, at positive lag p, which is

$$A_p \equiv \frac{\sum_{n=1}^{P-p} f_n f_{n+p}}{\sum_{n=1}^{P} f_n^2}. \tag{2.1}$$

Then we increase p and P without limit while preserving the ratio $p/P = t/T$.

Write $P = Mp + q$ where $M = \lfloor P/p \rfloor$ and $0 \leq q < p$, and $\lfloor \cdot \rfloor$ is 'the greatest integer not greater than' function. The sequence can be split into p interleaving subsequences each containing elements spaced p apart. The jth subsequence, $\{f_j, f_{j+p}, \ldots, f_{j+(L_j-1)p}\}$ has length L_j, where $L_j = M + 1$ for $1 \leq j \leq q$ and $L_j = M$ for $q + 1 \leq j \leq p$. Equation (2.1) can then be written as

$$A_p = \frac{a_1 + \ldots + a_p}{b_1 + \ldots + b_p}$$

where

$$a_j = \sum_{k=1}^{L_j} f_{j+(k-1)p} f_{j+kp} \qquad \text{and} \qquad b_j = \sum_{k=1}^{L_j} f^2_{j+(k-1)p}.$$

Evidently a_j/b_j is the autocorrelation of the jth subsequence with lag 1. Because each b_j is non-negative it follows that A_p is bounded between the maximum and minimum values of a_j/b_j, so

$$|A_p| \leq \max_j \{|a_j/b_j|\}.$$

Any of the subsequences can be relabelled $\{F_1, F_2, \ldots F_N\}$ where $N = M$ or $N = M + 1$ so the problem reduces to bounding the autocorrelation of this new sequence at lag 1, i.e. bounding

(2.2) $$A^* \equiv \frac{\sum_{n=1}^{N-1} F_n F_{n+1}}{\sum_{i=1}^{N} F_n^2}$$

for any sequence $\{F_n\}$ and choosing $N = M$ or $N = M + 1$ to give the least lower bound and greatest upper bound.

Let ρ be an extremum of A^* with respect to each element of $\{F_n\}$. Setting $\partial A^*/\partial F_n = 0$ gives

$$(F_{n-1} + F_{n+1}) \sum_{i=1}^{N} F_i^2 = 2F_n \sum_{i=1}^{N-1} F_i F_{i+1} \qquad n = 1, \ldots, N$$

if we define $F_0 \equiv 0$ and $F_{N+1} \equiv 0$. At this extremum the right-hand side of equation (2.2) is ρ, so this rearranges to the recurrence relation

(2.3) $$F_{n+1} = 2\rho\, F_n - F_{n-1}.$$

The general solution to equation (2.3) with $F_0 = 0$ is

$$F_n = K \sin(n \cos^{-1} \rho) \qquad n = 0, \ldots, N$$

where K is an arbitrary constant. This equation must be the general solution because it satisfies equation (2.3) and $F_0 = 0$ while keeping F_1 arbitrary. From the further condition $F_{N+1} = 0$ we identify possible values of ρ to be

(2.4) $$\rho = \cos\left(\tfrac{k\pi}{N+1}\right)$$

where k is an integer. So A^* takes its global maximum and minimum values of

(2.5) $$\rho_{\max} = \cos\left(\tfrac{\pi}{N+1}\right) \qquad \text{and} \qquad \rho_{\min} = -\cos\left(\tfrac{\pi}{N+1}\right)$$

with corresponding sequences

(2.6) $$F_n = K \sin\left(\tfrac{n\pi}{N+1}\right) \qquad \text{and} \qquad F_n = (-1)^n K \sin\left(\tfrac{n\pi}{N+1}\right),$$

whose elements are equally spaced samples of half-cycles of sinusoids.

An alternative derivation of equation (2.4) follows from writing the right-hand side of equation (2.2) as $(\mathbf{F}'\mathbf{C}\mathbf{F})/(\mathbf{F}'\mathbf{F})$ where $\mathbf{F}$ is the column vector $(F_1, F_2, \ldots, F_N)'$, and $\mathbf{C}$ is the $N \times N$ Toeplitz matrix with elements $1/2$ in the diagonals immediately either side of the leading diagonal (first super-diagonal and first sub-diagonal), and zero elsewhere. Differentiating with respect to $\mathbf{F}$ and setting the result to zero gives $\mathbf{CF} = \rho\, \mathbf{F}$, which is a re-expression of equation (2.3) with $F_0 = 0$ and $F_{N+1} = 0$. So the possible values of ρ and vectors $\mathbf{F}$ are the eigenvalues and eigenvectors of $\mathbf{C}$. The eigenvalues of an $N \times N$ matrix with elements c_0 in the

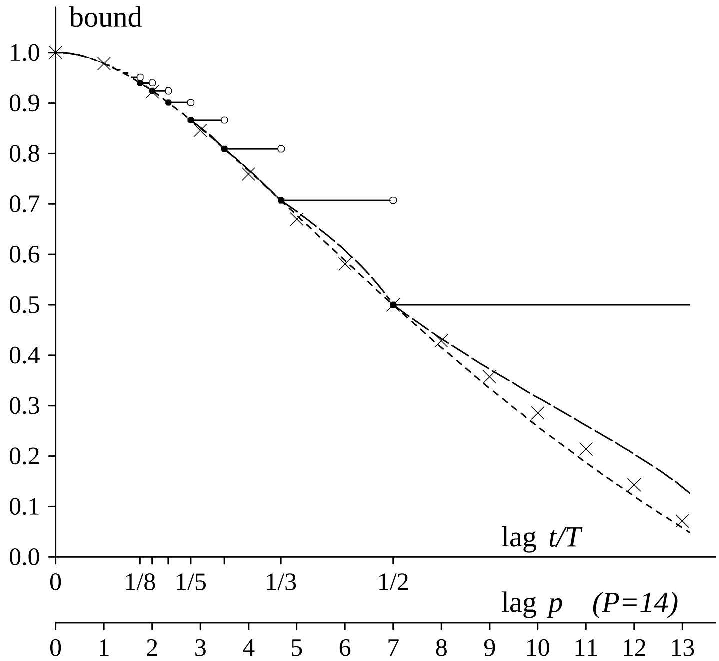

Figure 2.1: Bounds for the autocorrelation function. Solid line – magnitude of bound for all functions, $\cos(\pi/(\lceil T/t \rceil + 1))$*. Short-dashed line –* $\cos(\pi/(T/t+1))$*. Long-dashed line – apparent upper bound for unimodal functions. Cross marks – apparent upper bound for concave sequences of length 14. As* t/T *approaches zero each line approaches* $\cos(\pi t/T)$*.*

leading diagonal, c_1 in the first super-diagonal, c_2 in the first sub-diagonal and zero elsewhere are $c_0 + 2(c_1c_2)^{1/2}\cos(k\pi/(N+1)),\quad k = 1,\ldots,N$ [3, p. 284].

Both bounds in equation (2.5) are larger in magnitude when $N = M+1$ than when $N = M$. Also $M+1 = \lceil P/p \rceil$. Therefore the autocorrelation of the original sequence at lag p is bounded by

$$|A_p| \leq \cos\left(\frac{\pi}{\lceil P/p \rceil + 1}\right). \tag{2.7}$$

If p and P tend to infinity while maintaining the ratio $p/P = t/T$, the result is equation (1.2) for the bound on the autocorrelation defined by equation (1.3). To show this more rigorously define the stepwise function

$$f(x) = f_n \qquad \tfrac{(n-1)T}{P} < x \leq \tfrac{nT}{P} \qquad n = 1,2,\ldots,P,$$

which has fixed extent T and step length T/P, and define $f(x) = 0$ outside $(0,T]$. So for integer values of $k \geq 0$

$$f_{n+k} = f(x + kT/P) \qquad \tfrac{(n-1)T}{P} < x \leq \tfrac{nT}{P} \qquad n = 1,2,\ldots,P-k,$$

from which it follows that

$$f_n f_{n+k} = \frac{P}{T}\int_{(n-1)T/P}^{nT/P} f(x)f(x+kT/P)\,dx.$$

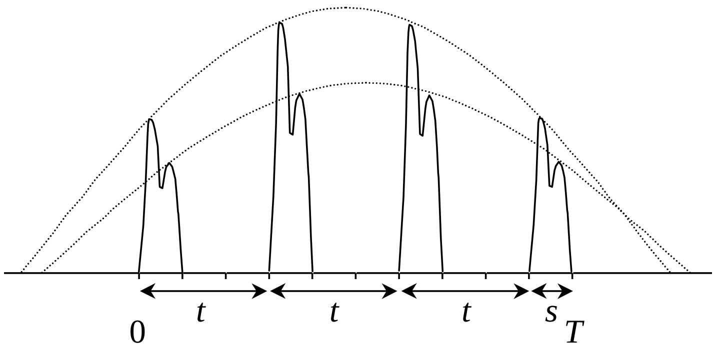

Figure 2.2: A function giving maximum correlation when T/t is not an integer. (Here $t/T = 0.3$.) The function comprises $\lceil T/t \rceil$ regions of duration $s = T - (\lceil T/t \rceil - 1)\, t$, each with identical arbitrary form but scaled so that corresponding points lie on a sinusoid, and $\lceil T/t \rceil - 1$ interleaving regions of zero.

Form the relevant sums $\sum f_n f_{n+k}$ appearing in equation (2.1) with $k = p$ in the numerator and $k = 0$ in the denominator, and set $t = pT/P$ to obtain

$$A_p = \frac{\int_0^{T-t} f(x)\, f(x+t)\, dx}{\int_0^T f^2(x)\, dx}.$$

Recall that T is fixed. Let p and P tend to infinity in a way that maintains the ratio $p/P = t/T$. No restrictions are placed on the f_n values so this enables $f(x)$ to approach any square-summable function, continuous or otherwise, which is zero outside $(0, T]$. Also the lags t corresponding to neighbouring values of p become arbitrarily close, so the result is valid for any t where $0 < t \leq T$. Therefore equation (2.1) becomes equation (1.3), and equation (2.7) becomes equation (1.2). Thus we have found bounds on the autocorrelation of any pulse at a lag which is a known proportion of the pulse duration.

If the condition that $f(x) = 0$ outside $(0, T]$ is relaxed then equation (1.3) no longer defines the autocorrelation function but describes a more general situation. The equations derived will still be true, as they do not require anything of $f(x)$ outside that interval, and by a shift of origin, with $T = b - a$ (for any real a, b, with $a < b$), equation (1.2) generalises to the more fundamental result that is equation (1.1).

The bound in equation (1.2) is given by the stepwise solid line in Figure 2.1. The left end-points of the pieces of this function correspond to integer values of T/t. For non-integer T/t the positive upper bound is only reached by functions such as that in Figure 2.2, (where, for example, $t/T = 0.3$). The function can only be non-zero in $\lceil T/t \rceil\ (= 4)$ regions each of duration $s = T - (\lceil T/t \rceil - 1)\, t\ (= 0.1)$. The points in these regions correspond to elements in the longer subsequences of length $M + 1 (= 4)$, in the discrete formulation given above. The function in each of these regions is of identical arbitrary form, but with a different scale factor. Corresponding points lie on a half-cycle of a sinusoid, as drawn in Figure 2.2 for the two modal points, and in accordance with the first sequence in equation (2.6). The function must be zero in the interleaving $\lceil T/t \rceil - 1\ (= 3)$ regions, containing points corresponding to elements in the shorter subsequences of length $M (= 3)$. The negative lower bound is reached by such a function if every second non-zero region is inverted, as in the second sequence in equation (2.6). Examples like this can be constructed for any lag, which indicates that the bounds given by equation (1.2) and equation (1.1) are the best possible.

The short-dashed line on Figure 2.1 gives the quantity $\cos\left(\pi/(T/t + 1)\right)$ which is the bound in equation (1.2) without application of the $\lceil \cdot \rceil$ function. As the lag approaches zero the bound

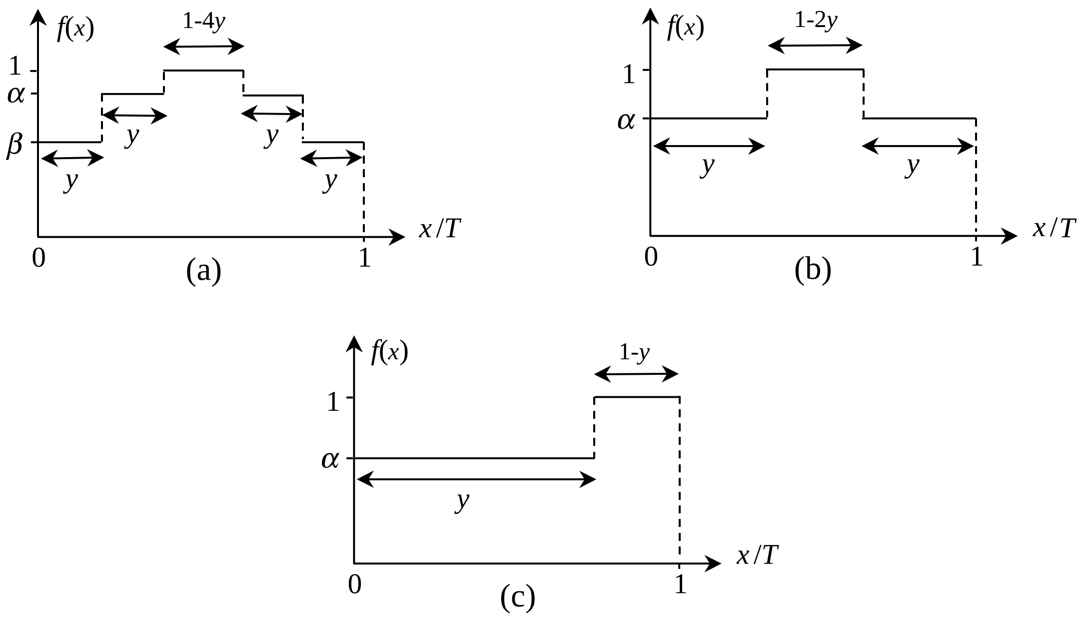

Figure 3.1: Unimodal functions maximising the autocorrelation at lag $y \equiv t/T$. (a) $1/6 \le y < 1/4$ (b) $1/4 \le y < 1/2$ (c) $1/2 \le y < 1$

approaches this quantity, which itself approaches $\cos(\pi t/T)$. Also, using equation (2.6) for sequences with increasing length, the shape of the pulse maximising the correlation approaches in some sense the half-cycle $\sin(\pi x/T)$ for $0 < x \le T$.

3. UNIMODAL FUNCTIONS

Consider the subset of pulses which are non-negative and unimodal, i.e. with a single modal point or plateau that might contain either extreme point 0 or T. An example is the central lobe of a sinc, i.e. $(\sin x)/x$, or sinc-squared function, as might be the form of $f(x)$ in the medical ultrasonics example. The bound of interest is the upper bound.

The discussion in the previous section suggests that for such functions the upper bound given in equation (1.2) is only reached when T/t is an integer and the function comprises T/t level sections each of length t with heights which are equally spaced samples of a half-sinusoid.

For general values of T/t a Monte Carlo technique was used to find the unimodal pulse shape maximising the autocorrelation for a given lag. (An analytical derivation was not found.) Random unimodal sequences of length $N = 8$ were created by the cumulative summation of uniform random numbers either side of a randomly selected mode. This was carried out 10^9 times, and for each lag the sequence giving the maximum autocorrelation was recorded. For $y \le 1/2$ the sequences suggested were symmetric, so 10^9 symmetric sequences of both $N = 13$ and $N = 16$ were studied. The sequences and functions strongly suggested by this technique are stepwise. Consider the autocorrelation as a function of the lag proportion $y \equiv t/T$. For $y < 1/2$ the function suggested is symmetric, stepwise and of the family including Figures 3.1a and 3.1b. For $y \ge 1/2$ the function suggested is of the form shown in Figure 3.1c (or its reflection about the axis $x/T = 1/2$).

In Figure 3.1 the modal region in each case is scaled to have height 1. Assuming these forms are correct, expressions are found for the maximum autocorrelation by setting to zero the derivatives of the autocorrelation with respect to the unknown levels, α (and β), solving a polynomial equation to obtain these levels, and calculating the autocorrelation. Thus the autocorrelation for a unimodal function is bounded by

$$0 \le A(t) \le \begin{cases} \frac{5y-1+\sqrt{(1-2y-3y^2)}}{4y} & \frac{1}{5} \le y < \frac{1}{4} \\ \frac{y}{3y-1+\sqrt{(1-4y+5y^2)}} & \frac{1}{4} \le y < \frac{1}{3} \\ \frac{3y-1+\sqrt{(1+2y-7y^2)}}{4y} & \frac{1}{3} \le y < \frac{1}{2} \\ \frac{1}{2}\sqrt{(\frac{1}{y}-1)} & \frac{1}{2} \le y < 1. \end{cases} \tag{3.1}$$

The upper bound is continuous and is the long-dashed line in Figure 2.1.

The two dashed lines on Figure 2.1 are close between $y = 1/5$ and $y = 1/3$ and, by suggestion, will be close for lower values of y also. Therefore, for $y \le 1/3$ the short-dashed line provides an approximate bound, and for a unimodal function the inequality

$$0 \ \le \ A(t) \ <\approx \ \cos\left(\frac{\pi}{T/t+1}\right) \qquad 0 < \tfrac{t}{T} \le \tfrac{1}{3} \tag{3.2}$$

might be used in place of equation (3.1).

The levels α (and β) in the functions of Figure 3.1 attaining the upper bound in equation (3.1) are simply related to this bound. Let V be the upper bound listed in equation (3.1). Then $\alpha = V$ and $\beta = 1/2$ for $1/5 \le y < 1/4$, $\alpha = 1/(2V)$ for $1/4 \le y < 1/3$, $\alpha = V$ for $1/3 \le y < 1/2$ and $\alpha = 2V$ for $1/2 \le y < 1$.

As with equation (1.1) an inequality for square-summable functions which are non-negative and unimodal in $(a, b]$ can be written using equation (3.1) or equation (3.2).

4. CONCAVE FUNCTIONS

A similar Monte Carlo analysis was performed for the subset of non-negative unimodal pulses which are concave, i.e. have a second derivative which is zero or negative at all points in the interval. The stepwise forms of Figure 3.1 are then excluded. Preliminary results suggested that the concave pulse maximising the autocorrelation for any fixed lag is symmetric. Subsequently 10^9 random symmetric concave sequences of length $P = 14$ were generated. The observed maximum autocorrelations of these sequences for lags $p = 1, \dots, 13$ are marked on Figure 2.1 together with the trivial bound of 1 for lag zero. The results for $p \ge 7$ suggest that the maximum autocorrelation for lag $t/T \ge 1/2$ lies on the straight line '$\text{bound} = 1 - t/T$' and the maximising function is uniform. For $p < 7$ the maximum autocorrelations appear to lie just below the short-dashed line. The corresponding sequences suggest that the maximising function is of the form shown normalised in Figure 4.1, where outside a central curved section the function is linear. With an increase in y, the quantities γ and δ increase and the absolute slope of the linear regions decreases. As y decreases the function approaches a half-cycle of a sinusoid.

The cross marks lie close to the short-dashed line in Figure 2.1. This suggests that for a concave function the inequality

$$0 \ \le \ A(t) \ <\approx \begin{cases} \cos\left(\frac{\pi}{T/t+1}\right) & 0 < \frac{t}{T} < \frac{1}{2} \\ 1 - \frac{t}{T} & \frac{1}{2} \le \frac{t}{T} \le 1 \end{cases} \tag{4.1}$$

may be more useful than equation (1.2).

As with equation (1.1) an inequality for square-summable functions which are non-negative and concave in some interval $(a, b]$ can be written using this result.

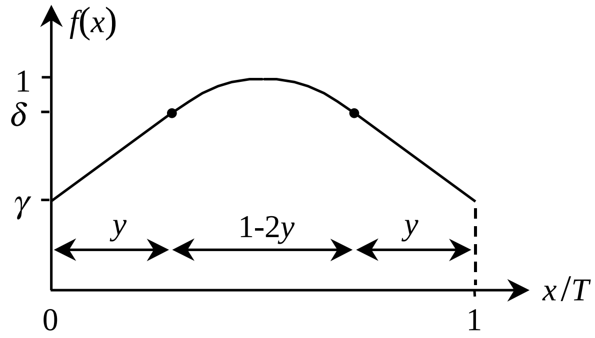

Figure 4.1: The concave function maximising the autocorrelation at lag $y \equiv t/T$ for $y < 1/2$.

5. SUMMARY

The autocorrelation of any square-summable pulse $f(x)$ of duration T at lag t is bounded by

$$\left| \frac{\int_0^{T-t} f(x)\, f(x+t)\, dx}{\int_0^T f^2(x)\, dx} \right| \le \cos\left(\frac{\pi}{\lceil T/t \rceil + 1} \right) \qquad 0 < t \le T,$$

which is equation (1.2). Similarly, the right-hand side is a bound on the autocorrelation of a pulse at a lag which is at least a proportion t/T of the pulse duration.

The magnitude of the bound is depicted by the stepwise solid line of Figure 2.1. If only non-negative and unimodal pulses are permitted then, using a Monte-Carlo method, suggested bounds are given by equation (3.1), the upper bound is shown by the long-dashed line of Figure 2.1 and approximate bounds are given by equation (3.2). If only pulses which are non-negative and concave are permitted then, using a Monte-Carlo method, approximate bounds are given by equation (4.1).

The importance of the sine and cosine functions in this analysis is evident. The pulses and sequences attaining the bounds are constrained by half-cycles of a sinusoid. As the lag approaches zero each upper bound approaches 1 according to $\cos(\pi t/T)$ and the shape of pulse maximising the correlation approaches in some sense a half-cycle of a sinusoid, which is unimodal and concave.

Each of these inequalities can be modified to apply to real functions square-summable on some interval. For any such function $f(x)$ and interval $(a, b]$ the appropriate inequality is

$$\left| \frac{\int_a^{b-t} f(x)\, f(x+t)\, dx}{\int_a^b f^2(x)\, dx} \right| \le \cos\left(\frac{\pi}{\lceil \frac{b-a}{t} \rceil + 1} \right) \qquad 0 < t \le b-a,$$

which is equation (1.1).

In addition, bounds on the autocorrelation of any real sequence $\{f_n\}$ of length P at lag p are given by

$$\left| \frac{\sum_{n=1}^{P-p} f_n f_{n+p}}{\sum_{n=1}^{P} f_n^2} \right| \le \cos\left(\frac{\pi}{\lceil P/p \rceil + 1} \right),$$

which is equation (2.7). For $p = 1$ the extreme correlations are given by equation (2.5) and the corresponding sequences by equation (2.6).

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