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NECESSARY AND SUFFICIENT CONDITIONS FOR JOINT LOWER AND UPPER ESTIMATES

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ABSTRACT. Necessary and sufficient conditions are given for joint lower and upper estimates of the following sums $\sum_1^m \gamma_n n^{-1}$ and $\sum_m^\infty \gamma_n n^{-1}$. An application of the results yields necessary and sufficient conditions for a pleasant and useful lemma of Sagher.

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1. INTRODUCTION

It seems to be widely accepted that certain estimates given for sums or integrals play key roles in proving theorems. Therefore it is always a crucial task to give those conditions, especially necessary and sufficient ones, that imply these estimates. Namely, to check the fulfillment of the given conditions, generally, is easier than examining the realization of the estimates.

Now we recall only one result of [1], namely we want to broaden the estimations given in the cited paper, more precisely to give lower estimations for the sums estimated there only from above; and to establish the necessary and sufficient conditions implying these lower estimates.

Our previous result reads as follows.

For notions and notations, please, consult the third section.

Theorem 1.1 ([1, Corollary 2]). *A positive sequence $\{\gamma_n\}$ bounded by blocks is quasi β-power-monotone increasing (decreasing) with a certain negative (positive) exponent β if and only if the inequality*

$$\sum_{n=1}^{m} \gamma_n n^{-1} \leqq K\gamma_m, \quad \left(\sum_{n=m}^{\infty} \gamma_n n^{-1} \leqq K\gamma_m\right), \qquad K := K(\gamma),$$

holds for any natural number m.

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We discovered this problem whilst reading the new book of A. Kufner, L. Maligranda and L.E. Persson [2], which gives a comprehensive survey on the fascinating history of the Hardy inequality, and on p. 94 we found the so-called Sagher-lemma [3], unfortunately just too late. This lemma is of independent interest and reads as follows:

Theorem 1.2 ([3, Lemma]). *Let $m(t)$ be a positive function. Then*

$$\int_0^r m(t)\frac{dt}{t} \asymp m(r) \tag{1.1}$$

if and only if

$$\int_r^\infty \frac{1}{m(t)}\frac{dt}{t} \asymp m(r)^{-1}. \tag{1.2}$$

It is easy to see that our inequalities are discrete (pertaining to sequences) analogies of Sagher's upper estimates. However, there are two important differences:

(1) Sagher's lemma also gives (or claims) joint lower estimates.
(2) It does not give conditions on the function $m(t)$ implying the realization of these inequalities.

These two things have raised the challenges of providing lower estimates for our sums assuming that our upper estimates also hold; and thereafter establishing conditions for $m(t)$ ensuring the fulfillment of Sagher's inequalities.

We shall see that the necessary and sufficient conditions for (1.1) are the same as that of (1.2), consequently we get a new proof of the equivalence of statements (1.1) and (1.2).

2. Results

Now we prove the following theorem.

Theorem 2.1. *If $\{\gamma_n\}$ is a positive sequence, then the inequalities*

$$\alpha_1\gamma_m \leqq \sum_{n=1}^{m} \gamma_n n^{-1} \leqq \alpha_2\gamma_m, \quad 0<\alpha_1\leqq\alpha_2<\infty, \tag{2.1}$$

hold jointly if and only if $\{\gamma_n\}$ is quasi β-power-monotone increasing with some negative β, and quasi $\overline{\beta}$-power-monotone decreasing with some $\overline{\beta} \leqq \beta$.

Furthermore,

$$\alpha_1\gamma_m \leqq \sum_{n=m}^{\infty} \gamma_n n^{-1} \leqq \alpha_2\gamma_m \tag{2.2}$$

holds jointly if and only if $\{\gamma_n\}$ is quasi β-power-monotone decreasing with some positive β, and quasi $\overline{\beta}$-power-monotone increasing with some $\overline{\beta} \geqq \beta$.

Remark 1. It is quite obvious that if we extend the given definitions from sequences to functions implicitly, then an analogous theorem for functions would also be valid.

Remark 2. It is easy to see that if $\{\gamma_n\}$ satisfies the conditions needed in (2.2), then the sequence $\{\gamma_n^{-1}\}$ satisfies the conditions required in (2.1). Consequently we observe that (1.1) and (1.2) have the same necessary and sufficient conditions, thus their equivalence follows from Theorem 2.1; more precisely, from its analogue for functions. These conditions are the following: There exist $\overline{\beta} \geqq \beta > 0$ such that $m(t)t^{-\beta}\uparrow$ and $m(t)t^{-\overline{\beta}}\downarrow$.

Corollary 2.2. *Let $m(t)$ be a positive function. Then (1.1) and (1.2) hold if and only if there exist $\overline{\beta} \geqq \beta > 0$ such that $m(t)t^{-\beta}\uparrow$ and $m(t)t^{-\overline{\beta}}\downarrow$.*

3. NOTIONS AND NOTATIONS

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called *quasi β-power-monotone* increasing (decreasing) if there exists a constant $K = K(\beta, \gamma) \geqq 1$ such that

$$Kn^\beta \gamma_n \geqq m^\beta \gamma_m \quad (n^\beta \gamma_n \leqq K m^\beta \gamma_m)$$

holds for any $n \geqq m$. These properties of γ will be denoted by $n^\beta \gamma_n \uparrow$ and $n^\beta \gamma_n \downarrow$, respectively.

We shall say that a sequence $\gamma := \{\gamma_n\}$ is *bounded by blocks* if the inequalities

$$\alpha_1 \Gamma_m^{(k)} \leqq \gamma_n \leqq \alpha_2 \Gamma_M^{(k)}, \quad 0 < \alpha_1 \leqq \alpha_2 < \infty$$

hold for any $2^k \leqq n \leqq 2^{k+1},\ k = 1, 2, \dots,$ where

$$\Gamma_m^{(k)} := \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \quad \text{and} \quad \Gamma_M^{(k)} := \max(\gamma_{2^k}, \gamma_{2^{k+1}}).$$

We shall use the notation $L \ll R$ in inequalities when there exists a positive constant K such that $L \leqq KR$; but where K may be different in different occurrences of "$\ll$". Naturally, the notation $\ll$ between the terms of sequences, e.g. $a_n \ll b_n$, means that $a_n \leqq K b_n$ holds with the same constant for every n.

If $L \ll R$ and $R \ll L$ hold simultaneously, then we shall write $L \asymp R$.

The capital letters K and K_i, above and later on, denote positive constants $(\geqq 1)$.

4. PROOF OF THEOREM 2.1

First we show that if γ_n satisfies (2.1) or (2.2) then it is a sequence bounded by blocks.

Let Γ_m denote either

$$\sum_{n=1}^{m} \gamma_n n^{-1} \quad \text{or} \quad \sum_{n=m}^{\infty} \gamma_n n^{-1}.$$

Then (2.1) and (2.2) each imply that

$$\alpha_2^{-1} \Gamma_m \leqq \gamma_m \leqq \alpha_1^{-1} \Gamma_m.$$

Hence, utilizing the monotonicity of $\{\Gamma_m\}$, (2.1) and (2.2), respectively, we get for any $2^{\mu-1} \leqq m \leqq 2^\mu$, if Γ_n is increasing, that

$$\frac{\alpha_1}{\alpha_2} \gamma_{2^{\mu-1}} \leqq \alpha_2^{-1} \Gamma_{2^{\mu-1}} \leqq \gamma_m \leqq \alpha_1^{-1} \Gamma_{2^\mu} \leqq \frac{\alpha_2}{\alpha_1} \gamma_{2^\mu}$$

holds. If Γ_n is decreasing then $2^{\mu-1}$ is substituted in place of 2^μ and the modified inequality holds. Herewith our assertion is verified.

Taking account of this fact, the assertions pertaining to the upper estimates given in (2.1) and (2.2) have been proved by Theorem 1.1.

Consequently in the proofs of the lower estimates we can implicitly use that $\gamma_n n^{-\beta} \uparrow$ or $\gamma_n n^\beta \downarrow$ with $\beta > 0$.

Next we show that the first inequality of (2.1) holds if and only if $\gamma_n n^{-\overline{\beta}} \downarrow$ with certain $\overline{\beta} > 0$.

If $\gamma_n n^{-\overline{\beta}} \downarrow$ $(\overline{\beta} > 0)$ then it is plain that

$$\sum_{n=1}^{m} \gamma_n n^{-1} \geqq \gamma_m m^{-\overline{\beta}} \sum_{n=1}^{m} n^{-1+\overline{\beta}} \gg \gamma_m$$

holds. Conversely, if

$$\gamma_m \ll \sum_{n=1}^{m} \gamma_n n^{-1} \ll \gamma_m, \tag{4.1}$$

then we show there exists $\overline{\beta} > 0$ such that $\gamma_n n^{-\overline{\beta}} \downarrow$. To verify this we first show that

$$\gamma_{2m} \ll \gamma_m \tag{4.2}$$

holds from (4.1).

Since $\gamma_n n^{-\beta} \uparrow$ with $\beta > 0$, therefore an elementary calculation shows that

$$\sum_{n=1}^{m} \frac{\gamma_n}{n} \ll \sum_{n=m+1}^{2m} \frac{\gamma_n}{n}.$$

Thus it suffices to show that

$$\gamma_{2m} \ll \frac{1}{m} \sum_{n=m+1}^{2m} \gamma_n =: \frac{1}{m}\sigma_m \tag{4.3}$$

implies (4.2).

Denote μ as the smallest positive integer such that

$$\sum_{n=2m-\mu+1}^{2m} \gamma_n \geqq \frac{\sigma_m}{2}. \tag{4.4}$$

Then, by $\gamma_n \uparrow$, (4.3) and (4.4), we obtain that

$$\mu\gamma_{2m} \gg \frac{\sigma_m}{2} \gg m\gamma_{2m},$$

whence

$$\mu \geqq m/K_1 \tag{4.5}$$

follows. Furthermore we know that

$$(m+1-\mu)\gamma_{2m-\mu+1} \gg \sum_{n=m+1}^{2m-\mu+1} \gamma_n \geqq \frac{\sigma_m}{2} \gg m\gamma_{2m},$$

thus

$$\gamma_{2m} \leqq K_2 \gamma_{2m-\mu+1}. \tag{4.6}$$

Without loss of generality we can assume that K_1 and K_2 are both greater than 4. Then let $K \geqq \max(K_1, K_2, K(\beta, \gamma))$ be an integer.

Hereafter, by (4.5) and (4.6), an easy consideration gives, repeating K-times the estimate given in (4.6), that

$$\gamma_{2m} \leqq K^K \gamma_m \tag{4.7}$$

holds, that is, (4.2) is proved.

Now let $2^{\overline{\beta}} := K^K$ and $K(\overline{\beta}, \gamma) := K^2 2^{2\overline{\beta}}$. We shall show that

$$K(\overline{\beta}, \gamma)\gamma_m m^{-\overline{\beta}} \geqq \gamma_n n^{-\overline{\beta}}, \quad \text{for any} \quad n \geqq m \tag{4.8}$$

holds, that is, $\gamma_n n^{-\overline{\beta}} \downarrow$ as required in Theorem 2.1.

Since inequality (4.7) gives

$$\gamma_{2n} \leqq 2^{\overline{\beta}} \gamma_n,$$

we obtain that

$$\frac{\gamma_{2n}}{(2n)^{\overline{\beta}}} \leqq \frac{2^{\overline{\beta}}\gamma_n}{(2n)^{\overline{\beta}}} = \frac{\gamma_n}{n^{\overline{\beta}}}. \tag{4.9}$$

If

$$2^{k+\nu+1} \geqq n \geqq 2^{k+\nu} \geqq 2^k \geqq m \geqq 2^{k-1},$$

then, using elementary estimates, (4.9) and the fact that $K\gamma_n \geqq K(\beta,\gamma)\gamma_n \geqq \gamma_m$, if $n \geqq m$, we obtain

$$\frac{\gamma_n}{n^{\overline{\beta}}} \leqq \frac{K\gamma_{2^{k+\nu+1}}}{(2^{k+\nu})^{\overline{\beta}}} \leqq \frac{2^{\overline{\beta}}K\gamma_{2^{k+\nu+1}}}{(2^{k+\nu+1})^{\overline{\beta}}} \leqq \frac{2^{\overline{\beta}}K\gamma_{2^{k-1}}}{(2^{k-1})^{\overline{\beta}}} \leqq \frac{2^{2\overline{\beta}}K^2\gamma_m}{m^{\overline{\beta}}},$$

whence (4.8) follows.

Now we can turn to the proof of the lower estimate of (2.2). This proof is similar to the proof given for the lower estimate of (2.1).

If $\gamma_n n^{\overline{\beta}} \uparrow \ (\overline{\beta} > 0)$, then

$$\sum_{n=m}^{\infty} \gamma_n n^{-1} \geqq \gamma_m m^{\overline{\beta}} \sum_{n=m}^{\infty} n^{-1-\overline{\beta}} \gg \gamma_m$$

clearly holds.

Conversely, if

$$\gamma_m \ll \sum_{n=m}^{\infty} \gamma_n n^{-1} \ll \gamma_m$$

holds, then first we show that

$$\gamma_m \ll \gamma_{2m}, \quad m \in \mathbb{N}. \tag{4.10}$$

Using the assumption $\gamma_n n^{\beta} \downarrow$ with some $\beta > 0$, we can easily show that

$$\sum_{n=2m+1}^{\infty} \gamma_n n^{-1} \ll \sum_{n=m}^{2m} \gamma_n n^{-1}$$

holds. Thus it suffices to prove that

$$\gamma_m \ll \frac{1}{m} \sum_{n=m}^{2m} \gamma_n =: \frac{1}{m}\sigma_m \tag{4.11}$$

implies (4.10).

Henceforth we can proceed likewise as above. Denote by μ the smallest positive integer such that

$$\sum_{n=m}^{m+\mu+1} \gamma_n \geqq \frac{\sigma_m}{2}. \tag{4.12}$$

Then, $\gamma_n n^{\beta} \downarrow, \ \beta > 0$; (4.11) and (4.12) imply that

$$m\gamma_m \ll \gamma_m(\mu + 2),$$

whence

$$\mu \geqq \frac{m}{K_1} \tag{4.13}$$

follows. Since

$$\sum_{n=m+\mu+2}^{2m} \gamma_n \leqq \frac{\sigma_m}{2} \leqq \sum_{n=m+\mu+1}^{2m} \gamma_n \ll \gamma_{m+\mu}(m-\mu),$$

thus, by (4.11), we get that

$$m\gamma_m \ll \gamma_{m+\mu}(m-\mu),$$

that is,

$$\gamma_m \leqq K_2\gamma_{m+\mu}.$$

Arguing as above, by (4.13), we can give a constant K such that

$$\gamma_m \leqq K\gamma_{2m} =: 2^{\overline{\beta}}\gamma_{2m} \tag{4.14}$$

holds.

Proceeding as before, we can show that with the $\overline{\beta}$ defined in (4.14), the sequence $\{\gamma_n\}$ is quasi $\overline{\beta}$-power-monotone increasing.

Herewith the proof of Theorem 2.1 is complete. □

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