Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 5, Article 79, 2002

POINTWISE ERROR ESTIMATE FOR A NONCOERCIVE SYSTEM OF QUASI-VARIATIONAL INEQUALITIES RELATED TO THE MANAGEMENT OF ENERGY PRODUCTION

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Received 27 May, 2002; accepted 24 July, 2002
Communicated by R. Verma

ABSTRACT. This paper is devoted to the approximation by a piecewise linear finite element method of a noncoercive system of elliptic quasi-variational inequalities arising in the management of energy production. A quasi-optimal L^∞ error estimate is established, using the concept of subsolution.

Key words and phrases: Quasi-Variational Inequalities, Subsolutions, Finite Elements, L^∞-Error Estimate.

2000 *Mathematics Subject Classification.* 49J40, 65N30, 65N15.

1. INTRODUCTION

A lot of results on error estimates in the L^∞ norm for the classical obstacle problem in particular and variational inequalities (VIs) in general have been achieved in the last three decades. (cf., e.g [6], [7], [8], [9]). However, very few works are known in this area when it comes to quasi-variational inequalities (QVIs) (cf., [10], [11]), and especially the case of systems which is the subject of this paper.(cf. e.g [3]

Indeed, we are concerned with the numerical approximation in the L^∞ norm for the noncoercive problem associated with the following system of QVIs: Find $U = (u^1, \dots, u^J) \in (H_0^1(\Omega))^J$ satisfying

$$\begin{cases} a^i(u^i, v - u^i) \geqq (f^i, v - u^i) & \forall v \in H_0^1(\Omega) \\ u^i \leq Mu^i;\ u^i \geq 0; & v \leq Mu^i \end{cases} \tag{1.1}$$

ISSN (electronic): 1443-5756

058-02

in which, Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, $a^i(\cdot,\cdot)$ are J- elliptic bilinear forms continuous on $H^1(\Omega) \times H^1(\Omega)$, assumed to be noncoercive, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ and f^i are J- regular functions.

This system arises in the management of energy production problems, where J-units are involved (see e.g. [1], [2] and the references therein). In the case studied here, Mu^i represents a "cost function" and the prototype encountered is

$$Mu^i = k + \inf_{\mu \neq i} u^\mu. \tag{1.2}$$

In (1.2) k represents the switching cost. It is positive when the unit is "turned on" and equal to zero when the unit is "turned off". Note also that operator M provides the coupling between the unknowns $u^1, \ldots, u^J$.

The L^∞-error estimate for the proposed system is a challenge not only for the practical motivation behind the problem, but also due to the inherent difficulty of convergence in this norm. Moreover, the interest in using such a norm for the approximation of VI and QVIs is that they are types of free boundary problems (cf. [4], [5]).

The coercive version of (1.1) is mathematically well understood. The numerical analysis study has also been considered in [3] and a quasi-optimal L^∞-error estimate established.

In this paper we propose to demonstrate that the standard finite element approximation applied to the noncoercive problem corresponding to system (1.1) is quasi-optimally accurate in $L^\infty(\Omega)$. For that purpose we shall develop an approach mainly based on both the L^∞- stability of the solution with respect to the right hand side and its characterization as the least upper bound of the set of subsolutions.

It is worth mentioning that the method presented in this paper is entirely different from the one developed for the coercive problem.

The paper is organized as follows. In Section 2 we state the continuous problem and study some qualitative proerties. In Section 3 we consider the discrete problem and achieve an analogous result to that of the continuous problem. In Section 4, we prove the main result.

2. THE CONTINUOUS PROBLEM

2.1. **Notations, Assumptions.** We are given functions $a^i_{jk}(x)$ in $C^{1,\alpha}(\bar{\Omega})$, $a^i_k(x)$, $a^i_0(x)$ in $C^{0,\alpha}(\overline{\Omega})$ such that:

$$\sum_{1 \leq j,k \leq N} a^i_{jk}(x)\xi_j\xi_k \geqq \alpha \left|\zeta\right|^2; \ \zeta \in \mathbb{R}^N; \ \alpha > 0, \tag{2.1}$$

$$a^i_0(x) \geqq \beta > 0 \ \ (x \in \Omega). \tag{2.2}$$

We define the second order differential operators

$$\mathcal{A}^i\varphi = \sum_{1 \leq j,k \leq N} \frac{\partial}{\partial x_j} a^i_{jk} \frac{\partial \varphi}{\partial x_k} + \sum_{k=1}^N a^i_k \frac{\partial \varphi}{\partial x_k} + a^i_0 \varphi \tag{2.3}$$

and the associated variational forms: for any $u, v \in H^1_0(\Omega)$

$$a^i(u,v) = \int_\Omega \left(\sum_{1 \leq j,k \leq N} a^i_{jk}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_k} + \sum_{k=1}^N a^i_k(x) \frac{\partial u}{\partial x_k} v + a^i_0(x) uv)dx \right). \tag{2.4}$$

We are also given right hand side $f^1, \ldots, f^{\ J}$ such that

$$f^i \in C^{0,\alpha}(\overline{\Omega}); \ f^i \geq f^0 > 0. \tag{2.5}$$

Throughout the paper $U = \partial(F, MU)$ will denote the solution of system (1.1) where $F = (f^1, \ldots, f^J)$ and $MU = (Mu^1, \ldots, Mu^J)$.

2.2. **Existence, Uniqueness and Regularity.** To solve the noncoercive problem, we transform (1.1) into the following auxiliary system: find $U = (u^1, \ldots, u^J) \in (H_0^1(\Omega))^J$ such that:

$$\begin{cases} b^i(u^i, v - u^i) \geqq (f^i + \lambda u^i, v - u^i) \quad \forall v \in H_0^1(\Omega) \\ u^i \le Mu^i;\ u^i \ge 0;\ v \le Mu^i\,, \end{cases} \tag{2.6}$$

where

$$b^i(u, v) = a^i(u,\ v) + \lambda(v, v) \tag{2.7}$$

and $\lambda > 0$ is large enough such that:

$$b^i(v,\ v) \ge \gamma \|v\|^2_{H^1(\Omega)} \ \ \gamma > 0;\ \ \forall v \in H^1(\Omega). \tag{2.8}$$

Let us recall just the main steps leading to the existence of a unique solution to system (1.1). For more details, we refer the reader to ([1]).

Let $\mathbb{H}^+ = (L_+^\infty(\Omega))^J = \{V = (v^1, \ldots, v^J)$ such that $v^i \in L_+^\infty(\Omega)\}$, equipped with the norm:

$$\|V\|_\infty = \max_{1 \le i \le J} \left\|v^i\right\|_{L^\infty(\Omega)}, \tag{2.9}$$

where $L_+^\infty(\Omega)$ is the positive cone of $L^\infty(\Omega)$. We introduce the following mapping

$$\begin{aligned} T : \mathbb{H}^+ &\longrightarrow \mathbb{H}^+ \\ W &\longrightarrow TW = (\zeta^1, \ldots, \zeta^J) \end{aligned} \tag{2.10}$$

where $\forall\ i = 1, \ldots, J,\ \ \zeta^i = \sigma(f^i + \lambda w^i;\ Mw^i)$ is solution to the following VI:

$$\begin{cases} b^i(\zeta^i, v - \zeta^i) \geqq (f^i + \lambda w^i, v - \zeta^i) & \forall v \in H_0^1(\Omega) \\ \zeta^i \le Mw^i\,, & v \le Mw^i \end{cases}. \tag{2.11}$$

Problem (2.11), being a coercive variational inequality, thanks to [12], it has a unique solution.

Let us also define the vector $\hat{U}^0 = (\hat{u}^{1,0}, \ldots, \hat{u}^{J,0})$, where $\forall i = 1, \ldots, J,\ \ \hat{u}^{i,0}$ is solution to the equation

$$a^i(\hat{u}^{i,0}, v) = (f^i, v)\ \forall v \in H_0^1(\Omega). \tag{2.12}$$

Since $f^i \ge 0$, there exists a unique positive solution to problem (2.12). Moreover, $\hat{u}^{i,0} \in W^{2,,p}(\Omega),\ p < \infty$ (Cf. e.g., [1]).

Proposition 2.1. *(Cf.* [1]*) Under the preceding notations and assumptions, the mapping T is increasing, concave and satisfies: $TW \le \hat{U}^0$, $\forall\ W \in \mathbb{H}^+$ such that $W \le \hat{U}^0$.*

The mapping T clearly generates the following iterative scheme.

2.3. **A Continuous Iterative Scheme.** Starting from $\hat{U}^0$ defined in (2.12) (resp. $\check{U}^0 = (0, \ldots, 0)$) , we define the sequences below

$$\hat{U}^{n+1} = T\hat{U}^n;\ n = 0, 1, \ldots \tag{2.13}$$

(resp.)

$$\check{U}^{n+1} = T\check{U}^n;\ n = 0, 1, \ldots. \tag{2.14}$$

Theorem 2.2. *(cf.* [1]*) Let* $\mathbb{C} = \{W \in \mathbb{H}^+ \text{ such that } W \leq \hat{U}^0\}$*. Then, under conditions of Proposition 2.1 the sequences* $(\hat{U}^n)$ *and* $(\check{U}^n)$ *remain in* $\mathbb{C}$ *. Moreover, they converge monotonically to the unique solution of system (1.1).*

Theorem 2.3. *(cf.* [1]*) Under the preceding assumptions, the solution* $(u^1, \ldots, u^J)$ *of system (1.1) belongs to* $(W^{2,p}(\Omega))^J$; $2 \leq p < \infty$.

In what follows, we shall give a monotonicity and an L^∞ stability property for the solution of system (1.1). These properties together with the notion of subsolution will play a crucial role in proving the main result of this paper.

2.4. **A Monotonicity Property.** Let $F = (f^1, \ldots, f^J)$; $\tilde{F} = (\tilde{f}^1, \ldots, \tilde{f}^J)$ be two families of right hands side and $U = \partial(F, MU) = (u^1, \ldots, u^J)$; $\tilde{U} = \partial(\tilde{F}, M\tilde{U}) = (\tilde{u}^1, \ldots, \tilde{u}^J)$ the corresponding solutions to system (1.1), respectively.

Theorem 2.4. *If* $F \geq \tilde{F}$ *then* $\partial(F, MU) \geq \partial(\tilde{F}, M\tilde{U})$.

Proof. We proceed by induction. For that let us associate with U and $\tilde{U}$ the following iterations

$$\hat{U}^n = (\hat{u}^{1,n}, \ldots, \hat{u}^{J,n}) \text{ and } \widetilde{\hat{U}}^n = (\widetilde{\hat{u}}^{1,n}, \ldots, \widetilde{\hat{u}}^{J,n})$$

respectively. Then, from (2.10), (2.11), (2.13) we clearly have

$$\hat{u}^{i,n+1} = \sigma(f^i + \lambda\hat{u}^{i,n},\, M\hat{u}^{i,n}) \text{ and } \widetilde{\hat{u}}^{i,n+1} = \sigma(f^i + \lambda\widetilde{\hat{u}}^{i,n},\, M\widetilde{\hat{u}}^{i,n}),$$

where $\hat{U}^0 = (\hat{u}^{1,0}, \ldots, \hat{u}^{J,0})$ and $\widetilde{\hat{U}^0} = (\widetilde{\hat{u}}^{1,0} \ldots, \widetilde{\hat{u}}^{J,0})$ are solutions to equation (2.12) with right hand sides F and $\tilde{F}$, respectively.

Clearly, $f^i \geq \tilde{f}^i$ implies $\hat{u}^{i,0} \geq \widetilde{\hat{u}}^{i,0}$. So, $f^i + \lambda\, \hat{u}^{i,0} \geq \tilde{f}^i + \lambda\widetilde{\hat{u}}^{i,0}$ and $M\hat{u}^{i,0} \geq M\widetilde{\hat{u}}^{i,0}$. Therefore, using standard comparison results in coercive variational inequalities, we get $\hat{u}^{i,1} \geq \widetilde{\hat{u}}^{i,1}$.

Now assume that $\hat{u}^{i,n-1} \geq \widetilde{\hat{u}}^{i,n-1}$. Then, as $f^i \geq \tilde{f}^i$, applying the same comparison argument as before, we get $\hat{u}^{i,n} \geq \widetilde{\hat{u}}^{i,n}$. Finally, by Theorem 2.2, making n tend to ∞, we get $U \geq \tilde{U}$. This completes the proof. □

2.5. **A Continuous L^∞ Stability Property.** Using the above notations we have the following result.

Theorem 2.5. *Under conditions of Theorem 2.4, we have*

$$\left\|\partial(F, MU) - \partial(\tilde{F}, M\tilde{U})\right\|_\infty \leq \frac{1}{\beta}\left\|F - \tilde{F}\right\|_\infty. \tag{2.15}$$

Proof. Let us denote by $u^i = \sigma(f^i, Mu^i)$; $\tilde{u}^i = \sigma(\tilde{f}^i, M\tilde{u}^i)$ the ith components of U and $\tilde{U}$, respectively. Then, setting $\Phi_i = \frac{1}{\beta}\left\|f^i - \tilde{f}^i\right\|_{L^\infty(\Omega)}$, using (2.2) it is easy to see that $\forall i = 1, 2, \ldots, J$

$$f^i \leq \tilde{f}^i + \left\|f^i - \tilde{f}^i\right\|_{L^\infty(\Omega)} \leq \tilde{f}^i + \frac{a_0^i(x)}{\beta}\left\|f^i - \tilde{f}^i\right\|_{L^\infty(\Omega)} \leq \tilde{f}^i + (a_0^i(x)\Phi_i).$$

Hence, making use of Theorem 2.4, it follows that

$$\begin{aligned}\sigma(f^i, Mu^i) &\leq \sigma(\tilde{f}^i + (a_0^i(x)\Phi_i, M(\tilde{u}^i + \Phi_i)) \\ &\leq \sigma(\tilde{f}^i, M\tilde{u}^i) + \Phi_i.\end{aligned}$$

Thus,

$$u^i - \tilde{u}^i \leq \Phi_i.$$

Interchanging the roles of f^i and $\tilde{f}^i$, we similarly get

$$\tilde{u}^i - u^i \leq \Phi_i.$$

This completes the proof. □

2.6. **Characterization of the solution of system (1.1) as the least upper bound of the set of sub-solutions.**

Definition 2.1. ([1]) $W = (w^1, .., w^J) \in (H_0^1(\Omega))^J$ is said to be a subsolution for the system of QVIs (1.1) if

$$\begin{cases} b^i(w^i, v) \leq (f + \lambda w^i, v) & \forall v \in H_0^1(\Omega)\ v \geq 0, \\ w^i \leq M w^i; & i = 1, \ldots, J. \end{cases} \tag{2.16}$$

Let $\mathbb{X}$ be the set of such subsolutions.

Theorem 2.6. *The solution of system of QVIs (1.1) is the maximum element of the set* $\mathbb{X}$.

Proof. It is a straightforward adaptation of ([1, p.358]) □

3. THE DISCRETE PROBLEM

Let Ω be decomposed into triangles and let τ_h denote the set of all those elements; $h > 0$ is the mesh size. We assume the family τ_h is regular and quasi-uniform.

Let V_h denote the standard piecewise linear finite element space and $\mathbb{B}^i,\ 1 \leq i \leq J$ be the matrices with generic entries:

$$(\mathbb{B}^i)_{ls} = b^i(\varphi_l, \varphi_s);\ 1 \leq l, s \leq m(h), \tag{3.1}$$

where $\varphi_s,\ s = 1, 2, \ldots m(h)$ are the nodal basis functions and r_h is the usual interpolation operator.

The discrete maximum principle assumption (dmp): We assume that the $\mathbb{B}^i$ are M-matrices (cf. [13]).

In this section, we shall see that the discrete problem below inherits all the qualitative properties of the continuous problem, provided the **dmp** is satisfied. Their respective proofs shall be omitted, as they are very similar to their continuous analogues.

Let $\mathbb{V}_h = (V_h)^J$. The noncoercive system of QVIs consists of seeking $U_h = (u_h^1, \ldots, u_h^J) \in \mathbb{V}_h$ such that

$$\begin{cases} a^i(u_h^i, v - u_h^i) \geqq (f^i, v - u_h^i) & \forall v \in V_h \\ u_h^i \leq r_h M u_h^i, & v \leq r_h M u_h^i, \end{cases} \tag{3.2}$$

or equivalently

$$\begin{cases} b^i(u_h^i, v - u_h^i) \geqq (f^i + \lambda w^i, v - u_h^i) & \forall v \in V_h \\ u_h^i \leq r_h M u_h^i, & v \leq r_h M u_h^i. \end{cases} \tag{3.3}$$

Let $\hat{U}_h^0$ be the piecewise linear approximation of $\hat{U}^0$ defined in (2.12):

$$a^i(\hat{u}_h^{i,0}, v) = (f^i, v)\ \forall v \in V_h;\quad 1 \leq i \leq J \tag{3.4}$$

and consider the following discrete mapping

$$\begin{aligned} T_h : \mathbb{H}^+ &\longrightarrow \mathbb{V}_h \\ W &\longrightarrow TW = (\zeta_h^1, \ldots, \zeta_h^J) \end{aligned} \tag{3.5}$$

where, $\forall\, i = 1, \ldots, J$, ζ_h^i is the solution of the following discrete VI:

$$\left\{\begin{array}{ll} b^i(\zeta_h^i, v - \zeta_h^i) \geqq (f^i + \lambda w^i, v - \zeta_h^i) & \forall v \in V_h, \\ \\ \zeta_h^i \leq r_h M w^i, & v \leq r_h M w^i. \end{array}\right. \tag{3.6}$$

Proposition 3.1. *Let the* ***dmp*** *hold. Then T_h is increasing, concave and satisfies $T_h W \leq \hat{U}_h^0$ $\forall W \in \mathbb{H}^+,\ W \leq \hat{U}_h^0$.*

3.1. **A Discrete Iterative Scheme.** We associate with the mapping T_h the following discrete iterative scheme: starting from $\hat{U}_h^0$ defined in (3.4), and $\check{U}_h^0 = 0$, we define:

$$\hat{U}_h^{n+1} = T_h \hat{U}_h^n \quad n = 0, 1, \ldots \tag{3.7}$$

and

$$\check{U}_h^{n+1} = T_h \check{U}_h^n \quad n = 0, 1, \ldots \tag{3.8}$$

respectively.

Similar to the continuous case, the following theorem establishes the monotone convergence of the above discrete sequences to the solution of system (3.2).

Theorem 3.2. *Let $\mathbb{C}_h = \{W \in \mathbb{H}^+$ such that $W \leq \hat{U}_h^0\}$. Then, under the* ***dmp,*** *the sequences $(\hat{U}_h^n)$ and $(\check{U}_h^n)$ remain in $\mathbb{C}_h$. Moreover, they converge monotonically to the unique solution of system (3.2).*

3.2. **A Discrete Monotonicity Property.** Let $F = (f^1, \ldots, f^J)$ and $\tilde{F} = (\tilde{f}^1, \ldots, \tilde{f}^J)$ be two families of right hand sides, and $U_h = \partial_h(F, MU_h)$, $\tilde{U}_h = \partial_h(\tilde{F}, M\tilde{U}_h)$ the corresponding solutions to system (3.2), respectively.

Theorem 3.3. *Under the* ***dmp****, if $F \geq \tilde{F}$ then $\partial_h(F, MU_h) \geq \partial_h(\tilde{F}, M\tilde{U}_h)$.*

3.3. **A Discrete L^∞ Stability Property.**

Theorem 3.4. *Under conditions of Theorem 3.3, we have*

$$\left\| \partial_h(F, MU_h) - \partial_h(\tilde{F}, M\tilde{U}_h) \right\|_\infty \leq \frac{1}{\beta} \left\| F - \tilde{F} \right\|_\infty. \tag{3.9}$$

3.4. **Characterization of the solution of system (3.2) as the least upper bound of the set of discrete sub-solutions.**

Definition 3.1. $W = (w_h^1, .., w_h^J) \in \mathbb{V}_h$ is said to be a subsolution for the system of QVIs (3.2) if

$$\left\{\begin{array}{l} b^i(w_h^i, \varphi_s) \leq (f^i + \lambda w_h^i, \varphi_s) \quad \forall \varphi_s;\ s = 1, \ldots, m(h); \\ \\ w_h^i \leq r_h M w_h^i. \end{array}\right. \tag{3.10}$$

Let $\mathbb{X}_h$ be the set of discrete subsolutions.

Theorem 3.5. *Under the* ***dmp,*** *the solution of system of QVIs (3.2) is the maximum element of the set $\mathbb{X}_h$.*

4. THE FINITE ELEMENT ERROR ANALYSIS

This section is dedicated to prove that the proposed method is quasi-optimally accurate in $L^\infty(\Omega)$, according to the approximation theory. To achieve that, we first introduce two auxiliary coercive systems of QVIs and give some intermediate error estimates.

4.1. **Definition of Two Auxiliary Coercive System of QVIs.**

1. A Continuous system of QVIs: Find $\bar{U}^{(h)} = (\bar{u}^{1(h)}, \ldots, \bar{u}^{J(h)}) \in (H_0^1(\Omega))^J$ solution to:

$$(4.1) \qquad \begin{cases} b^i(\bar{u}^{i(h)}, v - \bar{u}^{i(h)}) \geq (f^i + \lambda u_h^i, v - \bar{u}^{i(h)}) & \forall v \in H_0^1(\Omega); \\ \bar{u}^{i(h)} \leq M\bar{u}^{i(h)}; & v \leq M\bar{u}^{i(h)}, \end{cases}$$

where $U_h = (u_h^1, \ldots, u_h^J)$ is the solution of the discrete system of QVIs (3.2).

Lemma 4.1. *(cf.* [3]*)*

$$(4.2) \qquad \left\| \bar{U}^{(h)} - U_h \right\|_\infty \leq Ch^2 \left| Logh \right|^3.$$

2. A Discrete System of Coercive QVIs: Find $\bar{U}_h = \left(\bar{u}_h^1, \ldots, \bar{u}_h^J\right) \in \mathbb{V}_h$ solution to:

$$(4.3) \qquad \begin{cases} b^i(\bar{u}_h^i, v - \bar{u}_h^i) \geq (f^i + \lambda u^i, v - \bar{u}_h^i) & \forall v \in V_h; \\ u \leq r_h M\bar{u}_h^i; & v \leq r_h M\bar{u}_h^i, \end{cases}$$

where $U = (u^1, \ldots, u^J)$ is the solution of the continuous system of QVIs (1.1).

Lemma 4.2. *(cf.* [3]*)*

$$(4.4) \qquad \left\| \bar{U}_h - U \right\|_\infty \leq Ch^2 \left| Logh \right|^3.$$

4.2. L^∞**- Error Estimate For System (1.1).**

Theorem 4.3. *Let U and U_h be the solutions of the noncoercive problems (1.1) and (3.2), respectively. Then, then under conditions of Theorem 2.3, and Lemmas 4.1, 4.2, we have the error estimate*

$$(4.5) \qquad \left\| U - U_h \right\|_\infty \leq Ch^2 \left| Logh \right|^3.$$

Proof. The proof will be carried out in three steps.

Step 1. It consists of constructing a vector of continuous functions $\beta^{(h)} = (\beta^{1(h)}, \ldots, \beta^{J(h)})$ such that:

$$(4.6) \qquad \beta^{(h)} \leq U \quad \text{and} \quad \left\| \beta^{(h)} - U_h \right\|_\infty \leq Ch^2 \left| Logh \right|^3$$

Indeed, $\bar{U}^{(h)}$ being solution to system (4.1) it is easy to see that $\bar{U}^{(h)}$ is also a subsolution, i.e., $\forall i = 1, \ldots, J$

$$\begin{cases} b^i(\bar{u}^{i(h)}, v) \leq (f^i + \lambda u_h^i, v) & \forall v \in H_0^1(\Omega),\ v \geq 0, \\ \bar{u}^{i(h)} \leq M\bar{u}^{i(h)}; & v \leq M\bar{u}^{i(h)}. \end{cases}$$

This implies

$$\begin{cases} b^i\left(\bar{u}^{i(h)}, v\right) \leq \left(f^i + \lambda \left\| u_h^i - \bar{u}^{i(h)} \right\|_{L^\infty(\Omega)} + \lambda \bar{u}^{i(h)}, v\right) & \forall v \in H_0^1(\Omega),\ v \geq 0, \\ \bar{u}^{i\,(h)} \leq M\bar{u}^{i(h)}; & v \leq M\bar{u}^{i(h)}, \end{cases}$$

and, from Theorem 2.6, it follows that

$$(4.7) \qquad \bar{U}^{(h)} \leq \tilde{U} = \partial(\tilde{F}, M\tilde{U})$$

with $\tilde{F} = F + \lambda \left\| \bar{U}^{(h)} - U_h \right\|_\infty$. Therefore, using both the stability Theorem 2.5 and estimate (4.2) we get

$$(4.8) \qquad \left\| U - \tilde{U} \right\|_\infty \leq \lambda \left\| \bar{U}^{(h)} - U_h \right\|_\infty \leq Ch^2 \left| Logh \right|^3$$

which combined with (4.7) yields:

$$\bar{U}^{(h)} \leq U + Ch^2 \left| Logh \right|^3.$$

Finally, taking $\beta^{(h)} = \bar{U}^{(h)} - Ch^2 \left| Logh \right|^3$, (4.6) follows.

Step 2. Similarly to Step 1., we construct a vector of discrete functions $\alpha_h = (\alpha_h^1, \ldots, \alpha_h^J)$ satisfying

$$\alpha_h \leq U_h \text{ and } \left\| \alpha_h - U \right\|_\infty \leq Ch^2 \left| Logh \right|^3. \tag{4.9}$$

Indeed, $\bar{U}_h$ being solution to system (4.3), it is also a subsolution, i.e.

$$\begin{cases} b^i(\bar{u}_h^i, \varphi_s) \leq (f^i + \lambda u^i, \varphi_s) \; \forall \varphi_s; & s = 1, \ldots, m(h); \\ \\ u \leq M\bar{u}_h^i; & v \leq M\bar{u}_h^i, \end{cases}$$

which implies

$$\begin{cases} b^i(\bar{u}_h^i, \varphi_s) \leq (f^i + \lambda \left\| u^i - \bar{u}_h^i \right\|_{L^\infty(\Omega)} + \lambda \bar{u}_h^i, \varphi_s) \; \forall \varphi_s; & s = 1, \ldots, m(h) \\ \\ u \leq M\bar{u}_h^i; & v \leq M\bar{u}_h^i. \end{cases}$$

Hence, letting $\tilde{F} = F + \lambda \left\| \bar{U}_h - U \right\|_\infty$ and applying Theorem 3.5, we obtain that

$$\bar{U}_h \leq \tilde{U}_h = \partial_h(\tilde{F}, M\tilde{U}_h). \tag{4.10}$$

Therefore, using both Theorem 3.4 and estimate (4.4), we get

$$\left\| U_h - \tilde{U}_h \right\|_\infty \leq \lambda \left\| \bar{U}_h - U \right\|_\infty \leq Ch^2 \left| Logh \right|^3 \tag{4.11}$$

which combined with (4.10), yields

$$\bar{U}_h \leq U_h + Ch^2 \left| Logh \right|^3.$$

Finally, taking $\alpha_h = \bar{U}_h - Ch^2 \left| Logh \right|^3$, we immediately get (4.9).

Step 3. Now collecting the results of Steps 1 and 2., we derive the desired error estimate (4.5) as follows:

$$\begin{aligned} U_h &\leq \beta^{(h)} + Ch^2 \left| Logh \right|^3 \\ &\leq U + Ch^2 \left| Logh \right|^3 \\ &\leq \alpha_h + Ch^2 \left| Logh \right|^3 \leq U_h + Ch^2 \left| Logh \right|^3. \end{aligned}$$

Thus

$$\left\| U - U_h \right\|_\infty \leq Ch^2 \left| Logh \right|^3.$$

□

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